

Mahler measure and 3-hyperbolic volumes

Matilde N. Lalin

University of Texas at Austin

`mlalin@math.utexas.edu`

`http://www.ma.utexas.edu/users/mlalin`

Mahler measure and Lehmer's question

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

Look at $\frac{\Delta_{n+1}}{\Delta_n}$

$$\lim_{n \rightarrow \infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

For

$$P(x) = a \prod_i (x - \alpha_i)$$

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}$$

$$m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|$$

Kronecker's Lemma:

$$P \in \mathbb{Z}[x], P \neq 0,$$

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

Lehmer (1933)

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) \\ = \log(1.176280818\dots) = 0.162357612\dots$$

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

There exists $C > 0$, for all $P(x) \in \mathbb{Z}[x]$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

Is the polynomial above the best possible?

Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |e^{2\pi i\theta} - \alpha| d\theta = \log^+ |\alpha| = \log \max\{1, |\alpha|\}$$

recovers one-variable case.

Some properties

- $m(P \cdot Q) = m(P) + m(Q)$
- $m(P) \geq 0$ if P has integral coefficients.
- α algebraic number, and P_α minimal polynomial over \mathbb{Q} ,

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}] h(\alpha)$$

where h is the logarithmic Weil height.

- Boyd & Lawton : $P \in \mathbb{C}[x_1, \dots, x_n]$

$$\begin{aligned} & \lim_{k_2 \rightarrow \infty} \dots \lim_{k_n \rightarrow \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) \\ &= m(P(x_1, \dots, x_n)) \end{aligned}$$

Jensen's formula \longrightarrow simple expression in one-variable case.

Several-variable case?

Examples in several variables

Smyth (1981)

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

$$m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3)$$

$$L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s}$$

$$\chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \pmod{3} \\ -1 & n \equiv -1 \pmod{3} \\ 0 & n \equiv 0 \pmod{3} \end{cases}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Bloch – Wigner Dilogarithm

Dilogarithm:

$$\operatorname{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad x \in \mathbb{C}, \quad |x| < 1$$

$$\operatorname{Li}_2(x) := - \int_0^x \log(1-t) \frac{dt}{t} \quad x \in \mathbb{C} \setminus (1, \infty)$$

Bloch – Wigner Dilogarithm:

$$D(x) := \operatorname{Im}(\operatorname{Li}_2(x)) + \arg(1-x) \log|x|$$

real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$

Properties:

$$1. D(\bar{x}) = -D(x) \quad (\Rightarrow D|_{\mathbb{R}} \equiv 0)$$

$$2. D(x) = -D\left(\frac{1}{x}\right) = -D(1-x)$$

$$3. -2 \int_0^\theta \log |2 \sin t| dt = D(e^{2i\theta})$$

4. Five-term relation

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0$$

Hyperbolic volumes

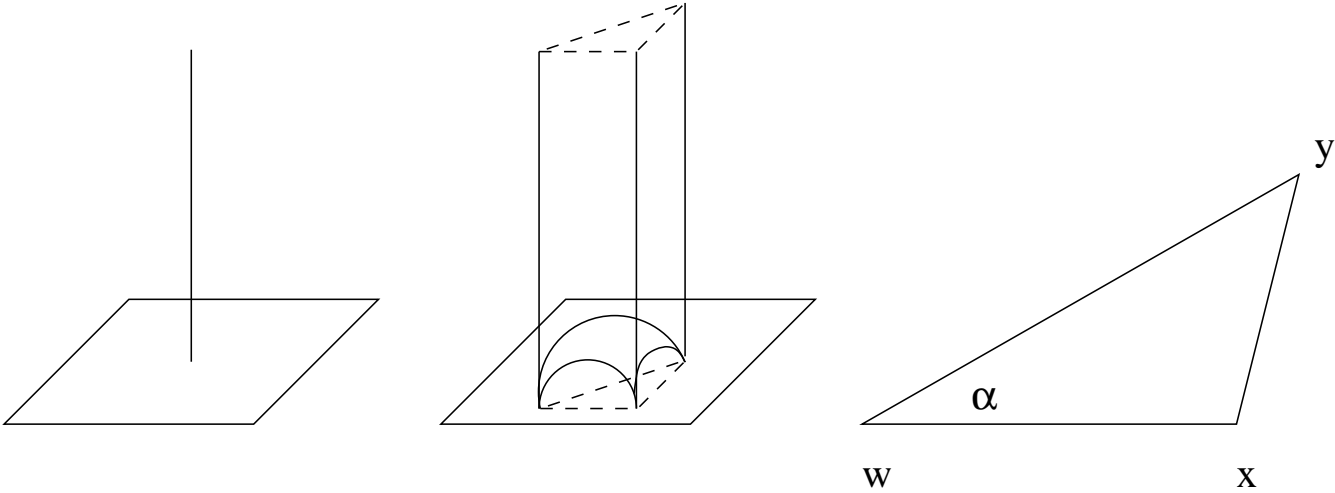
$$\mathbb{H}^3 \cong \mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}$$

Volume Element:

$$\frac{dx dy dz}{z^3}$$

Ideal Tetrahedron:

vertices in $\partial\mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$

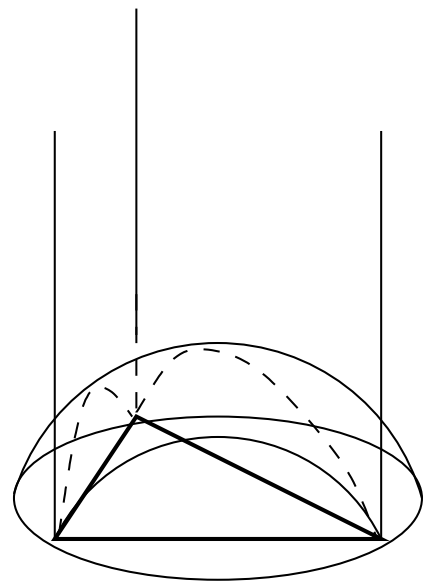
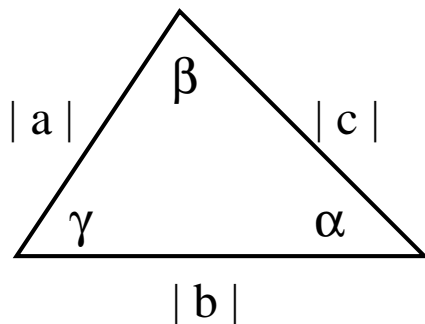


$$\text{Vol} \left(\pi^* \left(\triangle_{wxy} \right) \right) = D \left(\frac{y - w}{x - w} \right) = D \left(\left| \frac{y - w}{x - w} \right| e^{i\alpha} \right)$$

Cassaigne and Maillot's example

$$\pi m(a + bx + cy) =$$

$$\begin{cases} D\left(\left|\frac{a}{b}\right| e^{i\gamma}\right) + \alpha \log |a| + \beta \log |b| + \gamma \log |c| & \Delta \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \Delta \end{cases}$$



Results

1. Cassaigne and Maillot(2000):

$$y = \frac{ax + b}{c}$$

2. Vandervelde(2003):

$$y = \frac{bx + d}{ax + c}$$

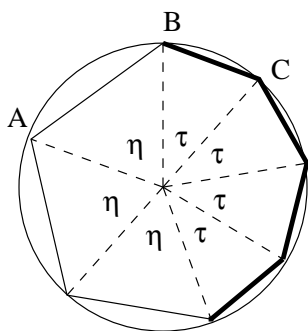
3. L(2004):

$$y = \frac{x^n - 1}{t(x^m - 1)} = \frac{x^{n-1} + \dots + 1}{t(x^{m-1} + \dots + 1)}$$

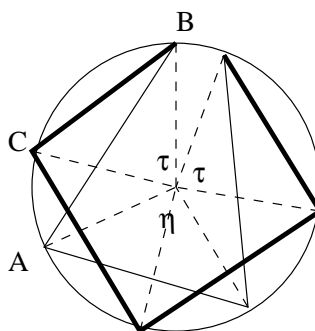
$$R_t(x, y) = t(x^m - 1)y - (x^n - 1)$$

P cyclic plane polygon is admissible of type (m, n) if

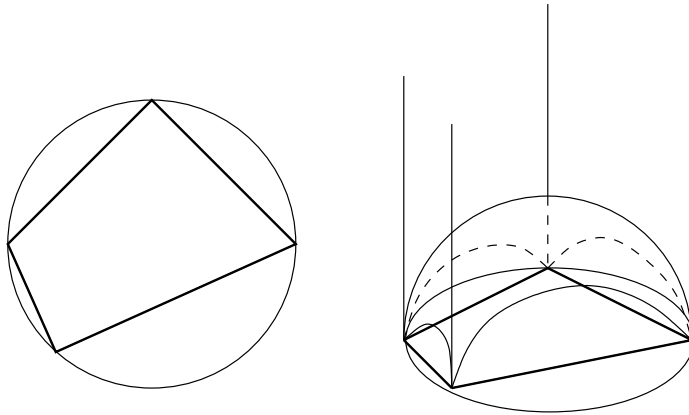
- P has m sides of length t (with $t \in \mathbb{R}_{>0}$) and n of length 1.
- Sides of length t wind around the center in the same direction and sides of length 1 wind around the center in the same direction, which may be opposite from the direction of the sides of length t .



a



b



Theorem 1

$$\pi m(R_t(x, y)) = \pi \log |t|$$

$$+ \frac{2}{mn} \sum \epsilon_k \text{Vol}(\pi^*(P_k)) + \epsilon \sum_{k=1}^N (-1)^k \log |t| \arg \alpha_k$$

$\epsilon, \epsilon_k = \pm 1$, P_k are all the admissible polygons of type (m, n) .

An example

$$y = \frac{x^3 - 1}{t(x^2 - 1)}$$

$$R_t(x, y) = t(x^2 - 1)y - (x^3 - 1)$$

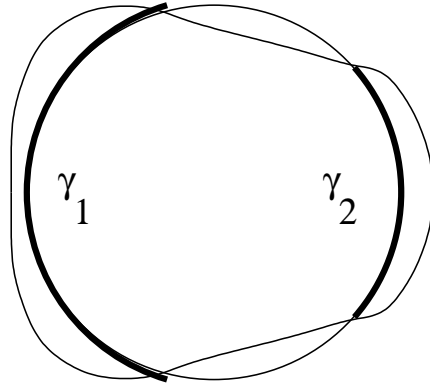
$$m(R_t) - \log |t|$$

$$= \begin{cases} \frac{2}{2 \cdot 3 \cdot \pi} (\epsilon_1 \text{Vol}(\pi^*(P_1)) + \epsilon_2 \text{Vol}(\pi^*(P_2))) \\ \quad + \frac{\sigma_1 - \sigma_2}{\pi} \log |t| & 0 < t < \frac{3}{2} \\ \frac{2}{2 \cdot 3 \cdot \pi} \epsilon_1 \text{Vol}(\pi^*(P_1)) + \frac{\sigma_1}{\pi} \log |t| & \frac{3}{2} \leq t \end{cases}$$

$$\begin{aligned}
& m(t(x^2 - 1)y - (x^3 - 1)) \\
&= \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log |x^3 - 1 - t(x^2 - 1)y| \frac{dx dy}{x y} \\
&= \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log |t(x^2 - 1)| \frac{dx}{x} \\
&+ \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log \left| \frac{x^3 - 1}{t(x^2 - 1)} - y \right| \frac{dx dy}{x y} \\
&= \log t + \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log^+ \left| \frac{x^3 - 1}{t(x^2 - 1)} \right| \frac{dx}{x}
\end{aligned}$$

by Jensen's formula

$$= \frac{1}{2\pi i} \int_{\gamma} \log \left| \frac{x^3 - 1}{t(x^2 - 1)} \right| \frac{dx}{x}$$



Determine for which points we have

$$\left| \frac{x^3 - 1}{t(x^2 - 1)} \right| = 1$$

$$\frac{x^3 - 1}{t(x^2 - 1)} \cdot \frac{x^{-3} - 1}{t(x^{-2} - 1)} = 1$$

$$x^4 + (2 - t^2)x^3 + (3 - 2t^2)x^2 + (2 - t^2)x + 1 = 0$$

Roots $\alpha_1, \alpha_1^{-1}, \alpha_2, \alpha_2^{-1}$

$$\operatorname{Re} \alpha_1 = \frac{t^2 - 2 - t\sqrt{t^2 + 4}}{4} \quad \text{for } 0 < t$$

$$\operatorname{Re} \alpha_2 = \frac{t^2 - 2 + t\sqrt{t^2 + 4}}{4} \quad \text{for } 0 < t < \frac{3}{2}$$

Let $\sigma_i = \arg \alpha_i$, $\operatorname{Im} \alpha_i \geq 1$

$$\pi > \sigma_1 > \frac{2\pi}{3}$$

$$\frac{2\pi}{3} > \sigma_2 > 0$$

$$\int_{\alpha}^{\beta} \log |x^n - 1| \frac{dx}{ix} = \frac{D(\alpha^n) - D(\beta^n)}{n}$$

For $0 < t < \frac{3}{2}$

$$\begin{aligned}
& m(t(x^2 - 1)y - (x^3 - 1)) - \log t \\
&= \frac{1}{2\pi i} \int_{\gamma_1 \cup \gamma_2} \log \left| \frac{x^3 - 1}{t(x^2 - 1)} \right| \frac{dx}{x} \\
&= \frac{D(\alpha_1^{-3}) - D(\alpha_1^3) + D(\alpha_2^3) - D(\alpha_2^{-3})}{3(2\pi)} \\
&\quad - \frac{D(\alpha_1^{-2}) - D(\alpha_1^2) + D(\alpha_2^2) - D(\alpha_2^{-2})}{2(2\pi)} \\
&\quad - \frac{2(\sigma_1 - \sigma_2)}{2\pi} \log t \\
&= \frac{3D(\alpha_1^2) - 2D(\alpha_1^3)}{6\pi} - \frac{3D(\alpha_2^2) - 2D(\alpha_2^3)}{6\pi} \\
&\quad - \frac{\sigma_1 - \sigma_2}{\pi} \log t
\end{aligned}$$

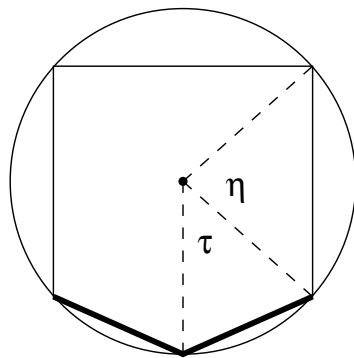
For $\frac{3}{2} \leq t$

$$\begin{aligned} & m(t(x^2 - 1)y - (x^3 - 1)) - \log t \\ &= \frac{3D(\alpha_1^2) - 2D(\alpha_1^3)}{6\pi} - \frac{\sigma_1}{\pi} \log t \end{aligned}$$

For α_1 ,

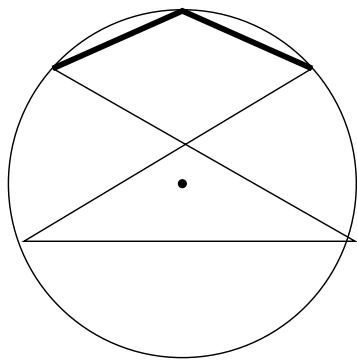
$$\pi > \sigma_1 > \frac{2\pi}{3} \longrightarrow \eta = 2\pi - 2\sigma_1, \quad \tau = 3\sigma_1 - 2\pi$$

$$3\eta + 2\tau = 2\pi$$

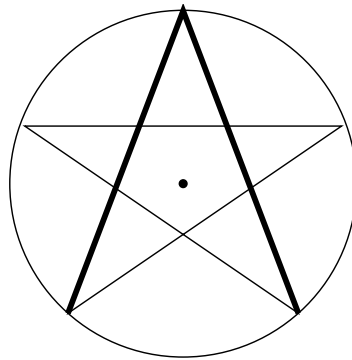


For α_2

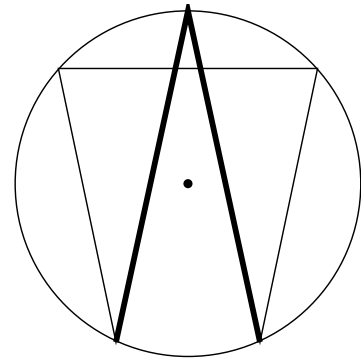
$0 < t < \frac{1}{\sqrt{2}}$	$\frac{2\pi}{3} > \sigma_2 > \frac{\pi}{2}$	$\eta = 2\pi - 2\sigma_2$ $\tau = 2\pi - 3\sigma_2$	$3\eta - 2\tau = 2\pi$
$\frac{1}{\sqrt{2}} < t < \frac{2}{\sqrt{3}}$	$\frac{\pi}{2} > \sigma_2 > \frac{\pi}{3}$	$\eta = 2\sigma_2$ $\tau = 2\pi - 3\sigma_2$	$3\eta + 2\tau = 4\pi$
$\frac{2}{\sqrt{3}} < t < \frac{3}{2}$	$\frac{\pi}{3} > \sigma_2 > 0$	$\eta = 2\sigma_2$ $\tau = 3\sigma_2$	$3\eta - 2\tau = 0$



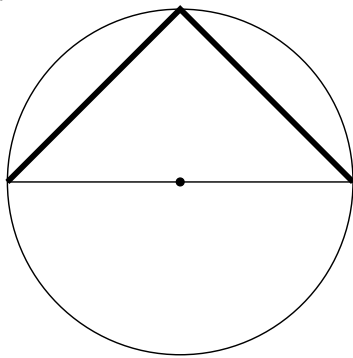
a



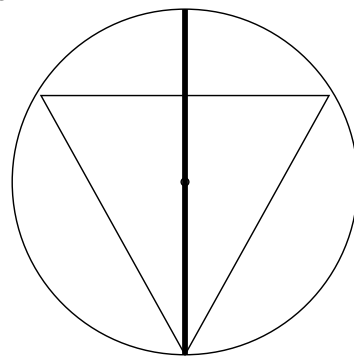
b



c



d



e

Analogies with the case of A -polynomials

M orientable, complete, one-cusped, hyperbolic manifold.

$$M = \bigcup_{j=1}^k \Delta(z_j)$$

- *Gluing equations.* ($r_{j,i}, r'_{j,i}$ integers depending on M).

$$\prod_{i=1}^k z_i^{r_{j,i}} (1-z_i)^{r'_{j,i}} = \pm 1 \quad \text{for } j = 1, \dots, k$$

- *Completeness equations.* (l_i, l'_i, m_i, m'_i integers depending on M).

$$\prod_{i=1}^k z_i^{l_i} (1-z_i)^{l'_i} = \pm 1$$

$$\prod_{i=1}^k z_i^{m_i} (1-z_i)^{m'_i} = \pm 1$$

$\text{Im } z_i > 0 \rightsquigarrow$ geometric solution

$$\text{Vol}(M) = \sum_{j=1}^k D(z_j)$$

Boyd

$$\pi m(A) = \sum V_i$$

A is the A -polynomial.

$V_0 = \text{Vol}(M)$ and the other V_i are pseudovolumes.

We have for $t = 1$

$$\pi m(R_1(x, y)) = \frac{2}{mn} \sum \epsilon_k \text{Vol}(\pi^*(P_k))$$

Replace completeness relations by

$$\prod_{i=1}^k z_i^{l_i} (1 - z_i)^{l'_i} = x^2$$

$$\prod_{i=1}^k z_i^{m_i} (1 - z_i)^{m'_i} = y^2$$

"deformation parameters" x and y

A -polynomial is obtained by eliminating z_1, \dots, z_k from this and gluing equations.

$$U = \begin{pmatrix} l_1 & \cdots & l_k & l'_1 & \cdots & l'_k \\ m_1 & \cdots & m_k & m'_1 & \cdots & m'_k \\ r_{1,1} & \cdots & r_{1,k} & r'_{1,1} & \cdots & r'_{1,k} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ r_{k,1} & \cdots & r_{k,k} & r'_{k,1} & \cdots & r'_{k,k} \end{pmatrix}$$

Theorem 2 (*Neumann–Zagier*)

$$U J_{2k} U^t = 2 \begin{pmatrix} J_2 & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$J_{2p} = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}$$

$$\pi m(R_1(x, y)) = \frac{1}{mn} \sum \epsilon_k \text{Vol}(\pi^*(P'_k))$$

$$w = e^{i\eta} \text{ and } z = e^{i\tau}$$

$$\left\{ \begin{array}{l} w_1^\alpha z_1^\beta = x^2 \\ w_1^{-mn(m+n)\alpha} z_1^{-mn(m+n)\beta} (1-w_1)^{2n} \dots (1-w_m)^{2n} \\ \quad \cdot (1-z_1)^{-2m} \dots (1-z_n)^{-2m} = y^2 \\ w_1 \dots w_m z_1 \dots z_n = 1 \\ w_1 w_2^{-1} = 1 \\ \vdots \\ w_1 w_m^{-1} = 1 \\ z_1 z_2^{-1} = 1 \\ \vdots \\ z_1 z_n^{-1} = 1 \end{array} \right.$$

for $n\alpha - m\beta = 1$, satisfies Neumann–Zagier.

One of the branches (the one with $w = x^{2n}, z = x^{-2m}$), is

$$y^2 = \left(\frac{x^n - x^{-n}}{x^m - x^{-m}} \right)^{2mn}$$

Consider

$$\tilde{R}(x, y) = (x^m - x^{-m})^{mn} y - (x^n - x^{-n})^{mn}$$

$$mn \cdot m(R_1) = m(\tilde{R})$$

Hence

$$\pi m(\tilde{R}(x, y)) = \sum \epsilon_k \text{Vol}(\pi^*(P'_k))$$

Mahler measure of several-variable polynomials

Matilde N. Lalín

University of Texas at Austin

`mlalin@math.utexas.edu`

`http://www.ma.utexas.edu/users/mlalin`

More examples in two variables

Boyd & Rodriguez-Villegas (1997)

$$m \left(x + \frac{1}{x} + y + \frac{1}{y} - k \right) \stackrel{?}{=} \frac{L'(E_k, 0)}{B_k} \quad k \in \mathbb{N}$$

$$m \left(x + \frac{1}{x} + y + \frac{1}{y} - 4 \right) = 2L'(\chi_{-4}, -1)$$

$$m \left(x + \frac{1}{x} + y + \frac{1}{y} - 4\sqrt{2} \right) = L'(A, 0)$$

$$A : y^2 = x^3 - 44x + 112$$

More examples in several variables

L (2003)

$$\pi^n m \left(1 + \left(\frac{1 - x_1}{1 + x_1} \right) \cdots \left(\frac{1 - x_n}{1 + x_n} \right) z \right)$$

= combination of $\zeta(\text{odd}) / L(\chi_{-4}, \text{even})$

$$\pi^n m \left(1 + x + \left(\frac{1 - x_1}{1 + x_1} \right) \cdots \left(\frac{1 - x_n}{1 + x_n} \right) (1 + y)z \right)$$

= combination of $\zeta(\text{odd}) / L(\chi_{-4}, \text{even})$,
polylogarithms

$$\pi^n m \left(1 + \left(\frac{1 - x_1}{1 + x_1} \right) \cdots \left(\frac{1 - x_n}{1 + x_n} \right) x \right)$$

$$+ \left(1 - \left(\frac{1 - x_1}{1 + x_1} \right) \cdots \left(\frac{1 - x_n}{1 + x_n} \right) \right) y$$

= combination of $\zeta(\text{odd})$

Examples

$$\begin{aligned} \pi^3 m \left(1 + \left(\frac{1-x_1}{1+x_1} \right) \left(\frac{1-x_2}{1+x_2} \right) \left(\frac{1-x_3}{1+x_3} \right) z \right) \\ = 24L(\chi_{-4}, 4) + \pi^2 L(\chi_{-4}, 2) \end{aligned}$$

$$\begin{aligned} \pi^4 m \left(1 + \left(\frac{1-x_1}{1+x_1} \right) \cdots \left(\frac{1-x_4}{1+x_4} \right) z \right) \\ = 62\zeta(5) + \frac{14}{3}\pi^2\zeta(3) \end{aligned}$$

$$\pi^4 m \left(1 + x + \left(\frac{1-x_1}{1+x_1} \right) \left(\frac{1-x_2}{1+x_2} \right) (1+y)z \right) = 93\zeta(5)$$

Polylogarithms

The k th polylogarithm is

$$\text{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$P_k(x) := \text{Re}_k \left(\sum_{j=0}^k \frac{2^j B_j}{j!} (\log |x|)^j \text{Li}_{k-j}(x) \right)$$

B_j is j th Bernoulli number, $\text{Li}_0(x) \equiv -\frac{1}{2}$,

$\text{Re}_k = \text{Re}$ or Im if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$,
continuous in $\mathbb{P}^1(\mathbb{C})$.

P_k satisfies lots of functional equations

$$P_k\left(\frac{1}{x}\right) = (-1)^{k-1} P_k(x) \quad P_k(\bar{x}) = (-1)^{k-1} P_k(x)$$

Bloch–Wigner dilogarithm ($k = 2$)

$$D(x) := \operatorname{Im}(\operatorname{Li}_2(x)) + \arg(1 - x) \log |x|$$

Five-term relation

$$D(x) + D(1 - xy) + D(y) + D\left(\frac{1 - y}{1 - xy}\right) + D\left(\frac{1 - x}{1 - xy}\right) = 0$$

Examples from the world of resultants

D'Andrea & L (2003).

- $m(\text{Res}_{\{0,m,n\}})$

$$= m(\text{Res}_t(x + yt^m + t^n, z + wt^m + t^n)) =$$
$$\frac{2}{\pi^2}(-mP_3(\varphi^n) - nP_3(-\varphi^m) + mP_3(\phi^n) + nP_3(\phi^m))$$

$$0 \leq \varphi \leq 1 \quad \text{root of } x^n + x^{n-m} - 1 = 0$$

$$1 \leq \phi \quad \text{root of } x^n - x^{n-m} - 1 = 0$$

- $m(\text{Res}_{\{(0,0),(1,0),(0,1)\}}) = m \left(\begin{vmatrix} x & y & z \\ u & v & w \\ r & s & t \end{vmatrix} \right)$

$$= m((1-x)(1-y) - (1-z)(1-w)) = \frac{9\zeta(3)}{2\pi^2}$$

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F number field)
- L-function ($L_X = \zeta_F$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)
- Regulator map

$$r : K \rightarrow \text{smooth differential forms}$$
$$(r = \log |\cdot|)$$

Conjecture: special value of $L_X \sim_{\mathbb{Q}^*} \int_{\gamma} r(\xi)$

(E.g. Dirichlet class number formula, F real quadratic, $\zeta'_F(0) \sim_{\mathbb{Q}^*} \log |\epsilon|$ $\epsilon \in \mathcal{O}_F^*$)

An algebraic integration for Mahler measure

Deninger (1997) : General framework.

Rodriguez-Villegas (1997) : $P(x, y) \in \mathbb{C}[x, y]$

$$P(x, y) = y + x - 1 \quad C = \{P(x, y) = 0\}$$

$$m(P) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log |y + x - 1| \frac{dx}{x} \frac{dy}{y}$$

by Jensen's equality:

$$\begin{aligned} &= \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log^+ |1 - x| \frac{dx}{x} \\ &= \frac{1}{2\pi i} \int_{\gamma} \log |y| \frac{dx}{x} = -\frac{1}{2\pi} \int_{\gamma} \eta(x, y) \end{aligned}$$

where

$$\gamma = C \cap \{|x| = 1, |y| \geq 1\}$$

and

$$\eta(x, y) = \log |x| d \arg y - \log |y| d \arg x$$

$$d\eta(x, y) = \operatorname{Im} \left(\frac{dx}{x} \wedge \frac{dy}{y} \right)$$

- $\eta(x, y) = -\eta(y, x)$
- $\eta(x_1 x_2, y) = \eta(x_1, y) + \eta(x_2, y)$

$$\eta(x, 1 - x) = dD(x)$$

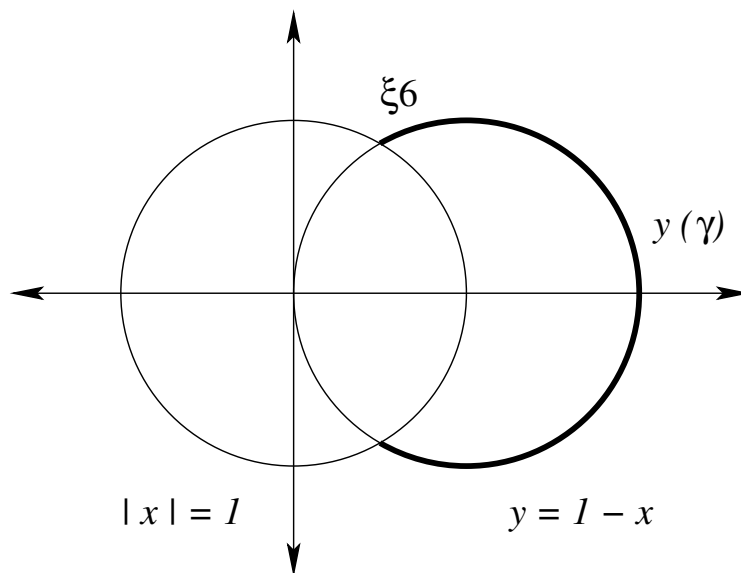
Use Stokes Theorem:

$$m(P) = -\frac{1}{2\pi} D(\partial\gamma)$$

$$x = e^{2\pi i\theta},$$

$$y(\gamma(\theta)) = 1 - e^{2\pi i\theta}, \quad \theta \in [1/6; 5/6]$$

$$\partial\gamma = [\bar{\xi}_6] - [\xi_6]$$



$$2\pi m(x + y + 1) = D(\xi_6) - D(\bar{\xi}_6)$$

$$= 2D(\xi_6) = \frac{3\sqrt{3}}{2}L(\chi_{-3}, 2)$$

In general,

$$P(x, y) \in \mathbb{C}[x, y]$$

$$m(P) = m(P^*) - \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$

In $\wedge^2(\mathbb{C}(C)^*) \otimes \mathbb{Q}$,

$$x \wedge y = \sum_j r_j z_j \wedge (1 - z_j)$$

$$\{x, y\} = 0 \text{ in } K_2(\mathbb{C}(C)) \otimes \mathbb{Q}.$$

Then

$$\int_{\gamma} \eta(x, y) = \sum r_j \int_{\gamma} \eta(z_j, 1 - z_j) = \sum r_j D(z_j)|_{\partial\gamma}.$$

Big picture

$$\cdots \rightarrow (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \rightarrow K_2(C, \partial\gamma) \rightarrow K_2(C) \rightarrow \cdots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

- $\eta(x, y)$ is exact, then $\{x, y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes' Theorem.

\rightsquigarrow dilogarithms \rightsquigarrow zeta function of a number field.

- $\partial\gamma = \emptyset$, then $\{x, y\} \in K_2(C)$. We have $\eta(x, y)$ is not exact.

\rightsquigarrow L-series of a curve.

We may get combinations of both situations.

The three-variable case

$$P(x, y, z) = (1-x) + (1-y)z \quad S = \{P(x, y, z) = 0\}$$

$$\begin{aligned} m(P) &= m(1-y) + \frac{1}{(2\pi i)^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{dx dy dz}{x y z} \\ &= \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{dx dy}{x y} \\ &= -\frac{1}{(2\pi)^2} \int_{\Gamma} \log |z| \frac{dx dy}{x y} \\ &= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, y, z) \end{aligned}$$

$$\Gamma = S \cap \{|x| = |y| = 1, |z| \geq 1\}$$

$$\begin{aligned}
\eta(x, y, z) = & \log |x| \left(\frac{1}{3} d \log |y| d \log |z| - d \arg y d \arg z \right) \\
& + \log |y| \left(\frac{1}{3} d \log |z| d \log |x| - d \arg z d \arg x \right) \\
& + \log |z| \left(\frac{1}{3} d \log |x| d \log |y| - d \arg x d \arg y \right)
\end{aligned}$$

$$d\eta(x, y, z) = \operatorname{Re} \left(\frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z} \right)$$

Theorem 3

$$\eta(x, 1-x, y) = d\omega(x, y)$$

where

$$\omega(x, y) = -D(x) d \arg y$$

$$+ \frac{1}{3} \log |y| (\log |1-x| d \log |x| - \log |x| d \log |1-x|)$$

$$\eta(x, y, z) = -\eta(x, 1 - x, y) - \eta(y, 1 - y, x)$$

$$m((1 - x) + (1 - y)z) = \frac{1}{4\pi^2} \int_{\gamma} \omega(x, y) + \omega(y, x)$$

Theorem 4

$$\omega(x, x) = dP_3(x)$$

Want to apply Stokes' Theorem again.

Maillot: if $P \in \mathbb{R}[x, y, z]$,

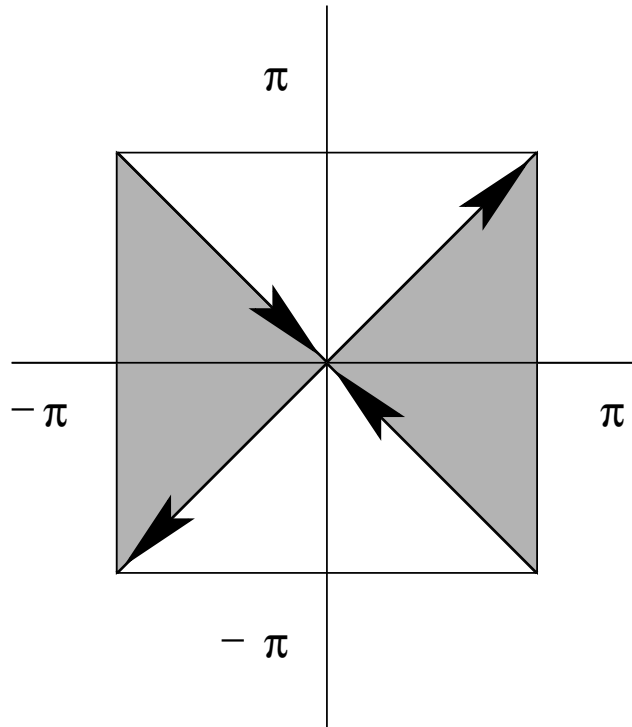
$$\partial\Gamma = \gamma = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$$

ω defined in

$$C = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\}$$

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m(P) = \frac{1}{4\pi^2} 8(P_3(1) - P_3(-1)) = \frac{7}{2\pi^2} \zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, y, z)$$

In $\wedge^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}$,

$$x \wedge y \wedge z = \sum r_i x_i \wedge (1 - x_i) \wedge y_i$$

$\{x, y, z\} = 0$ in $K_3^M(\mathbb{C}(S)) \otimes \mathbb{Q}$.

Then

$$\begin{aligned} \int_{\Gamma} \eta(x, y, z) &= \sum r_i \int_{\Gamma} \eta(x_i, 1 - x_i, y_i) \\ &= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i) \end{aligned}$$

Need

$$\{x\}_2 \otimes y = \sum r_i \{x_i\}_2 \otimes x_i$$

in $(\mathbf{B}_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$, where

$$\mathbf{B}_2(F) := \mathbb{Z}[\mathbb{P}_F^1]/\text{five term relation}$$

Then

$$\int_{\gamma} \omega(x, y) = \sum r_i \int_{\gamma} \omega(x_i, x_i) = \sum r_i P_3(x_i)|_{\partial\gamma}$$

The condition is $\{x_i\}_2 \otimes y_i$ is 0 "in"
 $K_4^{\{3\}}(\mathbb{C}(C))_{\mathbb{Q}}$.

Big picture II

$$\cdots \rightarrow K_4(\partial\Gamma) \rightarrow K_3(S, \partial\Gamma) \rightarrow K_3(S) \rightarrow \cdots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

$$\cdots \rightarrow (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \rightarrow K_4(C, \partial\gamma) \rightarrow K_4(C) \rightarrow \cdots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

In each step, we have the same two options as before.

