NONLINEAR MCKEAN-VLASOV DIFFUSIONS UNDER THE WEAK HÖRMANDER CONDITION WITH QUANTILE-DEPENDENT COEFFICIENTS

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Abstract. In this paper, the strong existence and uniqueness for a degenerate finite system of quantile-dependent McKean-Vlasov stochastic differential equations are obtained under a weak Hörmander condition. The approach relies on the a priori bounds for the density of the solution to time inhomogeneous diffusions. The time inhomogeneous Feynman-Fac formula is used to construct a contraction map for this degenerate system.

Keywords: Mckean-Vlasov equation, Langevin equation, weak Hörmander condition, Feynman-Kac formula, two-sided Gaussian estimates, quantile-dependent PDE.

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1. Introduction

Stochastic differential equations (SDEs) with coefficients depending on the probability distribution of the unknown, often referred to as mean-field or McKean-Vlasov SDEs, have become a hot research area in recent years. Their dynamics are of the form:

$$dX_t = F(t, \mathcal{L}(X_t), X_t)dt + \sigma(t, \mathcal{L}(X_t), X_t)dW_t, \tag{1.1}$$

where $(W_t)_{t\geq 0}$ is a d-dimensional Brownian motion and $\mathcal{L}(X_t) \in \mathcal{P}(\mathbb{R}^d)$ is the probability law of the unknown X_t. Here $\mathcal{P}(\mathbb{R}^d)$ denotes the space of Borel probability measures on $\mathbb{R}^d$. This type of dynamics naturally appears as the limit of a large system of interacting players. Each individual player in the system interacts through the empirical measure of the population. As the size of the population grows to infinity, due to the weak dependence nature and by the law of large number, the empirical measure will converge to the law of each individual, the limiting dynamic

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of each player depends only on its own law and not the measures of the others any more. This is the so-called propagation of chaos phenomenon, originally studied by McKean [20].

The well-posedness of equation (1.1) has been intensively investigated by many authors. Mishura and Veretennikov [22] proved the strong uniqueness under the usual Lipschitz condition and the assumption that σ is independent of $\mathcal{L}(X_t)$ and uniformly non-degenerate. Furthermore, assuming that F is Hölder continuous, σ is Lipschitz continuous in the second and third arguments, Chaudru de Raynal [7] showed that (1.1) admits a unique strong solution. Recently, Frikha, Konakov and Menozzi [14] extended Chaudru de Raynal's result to the non-linear Mckean-Vlasov SDEs driven by α-stable Lévy processes under mild Hölder regularity assumptions. Röchner and Zhang [26] showed the strong well-posedness of the above SDE when F and σ satisfy some integrability conditions in the third argument and are Lipschitz continuous in the second argument.

Notice that all the aforementioned results require the Lipschitz or Hölder continuity of coefficients. There are also some effort to lift these continuity assumptions. Let us mention only two works. In the work of Jourdain [17], the author studied the SDEs with F depending on the marginal of the solution at time t. The weak uniqueness was obtained without the continuity assumption of F on the third argument (but still having Lipschitz continuity in the second argument). In [19], Lacker proved the (weak and strong) uniqueness for the solutions to (1.1) when drift coefficient F is singular and diffusion coefficient σ is independent of the second argument. More precisely, the drift coefficient F in [19] is merely bounded and measurable, but still Lipschitz continuous in the second argument in the sense of the total variation distance. Let us point out that in all the above mentioned literature, both the drift and diffusion coefficients are assumed to be Lipschitz continuous with respect to the probability measure $\mathcal{L}(X_t)$. However, in reality it is too restrictive to assume the continuity of a function on the space of measures with respect to the Wasserstein metric. Thus, it is an interesting, natural, and challenging question to remove the restriction of continuity of the coefficients with respect to the probability measure $\mathcal{L}(X_t)$.

For example, in finance and other applications (e.g. [10]), the following quantile-dependent equation is introduced and studied:

$$dX_t = F(t, Q_\alpha(X_t), X_t)dt + \sigma(t, Q_\alpha(X_t), X_t)dW_t, \tag{1.2}$$

where $F : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ are continuous functions, $\alpha = (\alpha_1, \dots, \alpha_d) \in (0,1)^d$, and $Q_\alpha(X_t)$ is the α-quantile (vector) of the probability measure $\mathcal{L}(X_t)$ of X_t, namely,

$$(Q_\alpha(X_t))_j = (Q_\alpha(\mathcal{L}(X_t)))_j = \inf\Big\{ y_j \in \mathbb{R}, \int_{x\in\mathbb{R}^d, x_j \le y_j} \mathcal{L}(X_t)(dx) \ge \alpha_j \Big\}, \quad j = 1, \dots, d\,.$$

It is well-known that for any two real valued random variables X and Y with cumulative distributions F_X and F_Y, the p-Wasserstein distance is given by

$$I_p(X,Y) = \left(\int_0^1 |F_X^{-1}(\alpha) - F_Y^{-1}(\alpha)|^p d\alpha \right)^{1/p}.$$

From the above expression it is obvious that the coefficients in (1.2) are not continuous with respect to the Wasserstein distance for any finite $p \geq 1$. Hence, we need a completely different approach to study the existence and uniqueness problems for quantile-dependent equations.

The works [10] and [18] are among the first to study this type of equations. Crisan et al. [10] motivated such a model (1.2) from a financial viewpoint and proved the existence of a solution, but left open both the weak and strong uniqueness problems. Kolokoltsov [18] then established the strong uniqueness of (1.2) under some differentiable and Lipschitz conditions on σ and F. In particular, Kolokoltsov assumed the uniform ellipticity condition on $a := \sigma\sigma^*$, namely, there exists a constant $\Lambda > 0$, such that

$$\Lambda^{-1}|\xi|^2 \leq |\xi, a(t,y,x)\xi| \leq \Lambda|\xi|^2, \quad \forall\, (t,y,x) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d, \xi \in \mathbb{R}^d. \tag{1.3}$$

The main contribution of this paper is to remove the above uniform ellipticity condition (1.3) and we shall prove the pathwise uniqueness for the quantile-dependent SDE under the weak Hörmander condition (see e.g. [16, Page 355]). To control the quantile when proving the uniqueness, we require that the solution X_t (as a random vector) has a density (with respect to Lebesgue measure) and this density is strictly positive with a certain decay property. This problem of the existence of density is an important topic in probability theory and partial differential equations. The weak Hörmander condition imposed in the McKean-Vlasov equation ensures the existence of the density of its solution. Let us recall one such result on the following nd-dimensional Langevin-type stochastic differential equation:

$$\begin{cases} dX_t^1 = F_1\left(t, Q_\alpha(X_t), X_t^1, \cdots, X_t^n\right) dt + \sigma\left(t, Q_\alpha(X_t), X_t^1, \cdots, X_t^n\right) dW_t, \\ dX_t^2 = F_2\left(t, Q_\alpha(X_t), X_t^1, \cdots, X_t^n\right) dt, \\ dX_t^3 = F_3\left(t, Q_\alpha(X_t), X_t^2, \cdots, X_t^n\right) dt, \\ \qquad \vdots \\ dX_t^n = F_n\left(t, Q_\alpha(X_t), X_t^{n-1}, X_t^n\right) dt, \end{cases} \tag{1.4}$$

where d and n are positive integers; $(W_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion; $X^i, 1 \leq i \leq n$, are all d-dimensional processes, and $(X_t)_{t\geq 0} = (X_t^1, \ldots, X_t^n)_{t\geq 0}$; $F_1 : \mathbb{R}_+ \times \mathbb{R}^{nd} \times \mathbb{R}^{nd} \to \mathbb{R}^d$; $F_i : \mathbb{R}_+ \times \mathbb{R}^{nd} \times \mathbb{R}^{(n-i+2)d} \to \mathbb{R}^d$ for $i = 2, \cdots, n$; and $\sigma : \mathbb{R}_+ \times \mathbb{R}^{nd} \times \mathbb{R}^{nd} \to \mathbb{R}^d \otimes \mathbb{R}^d$ are continuous functions. Denote by I_d and 0_d the $d \times d$ identity and zero matrices respectively. Introducing $D = (I_d, 0_d, \cdots, 0_d)^T \in \mathbb{R}^{nd\times d}$, and letting $F = (F_1, \cdots, F_n)^T$, we can rewrite (1.4) in the following abbreviated form

$$dX_t = F(t, Q_\alpha(X_t), X_t)dt + D\sigma(t, Q_\alpha(X_t), X_t)dW_t. \tag{1.5}$$

The system of equation (1.4) (or (1.5)) is highly degenerate if $n \geq 2$ and the ellipticity condition (1.3) is obviously not satisfied. Still, in the special case that F and σ in (1.5) are independent of the quantile, namely, when (1.5) is reduced to

$$dX_t = \bar{F}(t, X_t)dt + D\bar{\sigma}(t, X_t)dW_t\,, \tag{1.6}$$

the existence of the density, its derivatives and its two-sided Gaussian bounds have been obtained in [9, 11, 21, 23], which are critical to this work.

The degenerate stochastic differential equations of the form (1.6) have attracted more and more attention in the past years (see e.g. [24, 29, 30]). When a Newton equation $\ddot{x}(t) = F(t, x(t), \dot{x}(t))$ is under influence of some uncertainty, the corresponding stochastic differential equation could be $\ddot{x}(t) = F(t, x(t), \dot{x}(t)) + G(t, x(t), \dot{x}(t))\dot{W}(t)$. This equation is of the form (1.6) if we let $x_2(t) = x(t)$ and $x_1(t) = \dot{x}(t)$, namely, $dx_1(t) = F(t, x_2(t), x_1(t))dt + G(t, x_2(t), x_1(t))dW(t)$ and $dx_2(t) = x_1(t)dt$. This type of equations naturally appears in various scientific contexts. For example, in physics, equation (1.6) corresponds to the dynamics of a finite-dimensional nonlinear Hamiltonian system (a chain of anharmonic oscillators) coupled with two heat reservoirs at different temperatures, which was used by Eckmann et al. [12] (see also [15, 27, 28]) to study the statistical mechanics of such system. Rey-Bellet and Thomas [25] considered the low-temperature asymptotic behavior of the invariant measure in the framework of (1.6). In mathematical finance, there are some applications of the Langevin-type equation in pricing Asian options (see e.g. [2]). In fact, many SDEs including some used in financial markets fail to satisfy the uniform ellipticity condition so alternative conditions like ours, which require substantially different proofs, are important. We hope that our extension to the quantile-dependent SDEs would bring more applications to various fields.

To obtain the existence and uniqueness of the solution of the PDE (2.7) which is associated to equation (1.5), we use the fixed point theorem (see the proof of Proposition 4.4). But, to apply the fixed point theorem, we need to bound a certain distance between $Q_\alpha(h_1)$ and $Q_\alpha(h_2)$ by a certain distance of h_1 and h_2 (see (4.1)). This was already done in [18]. We also need to bound the distance between $u^{(1)}$ and $u^{(2)}$ by the distance between $\omega^{(1)}$ and $\omega^{(2)}$, where each $u^{(i)}$ $(i = 1, 2)$ is the density of $X_t^{(i)}$ in (1.5) when $Q_\alpha(X_t^{(i)})$ is replaced by $\omega^{(i)}$ (see Proposition 4.2). This is relatively complicated and requires the fact that the density u of the solution X_t of (1.5) is characterized by the corresponding Fokker-Planck equation. Thus, the above problem of controlling the distance between $u^{(1)}$ and $u^{(2)}$ by the distance between $\omega^{(1)}$ and $\omega^{(2)}$ is reduced to investigating the regularity with respect to the non-linearities.

However, because of the degeneracy, it is hard to use the PDE approach as in [18]. Instead, we shall use a time-dependent Feynman-Kac formula.

The paper is organized as follows. In Section 2, we present the main hypotheses and main results of this paper. We also fix some notations in this section. Some useful a priori estimates on the density of the solution to the SDE (1.2) including the tail estimates and lower bounds are given in Section 3. We will also recall the time-dependent Feynman-Kac formula in this section. These two-sided bounds of the density and the Feynman-Kac formula play central roles in our approach. We give in Section 4 the proof of our main results.

2. Main results

For any $x \in \mathbb{R}^{nd}$, we write $x = (x_1, \cdots, x_n) = (x_1^1, \cdots, x_1^d; \cdots; x_n^1, \cdots, x_n^d)$, where for $i = 1, \cdots, n$, $j = 1, \cdots, d$, $x_i \in \mathbb{R}^d$, $x_i^j \in \mathbb{R}$. Let $|x_i|$ denote the Euclidean norm of x_i, that is, $|x_i| = \left(\sum_{j=1}^d |x_i^j|^2\right)^{\frac{1}{2}}$. Let $F_i : \mathbb{R}_+ \times \mathbb{R}^{nd} \times \mathbb{R}^{((n-i+2)\wedge n)d} \to \mathbb{R}^d$ be continuous mappings. For notational simplicity, we may consider F_i as a continuous mapping from $\mathbb{R}_+ \times \mathbb{R}^{nd} \times \mathbb{R}^{nd}$ to $\mathbb{R}^d$ and write $F = (F_1; \cdots; F_n) = (F_1^1, \cdots, F_1^d; \cdots; F_n^1, \cdots, F_n^d)$ as well.

For any $d \times d$ matrix $a = (a_{ij})_{i,j=1}^d$, denote by $|a| = (\sum_{i,j=1}^d |a_{ij}|^2)^{\frac{1}{2}}$ its Hilbert-Schmidt norm. In what follows, we use $\|\cdot\|_p$ for the L^p norm on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any measurable function f on a Euclidean space, $|f|_{L^p}$ denotes the L^p norm of $|f|$ with respect to the Lebesgue measure. $C(\mathbb{R}^d)$ and $C_b(\mathbb{R}^d)$ denote the sets of continuous functions and bounded continuous functions on $\mathbb{R}^d$, respectively.

The notation ∇ stands for the gradient with respect to all space variables. Let $f \in C(\mathbb{R}^+ \times \mathbb{R}^{nd} \times \mathbb{R}^{nk}, \mathbb{R}^d)$, $k = 1, \cdots, n$. $\nabla_{x_i} f(t, y, x)$ denotes the gradient operator with respect to the ith space variable $x_i \in \mathbb{R}^d$, which is a $d \times d$ Jacobian matrix.

Fix a time horizon $[0, T]$. We will need the following hypotheses for coefficients F and σ and the initial condition X_0.

(H1) F is uniformly bounded at the origin of the third argument. That is, there exists a positive constant κ such that

$$\sup_{t \in [0,T], y \in \mathbb{R}^{nd}} |F(t, y, 0)| \leq \kappa < \infty.$$

(H2) The function $a := \sigma\sigma^*$ is uniformly elliptic, namely, (1.3) is satisfied.

(H3) F and σ are uniformly Lipschitz continuous in space variables with constant $\kappa > 0$, i.e., for all $y, \bar{y}, x, \bar{x} \in \mathbb{R}^{nd}$,

$$\sup_{t \in [0,T]} \big(|F(t, y, x) - F(t, \bar{y}, \bar{x})| + |\sigma(t, y, x) - \sigma(t, \bar{y}, \bar{x})| \big) \leq \kappa \left(|x - \bar{x}| + |y - \bar{y}| \right).$$

(H4) The function $x \mapsto F(t,y,x)$ is twice differentiable and function $x_1 \mapsto a(t,y,x_1,\dots,x_n)$ is three times differentiable. Moreover, the following inequalities hold true

$$\sup_{(t,y,x)\in[0,T]\times\mathbb{R}^{nd}\times\mathbb{R}^{nd}} \left| \sum_{j,k=1}^{d} \frac{\partial^2}{\partial_{x_1^j}\partial_{x_1^k}} a_{jk}(t,y,x) \right| \le \kappa,$$

$$\sup_{t\in[0,T]} \sum_{j,k=1}^{d} \left| \frac{\partial^2}{\partial_{x_1^k}\partial_{x_1^j}} a_{kj}(t,y,x) - \frac{\partial^2}{\partial_{x_1^k}\partial_{x_1^j}} a_{kj}(t,\bar{y},\bar{x}) \right| \le \kappa\left(|x-\bar{x}| + |y-\bar{y}|\right),$$

and

$$\sup_{t\in[0,T]} \sum_{i=1}^{n}\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_i^j} F_i^j(t,y,x) - \frac{\partial}{\partial x_i^j} F_i^j(t,\bar{y},\bar{x}) \right| \le \kappa\left(|x-\bar{x}| + |y-\bar{y}|\right),$$

for all $x,\bar{x},y,\bar{y}\in\mathbb{R}^{nd}$.

(H5) For any integer $i=2,\dots,n$, the derivative $\nabla_{x_{i-1}}F_i(t,y,x)$ is η-Hölder continuous in the first spatial variable x_{i-1} with constant κ, and there exists a closed convex subset ε_{i-1} contained in the set of invertible $d\times d$ matrices, such that for all $t\in[0,T]$ and $(x_{i-1},\cdots,x_n)\in\mathbb{R}^{(n-i+2)d}$, the matrix $\nabla_{x_{i-1}}F_i(t,y,x_{i-1},\cdots,x_n)$ belongs to ε_{i-1}.

(I) X_0 is a random variable independent of W. The probability law of X_0 has a continuously differentiable density $f>0$ satisfying the following integrability condition

$$U = \int_0^\infty \sup_{|z|\ge r} |f(z)|^2 (r^{4n-1+\varepsilon} + r^{n-1})dr + \int_0^\infty \left[\sup_{|z|\ge\lambda} |\nabla f(z)|^4\right] \left(\lambda^{4nd-1+\varepsilon} + \lambda^{nd-1}\right) d\lambda < \infty, \tag{2.1}$$

for some constant $\varepsilon>0$.

Remark 2.1. The most important hypotheses in this work are the hypotheses (H2) and (H5): the matrices $(\nabla_{x_{i-1}}F_i)_{2\le i\le n}$ have full rank, which imply a version of the (weak) Hörmander condition. It ensures the existence of the probability density of the solution to (1.5) (see [11, Theorem 1.1]). Let us point out that in (H2) we assume that $a=\sigma\sigma^*$ is uniformly elliptic. However, the diffusion coefficient $D\sigma\sigma^*D^*$ of the whole system (1.5) is highly degenerate.

Remark 2.2. Hypotheses (H3), (H4) are to guarantee the Lipschitz continuity of the function c defined in (2.6) below. In addition, they imply that c is bounded, which is needed in the application of the Feynman-Kac formula (see Theorem 3.10).

Remark 2.3. At the first look, hypothesis (I) seems a little complicated. However, Gaussian densities and many other functions satisfy this condition. Furthermore, it is worth mentioning that to prove Proposition 4.2 (i.e. the local existence and uniqueness), hypothesis (I) can be weakened to the following form:

$$\int_{\mathbb{R}^{nd}} f(y)^2 \left(|y|^{nd+\varepsilon}+1\right) dy + \int_0^\infty \left[\sup_{|z|\geq\lambda} |\nabla f(z)|^4\right] \left(\lambda^{4nd-1+\varepsilon}+\lambda^{nd-1}\right) d\lambda < \infty.$$

The condition (2.1) is used to guarantee the global existence and uniqueness of solutions to (1.5).

In the next theorem, we provide the existence and uniqueness result for equation (1.5), which is the main result of this paper.

Theorem 2.4. *Assume that hypotheses (H1)-(H5) and hypothesis (I) hold true. Then, there exists a unique strong solution to SDE (1.5) on $[0,T]$.*

The idea to prove the above theorem is to construct a contraction mapping associated to equation (1.5). To this end, we need to introduce an auxiliary equation. Given a continuous (deterministic control) function ω on $[0,T]$ with values in $\mathbb{R}^{nd}$, we consider the following stochastic differential equation

$$dX_t^\omega = F(t,\omega_t,X_t^\omega)dt + D\sigma(t,\omega_t,X_t^\omega)dW_t, \tag{2.2}$$

with initial condition X_0 satisfying hypothesis (I). Under hypotheses (H1)-(H5), equation (2.2) has a unique strong solution, whose density $u_t^\omega(x)$ satisfies the following Fokker-Planck equation:

$$\begin{aligned}\frac{\partial}{\partial t}u_t^\omega(x) =&\frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2}{\partial x_1^i \partial x_1^j}\left(a_{ij}(t,\omega_t,x)u_t^\omega(x)\right) - \sum_{i=1}^n\sum_{j=1}^d \frac{\partial}{\partial x_i^j}\left(F_i^j(t,\omega_t,x)u_t^\omega(x)\right)\\ =&\frac{1}{2}\sum_{i,j=1}^d a_{ij}(t,\omega_t,x)\frac{\partial^2}{\partial x_1^i \partial x_1^j}u_t^\omega(x) + \langle b(t,\omega_t,x),\nabla u_t^\omega(x)\rangle + c(t,\omega_t,x)u_t^\omega(x),\end{aligned} \tag{2.3}$$

with initial condition $u_0^\omega = f$, where

$$a = (a_{ij})_{i,j=1}^d = \sigma\sigma^*, \tag{2.4}$$

$b=(b_1,\dots,b_n)$ with $b_i=(b_i^1,\dots,b_i^d)$, $i=1,\dots,n$ and

$$b_i^j(t,y,x) = -F_i^j(t,y,x) + \mathbf{1}_{\{i=1\}}\sum_{k=1}^d \frac{\partial}{\partial x_1^k}a_{kj}(t,y,x), \tag{2.5}$$

and

$$c(t,y,x) = -\sum_{i=1}^n\sum_{j=1}^d \frac{\partial}{\partial x_i^j}F_i^j(t,y,x) + \frac{1}{2}\sum_{j,k=1}^d \frac{\partial^2}{\partial x_1^j \partial x_1^k}a_{jk}(t,y,x), \tag{2.6}$$

for any $(t, y, x) \in [0, T] \times \mathbb{R}^{nd} \times \mathbb{R}^{nd}$. It follows from [1, Theorem 1] that (2.3) is well-posed. Thus, the density of $u_t^\omega(x)$ is the unique solution to the equation (2.3).

Similarly, if (1.5) has a solution X_t with quantile $Q_\alpha(X_t)$ being continuous in time, then the law of X_t has a density u that is the solution to the following equation

$$\begin{aligned}\frac{\partial}{\partial t}u_t(x) =& \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t, Q_\alpha(u_t), x)\frac{\partial^2}{\partial x_1^i \partial x_1^j}u_t(x) + \langle b(t, Q_\alpha(u_t), x), \nabla u_t(x)\rangle \\ &+ c(t, Q_\alpha(u_t), x)u_t(x).\end{aligned} \tag{2.7}$$

It will be shown in Section 4 that the proof of Theorem 2.4 is reduced to proving that PDE (2.7) admits a unique solution. However, it is not easy to deal with such PDE whose coefficients depend on quantiles. We shall find an appropriate Banach space $\mathbb{B}$ and construct a mapping $\mathcal{M}$ on $\mathbb{B}$. The well-posedness of (2.7) is shown by proving that $\mathcal{M}$ is a contraction map on $\mathbb{B}$ in Proposition 4.4 (below).

In the next theorem, we prove the well-posedness of (2.7).

Theorem 2.5. *Let f be a continuous differentiable function on $\mathbb{R}^{nd}$ satisfying hypothesis (I). Assume hypotheses (H1)-(H5) hold true. Then, there exists a function u on $[0, T] \times \mathbb{R}^{nd}$, which is the unique solution to PDE (2.7) with initial condition f.*

The strong existence and uniqueness as well as density estimates of solutions to degenerate SDEs (independent of the probability measure) with Hölder continuous drifts and under the weak Hörmander condition has been investigated by Chaudru de Raynal [6]. In [18], to obtain the stability result the author uses the two-sided bounds of the density and its first order derivatives under the uniform ellipticity condition. In our hypoellipticity case, we encounter the following difficulties.

(1) For (1.6), Pigato [23] obtained upper bounds for the derivatives of transition density of any order. The first derivative with respect to the variable x_i^j $(i = 1, \cdots, n; j = 1, \cdots, d)$ is given by

$$|\partial_{x_i^j} p(t, x; 0, y)| \le \frac{C}{t^{(2\lfloor \frac{ij-1}{d} \rfloor + 1 + n^2 d)/2}} \exp\left(-\frac{|\mathcal{T}_t^{-1}(x - \theta_t(y))|^2}{C}\right),$$

where y is the initial position; C is a constant; $\lfloor \cdot \rfloor$ denotes the integer part function, $\mathcal{T}$ and θ_t are given by (3.1) and (3.2) below. As we see, this bound is more singular near $t = 0$ than that in the elliptic case.

(2) To overcome this problem, we assume that the initial condition f satisfies certain differentiability and integrability (over the whole $\mathbb{R}^{nd}$) conditions, in hope that the singularity difficulty can be absorbed in the initial condition. However, proceeding with this effort, we immediately encounter the difficulty that we do not know how to pass the gradient $\nabla_x p(t, x; s, y)$ to ∇f in the

following integral:

$$\int_{\mathbb{R}^{nd}} f(y)\nabla_x p(t,x;s,y)dy$$

since $p(t,x;s,y)$ is not the form of $p(t,s,x-y)$. To get around this difficulty, we apply the time inhomogeneous Feynman-Kac formula. This enables us to finish the stability analysis of the solution to (2.3) with respect to ω.

3. A priori estimates of the density

In the remainder of the paper, we assume that $d=1$ to simplify the presentation. The case $d>1$ can be treated analogously with only additional notational complexity. We use $C>0$ to denote a generic constant which may vary from occurrence to occurrence.

First, let us turn to (1.6). We need a result from [9]. To state this result, we need to introduce the following conditions on the coefficients $\bar{F}$ and $\bar{\sigma}$.

(C1): $\bar{F}(t,0)$ is bounded for all $t\in[0,T]$ and $\bar{a}=\bar{\sigma}\bar{\sigma}^*$ is uniformly elliptic with the positive constant Λ.

(C2): $\bar{\sigma}$ is globally Lipschitz in the space variable uniformly in time variable. For all $j=1,2,\cdots,n$ the functions $\bar{F}_i, i=1,\cdots,j$, are uniformly η_j-Hölder continuous in the jth spatial variable with $\eta_j\in(\frac{2j-2}{2j-1},1]$, uniformly in time and other spatial variables.

(C2'): The functions $\bar{F}_1,\cdots,\bar{F}_n$ and $\bar{\sigma}$ are uniformly Lipschitz and η-Hölder continuous ($\eta\in(0,1]$) with respect to the underlying space variables respectively.

(C3): For each integer $2\le i\le n$, $(t,x_i,\cdots,x_n)\in\mathbb{R}_+\times\mathbb{R}^{(n-i+1)d}$, the function $x_{i-1}\in\mathbb{R}^d\mapsto\bar{F}_i(t,x_{i-1},\cdots,x_n)$ is continuously differentiable and its derivative, denoted by $(t,x_{i-1},\cdots,x_n)\in\mathbb{R}_+\times\mathbb{R}^{(n-i+2)d}\mapsto\nabla_{x_{i-1}}\bar{F}_i(t,x_{i-1},\cdots,x_n)$, is η-Hölder continuous in the first space variable x_{i-1} with constant κ. Moreover, there exists a closed convex subset ε_{i-1} contained in the set of invertible $d\times d$ matrices, such that for all $t\ge 0$, $i=2,\cdots,n$ and $(x_{i-1},\cdots,x_n)\in\mathbb{R}^{(n-i+2)d}$, the matrix $\nabla_{x_{i-1}}\bar{F}_i(t,x_{i-1},\cdots,x_n)$ belongs to ε_{i-1}.

Remark 3.1. Conditions (C1), (C2), (C2') and (C3) can be easily verified by hypotheses (H1)-(H5). In fact, (C1) and (C3) are the same as (H1), (H2) and (H5). Additionally, hypotheses (H3) and (H4) imply (C2) and (C2').

Theorem 3.2 (see [9])**.** *Assume that (C1), (C2) and (C3) hold true. Then there exists a unique strong solution to SDE* (1.6)*.*

We also need a result about the Gaussian estimate for the density of the solution to (1.6). To state this result we introduce the scale matrix $\mathcal{T}$ and shift vector θ as follows. Fix $t \geq 0$ and $x \in \mathbb{R}^{nd}$. Let $\mathcal{T}_t$ denote the following $nd \times nd$ diagonal matrix:

$$\mathcal{T}_t = \begin{pmatrix} \mathcal{T}_t^1 & 0 & \dots & 0 \\ 0 & \mathcal{T}_t^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{T}_t^n \end{pmatrix} = \begin{pmatrix} t^{\frac{1}{2}} I_d & 0 & \dots & 0 \\ 0 & t^{\frac{3}{2}} I_d & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{n-\frac{1}{2}} I_d \end{pmatrix}. \tag{3.1}$$

Let $\theta_\cdot(x) : [0,T] \mapsto \mathbb{R}^{nd}$ be the solution to following (deterministic) ODE,

$$\begin{cases} \frac{d}{dt}\theta_t(x) = \bar{F}\left(t, \theta_t(x)\right), \\ \theta_0(x) = x. \end{cases} \tag{3.2}$$

Theorem 3.3 (see [11]). *Assume that (C1), (C2') and (C3) hold true. Let X be the solution to (1.6) with initial condition $X_0 = x$, where $x \in \mathbb{R}^{nd}$. Then, for any $t \in [0,T]$, the law of X_t admits a probability density, denoted by $p_t(\cdot, x)$. Moreover, there exists a constant $C_T \geq 1$, depending on T, n, d, Λ, η, the Lipschitz constants in (C1)-(C3), and $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n-1}$, such that for any $y \in \mathbb{R}^{nd}$,*

$$\frac{1}{C_T t^{n^2 d/2}} \exp\left(-C_T |\mathcal{T}_t^{-1}(\theta_t(x) - y)|^2\right) \leq p_t(y,x) \leq \frac{C_T}{t^{n^2 d/2}} \exp\left(-C_T^{-1} |\mathcal{T}_t^{-1}(\theta_t(x) - y)|^2\right), \tag{3.3}$$

where $\mathcal{T}_t$ and $\theta_t(x)$ are given by (3.1) and (3.2) respectively.

Remark 3.4. We still cite the theorem for general dimension d. However, we will continue to work on the case $d = 1$.

The above Theorem 3.3 suggests that the *ith* coordinate X_t^i of the system at time t oscillates around $\mathcal{T}_t$ with fluctuations of order $t^{i-\frac{1}{2}}$. In addition, the deterministic flow θ plays an important role in the density bounds of X_t for (1.6). As we will see in the proof of Proposition 3.7 that the density of X_t for equation (2.2) has analogous two-sided bounds of the form (3.3) under Hypotheses (H1)-(H5). So in order to make use of the existing density bounds, we define the other deterministic flow θ^ω associated to equation (2.2) as follows.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined as in Section 1 (with $d = 1$). For any continuous function $\omega : [0,T] \to \mathbb{R}^n$, and $x \in \mathbb{R}^n$, we define, analogously to (3.2), a function $\theta^\omega = (\theta_1^\omega, \dots, \theta_n^\omega)$ on $[0,T]$ with values in $\mathbb{R}^n$ by the following ODE

$$\begin{cases} \frac{d}{dt}\theta^\omega(t,x) = F\left(t, \omega_t, \theta^\omega(t,x)\right), t \in [0,T], \\ \theta^\omega(0,x) = x. \end{cases} \tag{3.4}$$

We have the following lemma about θ^ω.

Lemma 3.5. *Assume that hypotheses (H1)-(H5) hold true and assume that ω is a continuous function of $t \in [0,T]$. Let θ^ω satisfy (3.4). Then, for $0 \leq t \leq T$ and*

$x \in \mathbb{R}^n$, the following inequalities hold

$$e^{-\kappa t}|x| - \kappa t \le |\theta^\omega(t,x)| \le (|x| + \kappa t)\, e^{\kappa t} \tag{3.5}$$

and

$$e^{-n\kappa t} \le \det(\nabla \theta^\omega(t,x)) \le e^{n\kappa t}, \tag{3.6}$$

where κ is the positive constant that appeared in hypotheses (H1)-(H5).

Proof. By the Lipschitz property and uniformly boundedness (at the origin) of F, we see that

$$\begin{aligned} |\theta^\omega(t,x)| =& \Big| x + \int_0^t F(r,\omega_r,\theta^\omega(r,x))dr \Big| \\ \le& |x| + \int_0^t \left(|F(r,\omega_r,0)| + \kappa |\theta^\omega(r,x)| \right) dr \\ \le& |x| + \kappa t + \int_0^t \kappa |\theta^\omega(r,x)| dr. \end{aligned}$$

An application of Gronwall's inequality yields

$$|\theta^\omega(t,x)| \le (|x| + \kappa t)\, e^{\kappa t}. \tag{3.7}$$

This proves the second inequality in (3.5). To prove the first inequality, we consider the following backward ODE:

$$\begin{cases} \frac{d}{ds}\hat{\theta}_s = -F(s,\omega_s,\hat{\theta}), \quad 0 \le s < t, \\ \hat{\theta}_t = \xi \in \mathbb{R}^n. \end{cases} \tag{3.8}$$

Similar to (3.7), we can show that

$$|\hat{\theta}_s| \le (|\xi| + \kappa(t-s))\, e^{\kappa(t-s)},$$

for all $s \in [0,t]$. Notice that $\hat{\theta} = \{\theta^\omega(s,x)\,;\ s \in (0,t)\}$ is the solution to (3.8) with terminal condition $\hat{\theta}_t = \theta^\omega(t,x)$. Then, we have

$$|x| = \hat{\theta}_0 \le (|\theta^\omega(t,x)| + \kappa t)\, e^{\kappa t}.$$

The proof of inequality (3.5) is then completed.

Taking the derivative of the following equation with respect to x,

$$\theta^\omega(t,x) = x + \int_s^t F(r,\omega_r,\theta_r(x))dr,$$

we have

$$\nabla\theta^\omega(t,x) = I_d + \int_s^t \nabla F(r,\omega_r,\theta^\omega(r,x))\nabla\theta^\omega(r,x)dr.$$

By Liouville's formula, we can write

$$\det(\nabla\theta^\omega(t,x)) = \exp\left(\int_s^t \mathrm{tr}[\nabla F(r,\omega_r,\theta^\omega(r,x))]dr\right). \tag{3.9}$$

Now the hypothesis (H3) can be applied to obtain (3.6). The lemma is then proved. □

By Lemma 3.5 and the implicit function theorem, we have the following corollary.

Corollary 3.6. *Assume that hypotheses (H1)-(H5) hold true and that $\omega(t)$, $0 \le t \le T$ is a continuous function. Let θ^ω satisfy (3.4). Then, there exist a function $(\theta^\omega)^{-1}(t,\cdot)$ such that*

$$\theta^\omega\big(t,(\theta^\omega)^{-1}(t,x)\big) = (\theta^\omega)^{-1}\big(t,\theta^\omega(t,x)\big) = x,$$

for all $x \in \mathbb{R}^n$. Moreover, the gradient of $(\theta^\omega)^{-1}$ with respect to the spatial variable satisfies the following inequality:

$$e^{-n\kappa t} \le \det(\nabla(\theta^\omega)^{-1}(t,x)) \le e^{n\kappa t}. \tag{3.10}$$

The above Lemma 3.5 and Corollary 3.6 are served as estimates in the following Proposition 3.7 and Proposition 3.8, respectively. In the next proposition, we provide a tail estimate for the solution to (2.3).

Proposition 3.7. *Assume that hypotheses (H1)-(H5) hold true. Let f be a positive, continuous integrable function on $\mathbb{R}^n$. Then for any $\varepsilon > 0$, there exist $K > 0$ such that*

$$\int_{\mathcal{G}_K} |u_t^\omega(x)|dx \le \varepsilon, \tag{3.11}$$

for any $t \in [0,T]$ and for any continuous function $\omega : [0,T] \to \mathbb{R}^n$, where

$$\mathcal{G}_K = \Big\{x = (x_1,\dots,x_n) \in \mathbb{R}^n : \max_{1\le i\le n} |x_i| \ge K\Big\}.$$

Proof. For any $\varepsilon > 0$, due to integrability of f, we can choose $\bar{K}$ such that

$$\int_{|x|\ge\bar{K}} |f(x)|dx \le \varepsilon. \tag{3.12}$$

Denote by $p_t^\omega(x,y)$ the transition density of X^ω from y at time 0 to x at time t. Then, it is well-known that

$$u_t^\omega(x) = \int_{\mathbb{R}^n} p_t^\omega(x,\xi)f(\xi)d\xi.$$

Notice that hypotheses (H1)-(H5) ensure that functions $\bar\sigma$ and $\bar F$ given by

$$\bar\sigma(t,x) = \sigma(t,\omega_t,x), \text{ and } \bar F(t,x) = F(t,\omega_t,x),$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$, satisfy conditions (C1)-(C3) and (C2'). Additionally, the independence of t and x of the constant of κ in hypotheses (H1)-(H5) and Remark 3.1 imply that the constant C_T appearing in Theorem 3.3 is independent of the choice of ω. This allows us to apply Theorem 3.3 to obtain

$$\begin{aligned}\int_{\mathcal{G}_K} |u_t^\omega(x)| dx &= \int_{\mathcal{G}_K} \left| \int_{\mathbb{R}^n} p_t^\omega(x,\xi) f(\xi) d\xi \right| dx \\ &\le \int_{\mathcal{G}_K} \int_{\mathbb{R}^n} C_T t^{-n^2/2} \exp\left(-C_T^{-1} |\mathcal{T}_t^{-1}(x - \theta^\omega(t,\xi))|^2\right) |f(\xi)| d\xi dx \\ &= \int_{\mathcal{G}_K^1} \int_{\mathbb{R}^n} C_T t^{-n^2/2} \exp\left(-C_T^{-1} |\mathcal{T}_t^{-1}(y)|^2\right) |f(\xi)| d\xi dy,\end{aligned}$$

where θ^ω is defined by (3.4) and

$$\mathcal{G}_K^1 = \{y \in \mathbb{R}^n : \max_{1\le i\le n} |y_i + \theta_i^\omega(t,\xi)| > K\}.$$

Performing a change of variable $z = \mathcal{T}_t^{-1}(y)$, we have

$$\begin{aligned}\int_{\mathcal{G}_K} |u_t^\omega(x)| dx &\le \int_{\mathcal{G}_K^2} \int_{\mathbb{R}^n} C_T \exp\left(-\frac{|z|^2}{C_T}\right) f(\xi) d\xi dz \\ &\le \int_{\mathbb{R}^n} \int_{\{|\xi|>\bar{K}\}} C_T \exp\left(-\frac{|z|^2}{C_T}\right) f(\xi) d\xi dz + \int_{\mathcal{G}_K^2} \int_{\{|\xi|\le\bar{K}\}} C_T \exp\left(-\frac{|z|^2}{C_T}\right) f(\xi) d\xi dz,\end{aligned}$$

where

$$\mathcal{G}_K^2 = \left\{z \in \mathbb{R}^n : \max_{1\le i\le n} |t^{i-\frac{1}{2}} z_i + \theta_i^\omega(t,\xi))| > K\right\}.$$

As a consequence of (3.12), we have

$$\int_{\mathbb{R}^n} \int_{\{|\xi|>\bar{K}\}} C_T \exp\left(-\frac{|z|^2}{C_T}\right) f(\xi) d\xi dz \le (\pi C_T)^{\frac{n}{2}} C_T \epsilon.$$

On the other hand, by Lemma 3.5, we know that

$$|\theta^\omega(t,\xi)| \le (|\xi| + \kappa t)\, e^{\kappa t} \le \bar{K} e^{\kappa T} + \kappa T e^{\kappa T},$$

for all $|\xi| \le \bar{K}$ and $t \in [0,T]$. Then, we have

$$\mathcal{G}_K^2 \subseteq \left\{z \in \mathbb{R}^n : \max_{1\le i\le n} |t^{i-\frac{1}{2}} z_i| > K - \bar{K} e^{\kappa T} - \kappa T e^{\kappa T}\right\} =: \mathcal{G}_K^3.$$

Therefore, for any $\epsilon > 0$, there exists K sufficiently large such that

$$\int_{\mathcal{G}_K^2} \int_{\{|\xi|\le\bar{K}\}} C_T \exp\left(-\frac{|z|^2}{C_T}\right) f(\xi) d\xi dz \le \int_{\{|\xi|\le\bar{K}\}} C_T f(\xi) d\xi \int_{\mathcal{G}_K^3} \exp\left(-\frac{|z|^2}{C_T}\right) dz \le \epsilon.$$

The proof of this proposition is complete. □

Proposition 3.8. *Assume that hypotheses (H1)-(H5) hold true. Let f be a positive, continuously integrable function on $\mathbb{R}^n$. Then for any $K>0$ there exists $\delta>0$ such that*

$$\inf\left\{u_t^\omega(x) : \max_{1\le j\le n}|x_j|\le K\right\}\ge\delta, \tag{3.13}$$

for all $t\in[0,T]$ and for all continuous functions ω on $[0,T]$ with values in $\mathbb{R}^n$.

Proof. Fix $K>0$. For any $x\in\mathbb{R}$ with $|x|\le K$. By the lower bound of (3.3) we get

$$\begin{aligned} u_t^\omega(x) &= \int_{\mathbb{R}^n} p_t^\omega(x,\xi)f(\xi)d\xi \\ &\ge \int_{\mathcal{R}} p_t^\omega(x,\xi)f(\xi)d\xi \\ &\ge \frac{1}{C_T t^{n^2/2}}\int_{\mathcal{R}}\exp\left(-C_T|\mathcal{T}_t^{-1}(x-\theta^\omega(t,\xi))|^2\right)f(\xi)d\xi, \end{aligned}$$

where

$$\mathcal{R}=\{\xi\in\mathbb{R}^n : |x_1-\theta_1^\omega(t,\xi)|\le\sqrt{t},\cdots,|x_n-\theta_n^\omega(t,\xi)|\le t^{(2n-1)/2},\max_{1\le j\le n}|x_j|\le K\}.$$

Due to Lemma 3.5, we know that

$$e^{-\kappa t}|\xi|-\kappa t\le|\theta^\omega(t,\xi)|,$$

for all $t\in[0,T]$. Thus, we have

$$|\xi_i|\le|\xi|\le\left(|\theta^\omega(t,\xi)|+\kappa t\right)e^{\kappa t}.$$

For any $\xi=(\xi_1,\dots,\xi_n)\in\mathcal{R}$, it is easy to see

$$|\theta^\omega(t,\xi)|\le\left[\sum_{i=1}^n\left(t^{\frac{2i-1}{2}}+|x_i|\right)^2\right]^{\frac{1}{2}}\le\sqrt{2}|x|+\sqrt{2n}(T^{\frac{n-1}{2}}+1).$$

This means that

$$\mathcal{R}\subseteq\mathcal{R}^1:=\{\xi\in\mathbb{R}^n : |\xi|\le(\sqrt{2}nK+\sqrt{2}n(T^{\frac{n-1}{2}}+1)+\kappa T)e^{\kappa T}\}. \tag{3.14}$$

Recall that f is a continuous positive integrable function. Hence, there exists $\delta>0$ such that $f(\xi)\ge\delta$ on the set $\mathcal{R}^1\supseteq\mathcal{R}$. As a consequence, for any $x\in\mathbb{R}^n$ with $|x|\le K$, we have

$$u_t^\omega(x)\ge\frac{\delta}{C_T t^{n^2/2}}\int_{\mathcal{R}}\exp\left(-C_T|\mathcal{T}_t^{-1}(x-\theta^\omega(t,\xi))|^2\right)d\xi.$$

By change of variable $\mathcal{T}_t^{-1}(x-\theta^\omega(t,\xi))=y$ and then by Corollary 3.6, we have

$$\begin{aligned} u_t^\omega(x) &\geq \frac{\delta}{C_T}\int_{\{y\in\mathbb{R}^n:|y_i|\leq 1, i=1,\dots n\}} \exp\left(-C_T|y|^2\right)\det\left(\nabla(\theta^\omega)^{-1}\left(t,x-\mathcal{T}_t(y)\right)\right)dy \\ &\geq \frac{\delta}{C_T}e^{-(nC_T+n\kappa T)}\int_{\{y\in\mathbb{R}^n:|y_i|\leq 1, i=1,\dots n\}} dy \\ &= \frac{\delta}{C_T}2^n e^{-(nC_T+n\kappa T)}, \end{aligned}$$

which completes the proof of the proposition. ☐

Combining the Propositions 3.7 and 3.8, we arrive at the following result.

Proposition 3.9. *Assume that hypotheses (H1)-(H5) hold true and that $\omega:[0,T]\to\mathbb{R}^n$ is a continuous function. Let u^ω be the solution to (2.3) with initial condition $f\in C(\mathbb{R}^n)\cup L^1(\mathbb{R}^n)$. For any $\alpha\in(0,1)^n$ and $t\in[0,T]$, let $\hat\omega^\alpha=\hat\omega=(\hat\omega_1,\cdots,\hat\omega_n)=Q_\alpha(u_t^\omega)$ be the α-quantile of u_t^ω. Then, there exist $K,\delta,\varepsilon>0$, independent of t and ω, such that*

$$\int_{\left\{x\in\mathbb{R}^n\,:\,\max\limits_{1\leq j\leq n}|x_j|\geq K\right\}}|u_t^\omega(x)|dx\leq\varepsilon, \tag{3.15}$$

$$\max_{1\leq j\leq n}|\hat\omega_j|\leq K, \tag{3.16}$$

and

$$\inf\left\{u_t^\omega(x):\max_{1\leq j\leq n}|x_j|\leq K\right\}\geq\delta. \tag{3.17}$$

Fix $\alpha\in(0,1)^n$ and $K,\delta,\varepsilon>0$. Denote by $\mathcal{S}=\mathcal{S}_{\alpha,K,\delta,\varepsilon}$ the collection of density functions h on $\mathbb{R}^n$ such that

$$\int_{\left\{x\in\mathbb{R}^n\,:\,\max\limits_{1\leq j\leq n}|x_j|\geq K\right\}}|h|dx\leq\varepsilon,\qquad \max_{1\leq j\leq n}|\left(Q_\alpha(h)\right)_j|\leq K,$$
$$\text{and }\ \inf\left\{h(x):\max_{1\leq j\leq n}|x_j|\leq K\right\}\geq\delta. \tag{3.18}$$

Then, $\mathcal{S}$ is a convex set.

Proof. Choose

$$0<\varepsilon<\min\left(\alpha_1,\cdots,\alpha_n,1-\alpha_1,\cdots,1-\alpha_n\right).$$

Then, by Proposition 3.7, there exists $K>0$ such that (3.15) is true. Inequality (3.16) also holds true, due to the fact that

$$\int_{-\infty}^{-K}dx_j\int_{\mathbb{R}^{n-1}}u_t^\omega(x)\prod_{i\neq j,1\leq i\leq n}dx_i\leq\int_{\{x\in\mathbb{R}^n:\max_{1\leq j\leq n}|x_j|\geq K\}}|u_t^\omega(x)|dx\leq\varepsilon\leq\alpha_j,$$

and

$$\int_K^\infty dx_j \int_{\mathbb{R}^{n-1}} u_t^\omega(x) \prod_{i\neq j, 1\le i\le n} dx_i \le 1-\alpha_j,$$

for all $j = 1, \ldots, n$. The inequality (3.17) is a straightforward consequence of Proposition 3.8.

In the next step, we prove the convexity of set $\mathcal{S}$. Let $h_1, h_2 \in \mathcal{S}$. For any $\beta \in [0,1]$, $h = \beta h_1 + (1-\beta)h_2$ is still a density function, and the first and the last properties in (3.18) are trivial for h. It suffices to show that for any $\beta \in [0,1]$,

$$Q_\alpha(\beta h_1 + (1-\beta)h_2) \le K,$$

which is true, because

$$\begin{aligned}
&\int_{-\infty}^{-K} dx_j \int_{\mathbb{R}^{n-1}} (\beta h_1(x) + (1-\beta)h_2(x)) \prod_{i\neq j, 1\le i\le n} dx_i \\
=&\beta \int_{-\infty}^{-K} dx_j \int_{\mathbb{R}^{n-1}} h_1(x) \prod_{i\neq j, 1\le i\le n} dx_i + (1-\beta) \int_{-\infty}^{-K} dx_j \int_{\mathbb{R}^{n-1}} h_2(x) \prod_{i\neq j, 1\le i\le n} dx_i \\
\le&\beta\alpha_j + (1-\beta)\alpha_j = \alpha_j,
\end{aligned}$$

and

$$\int_K^\infty dx_j \int_{\mathbb{R}^{n-1}} (\beta h_1(x) + (1-\beta)h_2(x)) \prod_{i\neq j, 1\le i\le n} dx_i \le 1-\alpha_j,$$

for all $j = 1, \ldots, n$. The proof of this proposition is completed. □

In the following, we will present a version of the Feynman-Kac formula for time-inhomogeneous PDE cited from [13, pages 131–132, Theorem 2.2].

Theorem 3.10. *Assume that* $\sigma\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$, $b\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ *are continuous in* (t,x) *and are Lipschitz continuous in* $x \in \mathbb{R}^n$ *uniformly in* $t \in [0,T]$. *Suppose that* $c\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ *is continuous and bounded and that* $f\colon \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and of polynomial growth. Denote*

$$a_{ij}(t,x) = \sum_{k=1}^n \sigma_{ik}\sigma_{jk}(t,x) \qquad \textit{or we can write} \quad a = (a_{ij})_{i,j=1}^n := \sigma\sigma^*\,.$$

Then for any $(t,x) \in (0,T] \times \mathbb{R}^n$, *the following stochastic differential equation*

$$\begin{cases} dX_s^{t,x} = \sigma\left(t-s, X_s^{t,x}\right) dW_s + b\left(t-s, X_s^{t,x}\right) ds, \quad s \in [0,t], \\ X_0^{t,x} = x\,, \end{cases} \tag{3.19}$$

has a unique solution $\{X_s^{t,x}, 0 \le s \le t\}$. *Assume that the following PDE*

$$\begin{cases} \dfrac{\partial}{\partial t}u_t(x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}u_t(x) + \langle b(t,x), \nabla u_t(x)\rangle + c(t,x)u_t(x)\,, \\ u_0(x) = f(x)\,, \end{cases} \tag{3.20}$$

has a unique solution u *so that its first and second derivatives with respect to both time and spatial variables are all continuously bounded on bounded domain. Then*

$$u_t(x) = \mathbb{E}\left[f\left(X_t^{t,x}\right)\exp\left(\int_0^t c(t-s, X_s^{t,x})ds\right)\right]. \tag{3.21}$$

Note that the Hypotheses (H1)-(H5) imposed on coefficients F and σ in SDE (2.2) imply that all the conditions required on functions σ, b, and c in Theorem 3.10 hold true. Additionally, Hypothesis (I) implies that f is uniformly bounded on $\mathbb{R}^n$. In other words, we can write the next proposition immediately.

Proposition 3.11. *Assume Hypotheses (H1)-(H5) and (I). The density of the solution to SDE* (2.2)*, satisfying the PDE* (2.3)*, can be written by the Feynman-Kac formula* (3.21) *with* $\sigma(t,x)$ *and* $b(t,x)$ *replaced by* $D\sigma(t,\omega_t,x)$ *and* $F(t,\omega_t,x)$ *respectively, and* c *defined as in* (2.6).

Proposition 3.12. *Assume that hypotheses (H1)-(H5) hold true. Let* u^ω *be the solution to* (2.3) *with initial condition* f *satisfying hypothesis (I). Then, we have for any* $t_0 > 0$,

$$\begin{aligned} U' := \sup_{\substack{t\in[t_0,T] \\ \omega\in C_b([0,T];\mathbb{R}^n)}} \Bigg(& \int_{\mathbb{R}^n} \sup_{|y|\ge r} u_t^\omega(y)^2 \left(|r|^{4n+\varepsilon-1} + r^{n-1}\right) dr \\ & + \int_0^\infty \left[\sup_{|z|\ge\lambda} |\nabla u_t^\omega(z)|^4\right] \left(\lambda^{4n-1+\varepsilon} + \lambda^{n-1}\right) d\lambda \Bigg) < \infty. \end{aligned} \tag{3.22}$$

Proof. We shall show that the second term in (3.22) is uniformly bounded. The uniform boundedness of the first term can be proved similarly. Using Proposition 3.11, for any $t \in [0,T]$ and $x \in \mathbb{R}$, we can write

$$u_t^\omega(x) = \mathbb{E}\left(f(X_t^{\omega,t,x})\exp\left(\int_0^t c\left(t-s, \omega_{t-s}, X_s^{\omega,t,x}\right)ds\right)\right),$$

where $X^{\omega,t,x}$ is the solution to (3.19) with $\sigma(t,x)$ and $b(t,x)$ replaced by $D\sigma(t,\omega(t),x)$ and $F(t,\omega(t),x)$. Differentiating this expression with respect to x, we have

$$\begin{aligned}\nabla u_t^\omega(x) =&\mathbb{E}\Big[\exp\left(\int_0^t c\left(t-s,\omega,X_s^{\omega,t,x}\right)ds\right)\nabla f(X_t^{\omega,t,x})\nabla X_t^{\omega,t,x}\\&+f(X_t^{\omega,t,x})\exp\left(\int_0^t c\left(t-s,\omega_{t-s},X_s^{\omega,t,x}\right)ds\right)\\&\int_0^t \nabla c\left(t-s,\omega_{t-s},X_s^{\omega,t,x}\right)\nabla X_s^{\omega,t,x}ds\Big].\end{aligned}$$

Due to hypotheses (H4), we know that c, ∇c are both bounded functions. By Cauchy-Schwarz's and Minkowski's inequalities, we can show that

$$\left|\nabla u_t^\omega(x)\right| \le C\Big[\big\||\nabla f(X_t^{\omega,t,x})|\big\|_2\big\||\nabla X_t^{\omega,t,x}|\big\|_2+\big\|f(X_t^{\omega,t,x})\big\|_2\int_0^t\big\||\nabla X_s^{\omega,t,x}|\big\|_2 ds. \tag{3.23}$$

Note that for any $r\in(0,t)$, $\nabla X_r^{\omega,t,x}$ satisfies the following equation

$$\begin{aligned}\nabla X_r^{\omega,t,x} =&I_n+\int_0^r\nabla(D\sigma)\left(t-s,\omega_{t-s},X_s^{\omega,t,x}\right)\nabla X_s^{\omega,t,x}dW_s\\&+\int_0^r\nabla F\left(t-s,\omega_{t-s},X_s^{\omega,t,x}\right)\nabla X_s^{\omega,t,x}ds.\end{aligned}$$

From Burkholder-Davis-Gundy's and Jensen's and Minkowski's inequalities it follows that

$$\begin{aligned}\big\||\nabla X_r^{\omega,t,x}|\big\|_2^2 \le& n+T^{\frac12}\int_0^r\big\||\nabla F\left(t-s,\omega_{t-s},X_s^{\omega,t,x}\right)\nabla X_s^{\omega,t,x}|\big\|_2^2 ds\\&+\int_0^r\big\||\nabla(D\sigma)\left(t-s,\omega_{t-s},X_s^{\omega,t,x}\right)\nabla X_s^{\omega,t,x}|\big\|_2^2ds\\ \le& n+C(\kappa,T)\int_0^r\big\||\nabla X_s^{\omega,t,x}|\big\|_2^2ds.\end{aligned}$$

By Gronwall's inequality, we obtain

$$\big\||\nabla X_r^{\omega,t,x}|\big\|_2\le\sqrt{ne^{C(\kappa,T)r}}\le\sqrt{ne^{C(\kappa,T)T}}.$$

Inserting this inequality into (3.23), we obtain

$$\left|\nabla u_t^\omega(x)\right| \le C(\kappa,T)\big(\big\||\nabla f(X_t^{\omega,t,x})|\big\|_2+\big\|f(X_t^{\omega,t,x})\big\|_2\big).$$

This implies

$$\begin{aligned}
&\int_0^\infty \Big[\sup_{|z|\geq\lambda} |\nabla u_t^\omega(z)|^4\Big] \left(\lambda^{4n-1+\varepsilon}+\lambda^{n-1}\right) d\lambda \\
\leq &C\int_0^\infty \Big[\sup_{|z|\geq\lambda} \left(\big\||\nabla f(X_t^{\omega,t,z})|\big\|_2 + \big\|f(X_t^{\omega,t,z})\big\|_2\right)^4\Big] \left(\lambda^{4n-1+\varepsilon}+\lambda^{n-1}\right) d\lambda \\
\leq &C\Bigg(\int_0^\infty \Big[\sup_{|z|\geq\lambda} \Big(\int_{\mathbb{R}^n} |\nabla f(x)|^2 p_t^\omega(x,z)dx\Big)^2\Big] \left(\lambda^{4n-1+\varepsilon}+\lambda^{n-1}\right) d\lambda \\
&\quad + \int_0^\infty \Big[\sup_{|z|\geq\lambda} \Big(\int_{\mathbb{R}^n} |f(x)|^2 p_t^\omega(x,z)dx\Big)^2\Big] \left(\lambda^{4n-1+\varepsilon}+\lambda^{n-1}\right) d\lambda\Bigg) \\
:=&C(D_1+D_2),
\end{aligned} \tag{3.24}$$

where $p_t^\omega(\cdot,\xi)$ is the probability density of the solution to (2.2) with initial condition $X_0^\omega = \xi \in \mathbb{R}^n$. Applying Theorem 3.3 and Jensen's inequality, we can show that

$$\begin{aligned}
D_1 \leq &C\int_0^\infty \Big[\sup_{|z|\geq\lambda} \int_{\mathbb{R}^n} |\nabla f(x)|^4 \frac{C_T}{t^{n^2/2}} \exp\left(-\frac{|\mathcal{T}_t^{-1}(\theta^\omega(t,z)-x)|^2}{C_T}\right) dx\Big] \\
&\qquad\qquad \left(\lambda^{4n-1+\varepsilon}+\lambda^{n-1}\right) d\lambda.
\end{aligned} \tag{3.25}$$

Now that hypothesis (I) implies that

$$\int_{\mathbb{R}^n} |\nabla f(x)|^4 dx \leq \int_0^\infty \sup_{|x|\geq\lambda} |\nabla f(x)|^4 \lambda^{n-1} d\lambda \leq U \tag{3.26}$$

and

$$\sup_{|x|\geq\delta} |\nabla f(x)| \leq \delta^{-1}\int_0^\delta \sup_{|x|\geq\lambda} |\nabla f(x)| d\lambda \leq \delta^{-1} U, \tag{3.27}$$

for any $\delta > 0$. Using (3.26) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla f(x)|^4 \frac{C_T}{t^{n^2/2}} \exp\left(-\frac{|\mathcal{T}_t^{-1}(\theta^\omega(t,z)-x)|^2}{C_T}\right) dx \\
=&\int_{|x|\leq\delta} |\nabla f(x)|^4 \frac{C_T}{t^{n^2/2}} \exp\left(-\frac{|\mathcal{T}_t^{-1}(\theta^\omega(t,z)-x)|^2}{C_T}\right) dx \\
&+ \int_{|x|>\delta} |\nabla f(x)|^4 \frac{C_T}{t^{n^2/2}} \exp\left(-\frac{|\mathcal{T}_t^{-1}(\theta^\omega(t,z)-x)|^2}{C_T}\right) dx \\
\leq&\frac{C_T}{t^{n^2/2}} e^{\frac{(t^{-(2n-1)}\vee t^{-1})\delta^2}{C_T}} U \exp\left(-\frac{|\mathcal{T}_t^{-1}(\theta^\omega(t,z))|^2}{C_T}\right) \\
&+ C_T\int_{\mathbb{R}^n} \mathbf{1}_{\{|\theta^\omega(t,z)-\mathcal{T}_t(y)|>\delta\}} |\nabla f(\theta^\omega(t,z)-\mathcal{T}_t(y))|^4 \exp\left(-\frac{|y|^2}{C_T}\right) dy,
\end{aligned}$$

where in the second part of the last inequality we perform a change of variable $x \to y = \mathcal{T}_t^{-1}(\theta^\omega(t,z) - x)$.

By Lemma 3.5, we have that

$$|\theta^\omega(t,z)| \ge (e^{-\kappa T}|z| - \kappa T)\mathbf{1}_{\{|z|\ge e^{\kappa T}\kappa T\}}.$$

This implies that

$$\begin{aligned}
&\int_0^\infty \frac{C_T}{t^{n^2/2}} e^{\frac{(t^{-(2n-1)}\vee t^{-1})\delta^2}{C_T}} U\left[\sup_{|z|\ge\lambda} \exp\left(-\frac{|\mathcal{T}_t^{-1}(\theta^\omega(t,z))|^2}{C_T}\right)\right]\left(\lambda^{4n-1+\varepsilon} + \lambda^{n-1}\right) d\lambda \\
\le & \frac{C_T}{t^{n^2/2}} e^{\frac{(t^{-(2n-1)}\vee t^{-1})\delta^2}{C_T}} U \int_{e^{\kappa T}\kappa T}^\infty \exp\left(-\frac{(t^{-2n+1}\wedge t^{-1})(e^{-\kappa T}\lambda - \kappa T)^2}{C_T}\right) \\
&\qquad \left(\lambda^{4n-1+\varepsilon} + \lambda^{n-1}\right) d\lambda \\
&+ \frac{C_T}{t^{n^2/2}} e^{\frac{(t^{-(2n-1)}\vee t^{-1})\delta^2}{C_T}} U \int_0^{e^{\kappa T}\kappa T} \left(\lambda^{4n-1+\varepsilon} + \lambda^{n-1}\right) d\lambda \le C, \qquad (3.28)
\end{aligned}$$

for some C depending on $n, C_T, t_0, T, \epsilon, \kappa$ and U. Similarly, on the set

$$\{|z| \ge \lambda\} \cap \{|\theta^\omega(t,z) - \mathcal{T}_t(y)| > \delta\},$$

we can deduce that

$$|\theta^\omega(t,z) - \mathcal{T}_t(y)| \ge \delta \vee (e^{-\kappa T}\lambda - \kappa T - |\mathcal{T}_t(y)|).$$

Therefore, it follows from (3.27) that

$$\begin{aligned}
&\int_0^\infty \left[\sup_{|z|\ge\lambda} \int_{\mathbb{R}^n} \mathbf{1}_{\{|\theta^\omega(t,z)-\mathcal{T}_t(y)|>\delta\}} |\nabla f(\theta^\omega(t,z) - \mathcal{T}_t(y))|^4 \exp\left(-\frac{|y|^2}{C_T}\right) dy\right] \\
&\qquad \left(\lambda^{4n-1+\varepsilon} + \lambda^{n-1}\right) d\lambda \\
\le & \int_{\mathbb{R}^n} \int_0^\infty \left[\sup_{|\tilde z|\ge \delta\vee(e^{-\kappa T}\lambda - \kappa T - |\mathcal{T}_t(y)|)} |\nabla f(\tilde z)|^4\right] \left(\lambda^{4n-1+\varepsilon} + \lambda^{n-1}\right) e^{-\frac{|y|^2}{C_T}} d\lambda dy \\
\le & \int_{\mathbb{R}^n} \int_0^\infty \mathbf{1}_{\{e^{-\kappa T}\lambda - \kappa T - |\mathcal{T}_t(y)|<\delta\}} \left[\sup_{|\tilde z|\ge\delta} |\nabla f(\tilde z)|^4\right] \left(\lambda^{4n-1+\varepsilon} + \lambda^{n-1}\right) e^{-\frac{|y|^2}{C_T}} d\lambda dy \\
&+ \int_{\mathbb{R}^n} \int_0^\infty \mathbf{1}_{\{e^{-\kappa T}\lambda - \kappa T - |\mathcal{T}_t(y)|\ge\delta\}} \left[\sup_{|\tilde z|\ge e^{-\kappa T}\lambda - \kappa T - |\mathcal{T}_t(y)|} |\nabla f(\tilde z)|^4\right] \\
&\qquad \left(\lambda^{4n-1+\varepsilon} + \lambda^{n-1}\right) e^{-\frac{|y|^2}{C_T}} d\lambda dy \\
\le & \delta^{-1} U \int_{\mathbb{R}^n} \int_0^{e^{\kappa T}(\delta + \kappa T + |\mathcal{T}_t(y)|)} \left(\lambda^{4n-1+\varepsilon} + \lambda^{n-1}\right) d\lambda e^{-\frac{|y|^2}{C_T}} dy + e^{(4n+\varepsilon)\kappa T} \int_{\mathbb{R}^n} \int_0^\infty e^{-\frac{|y|^2}{C_T}} \\
&\quad \times \left[\sup_{|\tilde z|\ge\tau} |\nabla f(\tilde z)|^4\right] \left(|\tau + \kappa T + |\mathcal{T}_t(y)||^{4n-1+\varepsilon} + |\tau + \kappa T + |\mathcal{T}_t(y)||^{n-1}\right) d\tau dy \\
\le & C, \qquad (3.29)
\end{aligned}$$

where $C > 0$ depends on $n, C_T, t_0, T, \epsilon, \kappa$, U and δ. Combining (3.25), (3.28) and (3.29) we see that D_1 is bounded. Using a similar argument, we can prove that D_2 is bounded uniformly in $t \in [t_0, T]$ and $\omega \in C_b([0,T];\mathbb{R}^n)$. The proof of this proposition is then completed. □

Remark 3.13. The main difficulty in the above proof is to show the integrability over an unbounded domain with respect to λ. After (3.27) we divided the integral domain into $|x| \le \delta$ and $|x| > \delta$ is for simplicity because even if we use $|x| \le \delta\sqrt{t}$ and $|x| > \delta\sqrt{t}$, we cannot get rid of the t_0 in the statement (3.22).

4. Proof of the main results

Now, we are ready to prove Theorems 2.4 and 2.5. In the first subsection we shall prove the existence and uniqueness of the local solution to PDE (2.7) up to a small time t_0.

4.1. Local solution.

In this subsection, we aim to apply the Banach fixed point theorem to prove a local version of Theorem 2.5 (see Proposition 4.4 for a local version). First, we need to bound the distance of quantiles by the distance of distributions. The following lemma is known (see e.g. [18]). We rewrite a short proof for the sake of completeness.

Lemma 4.1. *Let $\alpha \in (0,1)^n$ and let K, δ, ε be positive constants. Denote by $\mathcal{S}$ the collection of density functions satisfying (3.18). Then, for any $h_1, h_2 \in \mathcal{S}$,*

$$|Q_\alpha(h_1) - Q_\alpha(h_2)| \le \sqrt{n}(2K)^{-(n-1)}\delta^{-1}|h_1 - h_2|_{L^1}. \tag{4.1}$$

Proof. Since $h_1, h_2 \in \mathcal{S}$, where $\mathcal{S}$ is a convex set, we know that for any $\beta \in (0,1)$,

$$h^\beta := \beta h_1 + (1-\beta)h_2 \in \mathcal{S}$$

as well. Write $\hat{\omega}(\beta) = (\hat{\omega}_1(\beta), \ldots, \hat{\omega}_n(\beta)) = Q_\alpha(h^\beta)$.

By definition of the quantile, for any $j = 1, \cdots, n$, we have

$$\int_{-\infty}^{\hat{\omega}_j(\beta)} dx_j \int_{\mathbb{R}^{n-1}} h^\beta(x) \prod_{k\neq j} dx_k = \alpha_j. \tag{4.2}$$

Differentiating both sides of (4.2) with respect to β yields

$$\hat{\omega}_j'(\beta) \int_{\mathbb{R}^{n-1}} h^\beta(x) \prod_{k\neq j} dx_k \Big|_{x_j=\hat{\omega}_j(\beta)} + \int_{-\infty}^{\hat{\omega}_j(\beta)} dx_j \int_{\mathbb{R}^{n-1}} (h_1(x) - h_2(x)) \prod_{k\neq j} dx_k = 0\,.$$

Thus

$$\hat{\omega}_j'(\beta) = -\Big[\int_{\mathbb{R}^{n-1}} h^\beta(x) \prod_{k\neq j} dx_k \Big|_{x_j=\hat{\omega}_j(\beta)}\Big]^{-1} \int_{-\infty}^{\hat{\omega}_j(\beta)} dx_j \int_{\mathbb{R}^{n-1}} (h_1(x) - h_2(x)) \prod_{k\neq j} dx_k.$$

It follows that

$$
\begin{aligned}
|Q_\alpha^j(h_1) - Q_\alpha^j(h_2)| &= |\hat{\omega}_j(1) - \hat{\omega}_j(0)| = \left| \int_0^1 \hat{\omega}_j'(\beta) d\beta \right| \\
&= \left| \int_0^1 \frac{\int_{-\infty}^{\hat{\omega}_j(\beta)} dx_j \int_{\mathbb{R}^{n-1}} (h_1(x) - h_2(x)) \prod_{k\neq j} dx_k}{\int_{\mathbb{R}^{n-1}} h^\beta(x) \prod_{k\neq j} dx_k \big|_{x_j = \hat{\omega}_j(\beta)}} d\beta \right| \\
&\leq |h_1 - h_2|_{L^1} \left| \int_0^1 \left[\int_{\mathbb{R}^{n-1}} h^\beta(x) \prod_{k\neq j} dx_k \bigg|_{x_j = \hat{\omega}_j(\beta)} \right]^{-1} d\beta \right|. \qquad (4.3)
\end{aligned}
$$

Recall that $h^\beta \in \mathcal{S}$. This implies that $\max_{1\leq j\leq n} |\hat{\omega}_j(\beta)| \leq K$, and thus by (3.18) we have

$$
\begin{aligned}
\int_{\mathbb{R}^{n-1}} h^\beta(x) \prod_{k\neq j} dx_k \bigg|_{x_j = \hat{\omega}_j(\beta)} &\geq \int_{[-K,K]^{n-1}} h^\beta(x) \prod_{k\neq j} dx_k \bigg|_{x_j = \hat{\omega}_j(\beta)} \\
&\geq \int_{[-K,K]^{n-1}} \delta \prod_{k\neq j} dx_k \geq (2K)^{n-1}\delta.
\end{aligned}
$$

As a consequence, we have

$$
|Q_\alpha^j(h_1) - Q_\alpha^j(h_2)| \leq (2K)^{-(n-1)} \delta^{-1} |h_1 - h_2|_{L^1},
$$

for all $j = 1, \ldots, n$, which yields the lemma. □

The next proposition describes the dependence of the solution of (2.3) with respect to ω. It will be used to bound the distance of distributions of the solutions to (1.5) by the quantiles.

Proposition 4.2. *Let the hypotheses (H1)-(H5) be satisfied. Let $u^{(1)} = u^{\omega^{(1)}}$ and $u^{(2)} = u^{\omega^{(2)}}$ be the solutions to equation (2.3) corresponding to the continuous functions $\omega = \omega^{(1)}$ and $\omega = \omega^{(2)}$ respectively and with the same initial condition f satisfying hypothesis (I). Then, the following inequality holds true*

$$
\sup_{s\in[0,t]} |u_s^{(1)} - u_s^{(2)}|_{L^1} \leq C_0 \left(t + \sqrt{t} \right) \sup_{s\in[0,t]} |\omega_s^{(1)} - \omega_s^{(2)}|, \quad \forall\, t \in [0, T], \qquad (4.4)
$$

where C_0 is a positive constant independent of $\omega^{(1)}$, $\omega^{(2)}$ and t.

Proof. Recall that the equation (2.3) has a unique solution. Then, by the Feynman-Kac formula (Theorem 3.10), for $i = 1$ and 2, we can write

$$
u_t^{(i)}(x) = \mathbb{E} \left(f(X_t^{(i),t,x}) \exp \left(\int_0^t c^{(i)} \left(t - s, X_s^{(i),t,x} \right) ds \right) \right),
$$

where $X^{(i),t,x} = X^{\omega^{(i)},t,x}$ is the solution to (3.19) with the initial condition $X_0^{(i),t,x} = x$ and the coefficients

$$a^{(i)}(t-s,x) = a(t-s,\omega_{t-s}^{(i)},x),\ \ b^{(i)}(t-s,x) = b(t-s,\omega_{t-s}^{(i)},x),$$
$$c^{(i)}(t-s,x) = c(t-s,\omega_{t-s}^{(i)},x),$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^n$ with a, b and c being defined by (2.4)-(2.6) respectively. Thus, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |u_t^{(1)}(x) - u_t^{(2)}(x)| dx \\
=&\int_{\mathbb{R}^n} \mathbb{E}\Big[f(X_t^{(1),t,x}) \exp\left(\int_0^t c^{(1)}\left(t-s, X_s^{(1),t,x}\right) ds \right) \\
&- f(X_t^{(2),t,x}) \exp\left(\int_0^t c^{(2)}\left(t-s, X_s^{(2),t,x}\right) ds \right) \Big] dx \\
=&\int_{\mathbb{R}^n} \mathbb{E}\Big\{ f(X_t^{(1),t,x}) \Big[\exp\left(\int_0^t c^{(1)}\left(t-s, X_s^{(1),t,x}\right) ds \right) - \exp\left(\int_0^t c^{(2)}\left(t-s, X_s^{(2),t,x}\right) ds \right) \Big] \Big\} dx \\
&+ \int_{\mathbb{R}^n} \mathbb{E}\Big[\left(f(X_t^{(1),t,x}) - f(X_t^{(2),t,x}) \right) \exp\left(\int_0^t c^{(2)}\left(t-s, X_s^{(2),t,x}\right) ds \right) \Big] dx \\
=& I_1 + I_2.
\end{aligned} \tag{4.5}$$

Due to hypothesis (H4), we know that the function c is uniformly bounded on $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ by 2κ, and Lipschitz continuous. Then, the first term of (4.5) is bounded by using the mean value theorem as follows:

$$\begin{aligned}
I_1 =& \int_{\mathbb{R}^n} \mathbb{E}\Big[f(X_t^{(1),t,x}) \Big(\exp\left(\int_0^t c^{(1)}\left(t-s, X_s^{(1),t,x}\right) ds \right) \\
&- \exp\left(\int_0^t c^{(2)}\left(t-s, X_s^{(2),t,x}\right) ds \right) \Big) \Big] dx \\
\le& e^{2\kappa T} \int_{\mathbb{R}^n} \mathbb{E}\Big[f(X_t^{(1),t,x}) \left(\int_0^t c^{(1)}\left(t-s, X_s^{(1),t,x}\right) ds - \int_0^t c^{(2)}\left(t-s, X_s^{(2),t,x}\right) ds \right) \Big] dx \\
=& e^{2\kappa T} \int_{\mathbb{R}^n} \mathbb{E}\Big[f(X_t^{(1),t,x}) \left(\int_0^t c\left(t-s, \omega_{t-s}^{(1)}, X_s^{(1),t,x}\right) - c\left(t-s, \omega_{t-s}^{(1)}, X_s^{(2),t,x}\right) ds \right) \Big] dx \\
&+ e^{2\kappa T} \int_{\mathbb{R}^n} \mathbb{E}\Big[f(X_t^{(1),t,x}) \left(\int_0^t c\left(t-s, \omega_{t-s}^{(1)}, X_s^{(2),t,x}\right) - c\left(t-s, \omega_{t-s}^{(2)}, X_s^{(2),t,x}\right) ds \right) \Big] dx \\
\le& c_{\kappa,T} \int_0^t |\omega_{t-s}^{(1)} - \omega_{t-s}^{(2)}| ds \int_{\mathbb{R}^n} \mathbb{E}[f(X_t^{(1),t,x})] dx \\
&+ c_{\kappa,T} \int_{\mathbb{R}^n} \mathbb{E}\Big[f(X_t^{(1),t,x}) \int_0^t |X_s^{(1),t,x} - X_s^{(2),t,x}| ds \Big] dx \\
=& c_{\kappa,T} \left(I_{11} + I_{12} \right),
\end{aligned} \tag{4.6}$$

where $c_{\kappa,T}$ is a positive constant depending on κ and T. For $i=1,2$, denote by $p_t^{(i)}(\cdot,x)$ the probability density of $X_t^{(i)}$ and write $\theta^{(i)}=\theta^{\omega^{(i)}}$ the solution to (3.4) with $\omega=\omega^{(i)}$. Then, by Theorem 3.3 and Corollary 3.6, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\mathbb{E}[f(X_t^{(1),t,x})]dx &= \int_{\mathbb{R}^{2n}} f(y)p_t^{(1)}(y,x)dydx\\ &\le C_T\int_{\mathbb{R}^{2n}} f(y)t^{-n^2/2}\exp\left(-C_T^{-1}\sum_{i=1}^n\left(\frac{\theta_i^{(1)}(t,x)-y_i}{t^{i-\frac{1}{2}}}\right)^2\right)dxdy\\ &\le C_T\int_{\mathbb{R}^{2n}} f(y)\exp\left(-C_T^{-1}|z|^2\right)\det\left(\nabla(\theta^{(1)})^{-1}\left(t,y-\mathcal{T}_t(z)\right)\right)dzdy\\ &\le C_Te^{n\kappa T}\int_{\mathbb{R}^n} f(y)dy\int_{\mathbb{R}^n}\exp\left(-C_T^{-1}|z|^2\right)dz\\ &\le C_Te^{n\kappa T}(C_T\pi)^{\frac{n}{2}}. \end{aligned}\tag{4.7}$$

Hence,

$$I_{11}\le C_1t\sup_{s\in[0,t]}|\omega_s^{(1)}-\omega_s^{(2)}|,\tag{4.8}$$

for some positive constant C_1 independent of $\omega^{(1)}$, $\omega^{(2)}$ and t. On the other hand, for any $p\ge 1$, we can deduce that, for some constant $c_{n,p}>0$ depending on n and p,

$$\begin{aligned}&\mathbb{E}\big|X_t^{(1),t,x}-X_t^{(2),t,x}\big|^{2p}\\ &\le c_{n,p}\Bigg[\sum_{i=1}^n\mathbb{E}\left(\int_0^t\left(F_i(t-s,\omega_{t-s}^{(1)},X_s^{(1),t,x})-F_i(t-s,\omega_{t-s}^{(2)},X_s^{(2),t,x})\right)ds\right)^{2p}\\ &\quad+\mathbb{E}\left(\int_0^t\left(\sigma(t-s,\omega_{t-s}^{(1)},X_s^{(1),t,x})-\sigma(t-s,\omega_{t-s}^{(2)},X_s^{(2),t,x})\right)dW_s\right)^{2p}\Bigg].\end{aligned}\tag{4.9}$$

By hypothesis (H1) and the Burkholder-Davis-Gundy and Jensen's inequalities, we have

$$
\begin{aligned}
&\mathbb{E}\big|X_t^{(1),t,x} - X_t^{(2),t,x}\big|^{2p} \\
&\le c_{n,p}\kappa\Big[t^{2p-1}\left(\int_0^t |\omega_{t-s}^{(1)} - \omega_{t-s}^{(2)}|^{2p}ds + \int_0^t \mathbb{E}|X_s^{(1),t,x} - X_s^{(2),t,x}|^{2p}ds\right) \\
&\quad + \mathbb{E}\left(\int_0^t \left(|\omega_{t-s}^{(1)} - \omega_{t-s}^{(2)}| + |X_s^{(1),t,x} - X_s^{(2),t,x}|\right)^2 ds\right)^p\Big] \\
&\le c_{n,p,\kappa}\Big[t^{2p-1}\left(\int_0^t |\omega_{t-s}^{(1)} - \omega_{t-s}^{(2)}|^{2p}ds + \int_0^t \mathbb{E}|X_s^{(1),t,x} - X_s^{(2),t,x}|^{2p}ds\right) \\
&\quad + t^{p-1}\int_0^t |\omega_{t-s}^{(1)} - \omega_{t-s}^{(2)}|^{2p}ds + t^{p-1}\int_0^t \mathbb{E}|X_s^{(1),t,x} - X_s^{(2),t,x}|^{2p}ds\Big] \\
&\le c_{n,p,\kappa}(t^p + t^{2p})\sup_{s\in[0,t]}|\omega_{t-s}^{(1)} - \omega_{t-s}^{(2)}|^{2p} + c_{n,p,\kappa}(t^{2p-1} + t^{p-1})\int_0^t \mathbb{E}|X_s^{(1),t,x} - X_s^{(2),t,x}|^{2p}ds.
\end{aligned}
$$

An application of Gronwall's inequality yields that

$$
\mathbb{E}\big|X_t^{(1),t,x} - X_t^{(2),t,x}\big|^{2p} \le c_{n,p,\kappa}(t^p + t^{2p})e^{c_{n,p,\kappa}(T^{2p}+T^p)}\sup_{s\in[0,t]}|\omega_s^{(1)} - \omega_s^{(2)}|^{2p}. \tag{4.10}
$$

By Fubini's theorem, Hölder's inequality and (4.10), we get that

$$
\begin{aligned}
I_{12} &= \int_0^t \int_{\mathbb{R}^n} \mathbb{E}\big[f(X_t^{(1),t,x})|X_s^{(1),t,x} - X_s^{(2),t,x}|\big]dxds \\
&\le \int_0^t \int_{\mathbb{R}^n} \|f(X_t^{(1),t,x})\|_2 \big\||X_s^{(1),t,x} - X_s^{(2),t,x}|\big\|_2 dxds \\
&\le c_{n,p,T}\sup_{s\in[0,t]}|\omega_s^{(1)} - \omega_s^{(2)}| \int_0^t \int_{\mathbb{R}^n} \|f(X_t^{(1),t,x})\|_2 dxds, \qquad (4.11)
\end{aligned}
$$

for some positive constant $c_{n,p,T}$ depending on n, p and T. Notice that by Theorem 3.3, Corollary 3.6 and Cauchy-Schwarz's inequality, we can deduce that

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \|f(X_t^{(1),t,x})\|_2 dx = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(y)^2 p_t^{(1)}(y,x)dy\right)^{\frac{1}{2}}dx \\
&\quad\le \frac{\sqrt{C_T}}{t^{n^2/4}}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(y)|^2\exp\left(C_T^{-1}|\mathcal{T}_t^{-1}(\theta^{(1)}(t,x) - y)|^2\right)dy\right)^{\frac{1}{2}}dx \\
&\quad\le \frac{\sqrt{C_T}}{t^{n^2/4}}\left(\int_{\mathbb{R}^{2n}}|f(y)|^2\exp\left(C_T^{-1}|\mathcal{T}_t^{-1}(\theta^{(1)}(t,x) - y)|^2\right)\left(|\theta^{(1)}(t,x)|^{\frac{n+\varepsilon}{2}} \vee 1\right)^2 dydx\right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_{\mathbb{R}^n}\left(|\theta^{(1)}(t,h)|^{\frac{n+\varepsilon}{2}} \vee 1\right)^{-2}dh\right)^{\frac{1}{2}}.
\end{aligned}
$$

By changing of variables $x \to z = \mathcal{T}_t^{-1}(\theta^{(1)}(t,x) - y)$ and $h \to l = \theta^{(1)}(t,h)$, we can write

$$\begin{aligned}
&\int_{\mathbb{R}^n} \|f(X_t^{(1),t,x})\|_2 dx \\
\leq & \sqrt{C_T}\left[\int_{\mathbb{R}^{2n}} \det\left(\nabla\left(\theta^{(1)}\right)^{-1}(t, y+\mathcal{T}_t(z))\right)|f(y)|^2 e^{-\frac{|z|^2}{C_T}}\left(|\mathcal{T}_t z + y|^{n+\varepsilon} \vee 1\right) dz dy\right]^{\frac{1}{2}} \\
&\times \left[\int_{\mathbb{R}^n} \det\left(\nabla\left(\theta^{(1)}\right)^{-1}(t,l)\right)\left(|l|^{-(n+\varepsilon)} \vee 1\right) dl\right]^{\frac{1}{2}} \\
\leq & c_{n,\epsilon,T}\sqrt{C_T} e^{n\kappa T}\left(\int_{\mathbb{R}^{2n}} f(y)^2 \exp\left(-C_T^{-1}|z|^2\right)\left(|z|^{n+\varepsilon} + |y|^{n+\varepsilon} + 1\right) dz dy\right)^{\frac{1}{2}} \\
&\times \left[\int_{\mathbb{R}^n} \left(|l|^{-(n+\varepsilon)} \vee 1\right) dl\right]^{\frac{1}{2}}. \qquad (4.12)
\end{aligned}$$

Recall that $f > 0$ is a probability density satisfying hypothesis (I). (4.12) tells us that

$$\int_{\mathbb{R}^n} \|f(X_t^{(1),t,x})\|_2 dx \leq C, \quad \forall t \in [0,T], \qquad (4.13)$$

where $C > 0$ depends on $C_T, n, p, \kappa, \epsilon, T$ and U. Combining inequalities (4.11) and (4.13), we finally obtain

$$I_{12} \leq C_1 t \sup_{s\in[0,t]} |\omega_s^{(1)} - \omega_s^{(2)}|, \qquad (4.14)$$

for some C_1 independent of $\omega^{(1)}, \omega^{(2)}$ and t.

In the next step, we estimate the term I_2 in (4.5). By Cauchy-Schwarz's inequality and the fact that c is uniformly bounded, we can write

$$\begin{aligned}
I_2 &\leq \int_{\mathbb{R}^n} \|f(X_t^{(1),t,x}) - f(X_t^{(2),t,x})\|_2 \left\| \exp\left(\int_0^t c^{(2)}\left(t-s, X_s^{(2),t,x}\right) ds\right)\right\|_2 dx \\
&\leq e^{2\kappa T}\int_{\mathbb{R}^n} \|f(X_t^{(1),t,x}) - f(X_t^{(2),t,x})\|_2 dx. \qquad (4.15)
\end{aligned}$$

To bound the above integral, we first claim the following version of the mean value theorem. For any $x, y \in \mathbb{R}^n$, the following inequality holds true:

$$|f(x) - f(y)| \leq 2 \sup_{|x|\wedge|y|\leq|z|\leq|x|\vee|y|} |\nabla f(z)||x-y|. \qquad (4.16)$$

Before proceeding with the proof of (4.16), let us make a brief remark for this inequality. Typically, by applying the first order Taylor expansion, one gets

$$|f(x)-f(y)| \le \int_0^1 |\nabla f(\lambda x+(1-\lambda)y)||x-y|d\lambda \\ \le \sup_{\lambda\in[0,1]} |\nabla f(\lambda x+(1-\lambda)y)||x-y|. \tag{4.17}$$

This estimate is not sharp enough compared with (4.16). For example, consider a situation that ∇f decrease to 0 at infinity, e.g. $f(x)=\exp(-|x|^2)$. Then, in case both x and y are far away from 0, the quantity $\sup_{|x|\wedge|y|\le|z|\le|x|\vee|y|}|\nabla f(z)|$ in (4.16) is very small. But in (4.17), one can only tell $\sup_{\lambda\in[0,1]}|\nabla f(\lambda x+(1-\lambda)y)|$ is bounded by a fixed constant, since the segment $\{\lambda x+(1-\lambda)y\colon \lambda\in[0,1]\}$ may be closed to 0 even if $|x|\wedge|y|\gg 0$. As in (4.15), we need to integrate with respect to x. Thus, the bound (4.16) and Condition (I) are applied for obtaining the desired result.

To establish (4.16), we consider a plane $\mathcal{P}$ such that $0,x,y\in\mathcal{P}$. Without loss of generality, suppose that $|x|\le|y|$. Let x' be the intersection of the straight line connecting 0 and y, and the circle $\mathcal{O}$ centered at 0 with radius $|x|$. Applying the fundamental theorem of calculus to the path integral of ∇f along the (shorter) arc $x\to x'$ on $\mathcal{O}$, and then along the straight line $x'\to y$, we obtain immediately,

$$|f(x)-f(y)| \le \sup_{|x|\le|z|\le|y|} |\nabla f(z)|(|\widehat{xx'}|+|y-x'|)\,, \tag{4.18}$$

where $\widehat{xx'}$ denotes the arc length. Since the angle between the ray $x'y$ and the line $x'x$ is greater than or equal to $\pi/2$, we see that both $\widehat{xx'}$ and $|y-x|$ are less than or equal to $|y-x|$. Thus, inequality (4.16) follows immediately from (4.18). It is worth noticing that we do not apply the mean value theorem on the straight line $x\to y$. Since if so, we have $|f(x)-f(y)|\le|\nabla f(\xi)||x-y|$, where the point $\xi=t_0x+(1-t_0)y$ for some $t_0\in[0,1]$. We can have $|\xi|\le|x|\vee|y|$. However, we cannot guarantee $|\xi|\ge|x|\wedge|y|$, which is critical in the following immediate application.

Using (4.16) and Cauchy-Schwarz's inequality, we can write

$$\|f(X_t^{(1),t,x})-f(X_t^{(2),t,x})\|_2 \\ \le\big\|g\big(|X_t^{(1),t,x}|\wedge|X_t^{(2),t,x}|\big)\big\|_4\big\|X_t^{(1),t,x}-X_t^{(2),t,x}\big\|_4, \tag{4.19}$$

where $g:\mathbb{R}_+\to\mathbb{R}$ is given by

$$g(\lambda):=\sup\{|\nabla f(z)|:|z|\ge\lambda\},\quad \forall\lambda\ge 0.$$

Notice that $g(\lambda_1\wedge\lambda_2)\le g(\lambda_1)+g(\lambda_2)$ for all $\lambda_1,\lambda_2\ge 0$. It follows that

$$\int_{\mathbb{R}^n}\big\|g\big(X_t^{(1),t,x}\wedge X_t^{(2),t,x}\big)\big\|_4 dx \\ \le\int_{\mathbb{R}^n}\big\|g\big(X_t^{(1),t,x}\big)\big\|_4 dx+\int_{\mathbb{R}^n}\big\|g\big(X_t^{(2),t,x}\big)\big\|_4 dx. \tag{4.20}$$

Therefore, proceeding with a similar argument to that in (4.12) and (4.13) and recalling hypothesis (I), we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \|g(X_t^{(1),t,x})\|_4 dx \\
\le& \frac{\sqrt{C_T}}{t^{n^2/4}} \left(\int_{\mathbb{R}^{2n}} |g(|y|)|^4 \exp\left(C_T^{-1}|\mathcal{T}_t^{-1}(\theta^{(1)}(t,x)-y)|^2\right) \left(|\theta^{(1)}(t,x)|^{\frac{3(n+\varepsilon)}{4}} \vee 1\right)^4 dydx \right)^{\frac{1}{4}} \\
&\times \left(\int_{\mathbb{R}^n} \left(|\theta^{(1)}(t,x)|^{\frac{3(n+\varepsilon)}{4}} \vee 1\right)^{-\frac{4}{3}} dx \right)^{\frac{3}{4}} \\
\le& c_{n,\epsilon}\sqrt{C_T} e^{n\kappa T} \left(\int_{\mathbb{R}^{2n}} |g(|y|)|^4 \exp\left(-C_T^{-1}|z|^2\right) \left(|z|^{3(n+\varepsilon)} + |y|^{3(n+\varepsilon)} + 1\right) dzdy \right)^{\frac{1}{2}} \\
&\times \left[\int_{\mathbb{R}^n} \left(|z|^{-(n+\varepsilon)} \wedge 1\right) dz \right]^{\frac{1}{2}} \le C, \qquad (4.21)
\end{aligned}$$

for some constant $C > 0$ depending on $C_T, n, p, \kappa, \epsilon$ and U. Combining inequalities (4.10), (4.15), (4.19) - (4.21), we get

$$I_2 \le C_2\sqrt{t+t^2} \sup_{s\in[0,t]} |\omega_s^{(1)} - \omega_s^{(2)}|. \qquad (4.22)$$

Therefore, inequality (4.4) follows by inserting inequalities (4.8), (4.14) and (4.22) into (4.5). As we can see from (4.13) and (4.21), the constant C_0 appearing in inequality (4.4) depends on the initial condition through U. □

Remark 4.3. In formulation (4.15), the function $c^{(2)}(t-s, X_s^{(2),t,x})$ is bounded because of the hypothesis (H4). This means that the integrability in x has to be guaranteed by that of the term $\|f(X_t^{(1),t,x}) - f(X_t^{(2),t,x})\|_2$. This is the reason that we assume the integrability hypothesis (I) on ∇f.

Proposition 4.4. *Assume that the conditions in Theorem 2.5 hold true. Then, there exists $t_0 > 0$ such that (2.7) with initial condition f has a unique solution on the interval $[0, t_0]$.*

Proof. For any continuous function $\omega \in C([0,T], \mathbb{R}^n)$ by a similar argument to that in Proposition 4.2, we have that

$$\lim_{s\to t} |u_t^\omega - u_s^\omega|_{L_1} = 0.$$

Then, it follow from (4.1) that

$$\lim_{s\to t} |Q_\alpha(u_t^\omega) - Q_\alpha(u_s^\omega)| \le \lim_{s\to t} \sqrt{n} K^{1-n}\delta^{-1}|u_t^\omega - u_s^\omega|_{L^1} = 0.$$

In other words, $Q_\alpha(u_t^\omega)$ is a continuous function in t.

We shall use the Banach fixed point theorem to prove the proposition. Fix a $t_0 > 0$ satisfying the condition given by (4.28) below. Let $C([0,t_0], \mathbb{R}^n)$ be the Banach space of all continuous functions with the sup norm. For any $\omega \in C([0,t_0],\mathbb{R}^n)$, let $u^\omega : [0,t_0]\times\mathbb{R}^n$ be the (unique) solution to (2.3) associated with ω. Define

$$\mathbb{B} = \{(\omega, u^\omega), \omega \in C([0,t_0],\mathbb{R}^n)\} \subseteq C([0,t_0],\mathbb{R}^n) \oplus C([0,t_0], L_1(\mathbb{R}^d)) \tag{4.23}$$

with the norm

$$\|(\omega,u^\omega)\|_{\mathbb{B}} = \sup_{0\le t\le t_0} |\omega(t)| + \sup_{0\le t\le t_0} |u^\omega_t|_{L_1}\,. \tag{4.24}$$

We claim that $\mathbb{B}$ is a closed set of the Banach space $C([0,t_0],\mathbb{R}^n)\oplus C([0,t_0],L_1(\mathbb{R}^d))$. In fact, if $(\omega^{(n)}, u^{\omega^{(n)}}) \in \mathbb{B}$ converges to $(\omega, v) \in C([0,t_0],\mathbb{R}^n)\oplus C([0,t_0],L_1(\mathbb{R}^d))$, then $\omega^{(n)} \to \omega$ in $C([0,t_0],\mathbb{R}^n)$ and $u^{\omega^{(n)}} \to v$ in $C([0,t_0],L_1(\mathbb{R}^d))$. Thus, $\omega \in C([0,t_0],\mathbb{R}^n)$. Solving (2.3) associated with ω, we obtain $u^\omega \in C([0,t_0],L_1(\mathbb{R}^d))$. By (4.4), we know that $u^{\omega^{(n)}} \to u^\omega$ in $C([0,t_0],L_1(\mathbb{R}^d))$. This implies that $v = u^\omega$. In other word, $\mathbb{B}$ is closed and hence it is also a Banach space.

Fix $\alpha = (\alpha_1,\cdots,\alpha_n) \in \mathbb{R}^n$. Let $K,\delta,\varepsilon > 0$ be defined in (3.15)-(3.17). Now, we define a mapping $\mathcal{M} : \mathbb{B} \to \mathbb{B}$ as follows

$$\mathcal{M}(\omega,u^\omega) = (\mathcal{M}_1(\omega,u^\omega), \mathcal{M}_2(\omega,u^\omega))\,, \tag{4.25}$$

where $(\omega,u^\omega) \in \mathbb{B}$ and

$$\begin{cases} \mathcal{M}_1(\omega,u^\omega) = Q_\alpha(u^\omega_\cdot)\,,\\ \mathcal{M}_2(\omega,u^\omega) = u^{Q_\alpha(u^\omega_\cdot)}\,. \end{cases}$$

Let $\omega^{(1)}$ and $\omega^{(2)}$ be continuous functions on $[0,t_0]$ with values in $\mathbb{R}^n$, and let $u^{(1)}$ and $u^{(2)}$ be the solutions to equation (1.5) associated with $\omega = \omega^{(1)}$ and $\omega = \omega^{(2)}$ respectively, and with the same initial condition f. Lemma 4.1 and Proposition 4.2 imply that

$$\sup_{0\le t\le t_0} |Q_\alpha(u^{\omega^{(1)}}_t) - Q_\alpha(u^{\omega^{(2)}}_t)| \le C_0\sqrt{n}(2K)^{1-n}\delta^{-1}\left(t_0+\sqrt{t_0}\right) \sup_{t\in[0,t_0]} |\omega^{(1)}_t - \omega^{(2)}_t| \tag{4.26}$$

and

$$\begin{aligned} &\sup_{0\le t\le t_0} |u_t^{Q_\alpha(u_\cdot^{\omega^{(1)}})} - u_t^{Q_\alpha(u_\cdot^{\omega^{(2)}})}|_{L_1}\\ \le & C_0\left(t_0+\sqrt{t_0}\right)\sup_{s\in[0,t_0]} |Q_\alpha(u^{\omega^{(1)}}_s) - Q_\alpha(u^{\omega^{(2)}}_s)|\\ \le & C_0\sqrt{n}(2K)^{1-n}\delta^{-1}\left(t_0+\sqrt{t_0}\right)\sup_{t\in[0,t_0]} |u^{\omega^{(1)}}_t - u^{\omega^{(2)}}_t|_{L_1}\,. \end{aligned} \tag{4.27}$$

Choose $t_0 > 0$ such that

$$C_0\sqrt{n}(2K)^{1-n}\delta^{-1}\left(t_0+\sqrt{t_0}\right) = L < 1\,. \tag{4.28}$$

Then, from (4.26)-(4.27) it follows that the mapping $\mathcal{M}$ defined by (4.25) is a contraction map on $\mathbb{B}$. It has then a fixed point $(\omega,u^\omega) \in \mathbb{B}$. By our construction, we see that u^ω satisfies (2.3) with $\omega = Q_\alpha(u^\omega_t)$. This means that $u = u^\omega$ satisfies (2.7).

To show the uniqueness, we assume v is another solution to (2.7). Letting $\omega' = Q_\alpha(v) = \{Q_\alpha(v_s)|s \in [0, t_0]\}$, replacing $Q_\alpha(v)$ by ω' in (2.7), we see that v is also a solution of (2.3) with ω'. Thus, (ω', v) is a fixed point of $\mathcal{M}$. By the uniqueness of the fixed point of map $\mathcal{M}$, we complete the proof of the proposition. □

4.2. **Global solution and proof of main result.** In the previous subsection, we proved that (2.7) has a unique solution u on $[0, t_0]$ when t_0 is small enough. A natural question is whether this solution can be uniquely extended to any time interval. A positive answer is given in this subsection by using Proposition 3.12.

Proof of Theorem 2.5. By Proposition 4.4, there exists t_0, such that (2.7) has a unique solution on $[0, t_0]$. Consider (2.7) with $t \geq t_0$ and with initial condition $f = u_{t_0}$. Proposition 3.12 can be applied to find that there exists $t_1 > 0$ depending on the initial condition $f = u_{t_0}$ only through U' given by (3.22) such that equation (2.7) has a unique solution on $[t_0, t_0 + t_1]$. Notice that U' is independent of $t \in [t_0, T]$. This allows us to extend the solution of (2.7) repeatedly to the interval $[0, t_0 + nt_1]$ until time $t_0 + nt_1 \geq T$. In other words, (2.7) has a unique solution on the whole time interval $[0, T]$. □

Proof of Theorem 2.4. Under the hypotheses (H1)-(H5) and (I), the Theorem 2.5 implies that the α-quantile of any weak solution to SDE (1.5) is the same function on $[0, T]$. Therefore, the existence of a unique strong solution of SDE (1.5) is a straightforward result of Theorem 3.2. □

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