Electron. Commun. Probab. **0** (2023), no. 0, 1–13.
DOI: 10.1214/YY-TN
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Stochastic Differential Equations with Local Interactions

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Abstract

An infinite system of stochastic differential equations for particle locations is considered. The particles exhibit local interactions through drift coefficients that depend upon other particles within a fixed distance. Strong existence and uniqueness is proved for this particle system with potentially discontinuous, local interactions.

Keywords: Stochastic Differential Equation; Particle System; Local Interactions;Uniqueness.
AMS MSC 2010: 60H10; 37L55.
Submitted to ECP on February 10, 2023, final version accepted on FIXME!.

1 Introduction

Many real particle systems exhibit local interactions. For example, celestial bodies experience gravitational pull and electrons face magnetic repulsion with forces decaying at rates $1/r^2$, where r is the distance between the particles. Further, individual motions in a species are generally affected by close neighbours and not by those at a distance. We consider extreme local interactions where particles only interact when within a distance r and establish a sensible model by proving strong existence and uniqueness.

Interacting particle systems have been considered in classical mechanics in the ordinary differential equation setting for over fifty years (see Lanford (1968)). Further, much work has already been done on systems of interacting stochastic differential equations (SDEs). We mention just a few examples here and refer the reader to the citations of these articles for more works on interacting SDEs. Lang (1977) initiated the study of a countable system of stochastic gradient differential equations of the form

$$dX_t^i = -\sum_{j\neq i} \nabla U(X_t^i - X_t^j)dt + \sigma dB_t^i, \tag{1.1}$$

where $\{B^i\}$ are independent standard Brownian motions and U is a superstable potential. Fritz (1987) furthered this work, proving existence, strong uniqueness and regularity for solutions of (1.1) with U chosen so the system had a finite radius interactions.

Kondraitiev et. al. (2006) consider well-posedness and scaling limits of the interacting diffusion dynamics $X(t) = \{x_t\}$

$$dx_t = \sqrt{2}\exp\left(\frac{1}{2}\sum_{y_t\in X(t), y_t\neq x_t} \phi(x_t - y_t)\right) dB_t^x \tag{1.2}$$

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on the space Γ of locally finite configurations of $\mathbb{R}^d$ using Dirichlet forms, where $\{B^x\}$ are independent standard Brownian motions. This equation is decidedly different from the one discussed above as there is no drift coefficient and the interaction is through the diffusion coefficient. Ma and Röckner (2000) and Röckner and Zhang (2004) also consider infinite systems of interacting diffusion processes using Dirichlet forms. The area remains very active (see e.g. Conrad et. al. (2013), Fradon and Roelly (1987), Tsai (2016) and Osada and Tanemura (2020) and their references).

Athreya et. al. (2006) show weak existence and uniqueness of the interacting SDEs

$$dX_t^i = -\lambda_i b_i(X_t) X_t^i dt + \sum_{j=1}^{\infty} \sigma_{ij}(X) dB_t^j \tag{1.3}$$

in any separable Hilbert space $(H, \langle \cdot, \cdot \rangle)$, where $\{B^j\}$ are independent standard Brownian motions and $X_t^i = \langle X_t, e_i \rangle$ with $\{e_i\}$ being a basis for H. Ellipticity, Hölder continuity and boundedness conditions are assumed and Strook-Varadhan techniques are used by treating the system as a perturbation of an H-valued Ornstein-Uhlenbeck process. To connect this system to the H-valued stochastic differential equation, they take $(\lambda_i b_i(x), e_i)$ to be an eigenvalue/vector pair of an operator $b(x)$ and $a_{ij}(x) = \langle e_i, a(x) e_j \rangle$ for another operator $a(x)$ for each i, j and $x \in H$ and then assume $\lambda_i \nearrow \infty$ at a certain rate. The well-posedness of the H-space equation is related to earlier work by Zambotti (2000).

Kurtz and Xiong (1999) showed existence and uniqueness for systems of weakly-interacting SDE particle and then ultimately for a class of stochastic partial differential equations (SPDE). The conditions they used to facilitate their SPDE goals preclude the type of strong local interactions we are interested in.

We consider the following system: The particle positions follow the $\mathbb{R}^d$-valued SDEs

$$X_t^i = x^i + \int_0^t \sigma(X_s^i) dB_s^i + \int_0^t b(X_s^i, N_s) ds + \int_0^t \alpha(X_s^i) dW_s, \ \forall i \in \mathbb{Z}, \tag{1.4}$$

where $\{B^i\}_{i \in \mathbb{Z}}$, W are all independent standard $\mathbb{R}^d$-valued Brownian motions and $N_s = \sum_i \delta_{X_s^i}$ is the configuration of the system of particles of mass 1 at time s. Our interests are in the situation where the particles only interact with each other when they are within a fixed radius r of each other. They could interact strongly within r and not at all outside of r creating discontinuous interactions through our drift coefficient b.

2 Motivation and Results

$|\cdot|$ will denote Euclidean distance. To motivate our conditions, re-consider (1.1) with the mindset of replacing $\nabla U(x-y)$ with some general drift coefficient b.

Example 2.1. Consider the case $r = 1$ for notational simplicity. Suppose U is a smooth function with derivatives bounded by K say, $U(x) = 0$ if $|x| > 1$ (as in Fritz (1987)) and

$$b(x, \nu) = -\int_{\mathbb{R}^d} \nabla U(x-y) \nu(dy)$$

so $b(X_s^i, N_s) = -\sum_{j \neq i} \nabla U(X_t^i - X_t^j)$. Then, it follows by the support property of U that

$$\left| b\left(x, \sum_j \delta_{z^j}\right) \right| = \left| \sum_{j : |x - z^j| \leq 1} \nabla U(x - z^j) \right| \leq K \sum_{j : z^j \neq x} 1_{[x-1, x+1]}(z^j),$$

which is a growth condition to replace the usual boundedness condition, and

$$\begin{aligned} &\left|b\left(x,\sum\delta_{z^i}\right)-b\left(\tilde{x},\sum\delta_{\tilde{z}^i}\right)\right| \qquad (2.1)\\ \le\ &\left|\sum_{j:|x-z^j|\le 1}\nabla U(x-z^j)-\sum_{j:|\tilde{x}-\tilde{z}^j|\le 1}\nabla U(\tilde{x}-\tilde{z}^j)\right|\\ \le\ &\sum_{j:|x-z^j|\le 1}\left|\nabla U(x-z^j)-\nabla U(\tilde{x}-\tilde{z}^j)\right|+\sum_{j:|\tilde{x}-\tilde{z}^j|\le 1}\left|\nabla U(x-z^j)-\nabla U(\tilde{x}-\tilde{z}^j)\right|\\ \le\ &K|x-\tilde{x}|\#\{j:|x-z^j|\le 1\}+K\sum_{|x-z^j|\le 1}|\tilde{z}^j-z^j|\\ +\ &K|x-\tilde{x}|\#\{j:|\tilde{x}-\tilde{z}^j|\le 1\}+K\sum_{|\tilde{x}-\tilde{z}^j|\le 1}|\tilde{z}^j-z^j|. \end{aligned}$$

Conditions (Lip, B) to follow hold with $r=1$ if α,σ are bounded and Lipschitz continuous.

We adapt our existence-uniqueness definitions and Yamada-Watanabe implication from (Karatzas and Shreve, 1988, Chapter 5). They use regular conditional probability, known for complete separable metric spaces, but the metric $q(z,y)=\sqrt{\sum_{i\in\mathbb{Z}}e^{-|x^i|}|z^i-y^i|^2}$ turns our particle state space $(\mathbb{R}^d)^{\mathbb{Z}}$ into such a space. Also, the notion of solution requires coefficient measurability so we consider N as a counting measure and the counting measures, $\mathcal{M}_c(\mathbb{R}^d)$, as Radon measures on $\mathbb{R}^d$ with the vague topology. Let $\mathcal{B}(\mathcal{X})$ denote the Borel σ-algebra on a topological space $\mathcal{X}$.

Definition 2.2. *Let $(\Omega,\mathcal{F},P)$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions and $W,\ \{B^i\}$ be independent and adapted standard Brownian motions. Suppose σ,α are Borel measurable and b is also $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathcal{M}_c(\mathbb{R}^d))/\mathcal{B}(\mathbb{R}^d)$ measurable. Then, an $(\mathbb{R}^d)^{\mathbb{Z}}$-valued process $\{X^i\}$ is a solution to (1.4) if:*

a) $\{X^i\}$ is $\{\mathcal{F}_t\}_{t\ge 0}$-adapted and has continuous paths,

b) $N=\sum_i \delta_{X^i}$,

c) $P\left(\int_0^t |b_j(X_s^i,N_s)|+\sigma^2_{j,k}(X_s^i)+\alpha^2_{j,k}(X_s^i)\,ds<\infty\right)=1$, $\forall i\in\mathbb{Z}$, $1\le j,k\le d$, $t>0$, and

d) the equation (1.4) holds a.s.

Definition 2.3. *$\{X^i\}$ is a weak solution to (1.4) if it is a solution and $(\Omega,\mathcal{F},P)$, $\{\mathcal{F}_t\}_{t\ge 0}$, $W,\ \{B^i\}$ are found as part of the solution.*

Definition 2.4. *$\{X^i\}$ is a strong solution to (1.4) if it is a solution with respect to given $W,\ \{B^i\}$ on a given $(\Omega,\mathcal{F},P)$ and $\mathcal{F}_t=\sigma\left(\mathcal{G}_t\cup\mathcal{N}\right)$, where $\mathcal{G}_t=\sigma\left(W_s,B_s^i;0\le s\le t,i\in\mathbb{Z}\right)$, $\mathcal{G}_\infty=\sigma\left(\bigcup_{t>0}\mathcal{G}_t\right)$ and $\mathcal{N}=\{N\subset\Omega:\exists\, G\in\mathcal{G}_\infty$ with $N\subset G$ and $P(G)=0\}$.*

Definition 2.5. *The pathwise uniqueness holds for (1.4) if for any two solutions $\{X^i\}$, $\{Y^i\}$ on the same probability space $(\Omega,\mathcal{F},P)$ with respect to the same initials $\{x^i\}$ and Brownian motions $W,\ \{B^i\}$, we have almost surely that $X_t^i=Y_t^i$ for all $t\ge 0$ and $i\in\mathbb{Z}$.*

It follows by the proof of the Yamada-Watanabe theorem in e.g. (Karatzas and Shreve, 1988, Proposition 5.3.20 and Corollary 5.3.23) that the weak existence and the pathwise uniqueness implies the existence of a unique strong solution that will have the form $X=h(x,W,B)$ for some measurable function h.

To consider the existence and the uniqueness for the solution to (1.4), we impose the following regularity conditions:

(**I**): $\sum_{i\in\mathbb{Z}} e^{-p\,|x^i|}<\infty$ for all $p>0$. Non-clumped initial configuration.

(**B**): There are constants $K,\ r>0$ such that, for any $x\in\mathbb{R}^d$, $\{z^i\}_{i\in\mathbb{Z}}\subset\mathbb{R}^d$

(Bσ) $\max_j|\sigma^j(x)|\le K$, where σ^j is the j^{th} row of σ

(Bα) $\max_j |\alpha^j(x)| \le K$, where α^j is the j^{th} row of α

(Bb) $\left| b\left(x, \sum_i \delta_{z^i}\right)\right| \le K \sum_j 1_{D(x,r)}(z^j)$.

Here and below, $D(x,r) = \{y \in \mathbb{R}^d : |y - x| \le r\}$ is the closed ball of radius $r > 0$.
(**Lip**): There are $K,\ r > 0$ such that, for any $x, \tilde{x} \in \mathbb{R}^d$ and $\{z^i\}_{i\in\mathbb{Z}}, \{\tilde{z}^i\}_{i\in\mathbb{Z}} \subset \mathbb{R}^d$

$$|\sigma(x) - \sigma(\tilde{x})| + |\alpha(x) - \alpha(\tilde{x})| \le K|x - \tilde{x}| \tag{2.2}$$

$$\begin{aligned} & \left| b\left(x, \sum \delta_{z^i}\right) - b\left(\tilde{x}, \sum \delta_{\tilde{z}^i}\right)\right| \\ \le\ & K|x - \tilde{x}| \left(1 + \#\left\{i:\ |z^i - x| \le r\right\} + \#\left\{i:\ |\tilde{z}^i - \tilde{x}| \le r\right\}\right) \\ & +K\left(\sum_{|z^i - x|\le r} |z^i - \tilde{z}^i| + \sum_{|\tilde{z}^i - \tilde{x}|\le r} |z^i - \tilde{z}^i|\right). \end{aligned} \tag{2.3}$$

While (B) stands for boundedness and (Lip) for Lipschitz, b satisfies a growth condition not a bounded one. Its Lipschitz condition has growth and is over the configuration space as well as the point x. Condition (Lip) restricts the interaction of the particle system through b to neighbors of distance no larger than r. b is said to be bounded if

$$\left| b\left(x, \sum_i \delta_{z^i}\right)\right| \le K,\ \forall x \in \mathbb{R}^d, \{z^i\}_{i\in\mathbb{Z}} \subset \mathbb{R}^d.$$

The maximum number of particles $\mathbb{M}_T^i$ in the ball of radius r around particle i up until time T will be important in stating and proving our existence and uniqueness result.

Definition 2.6. *Mass functions $\{p_m^i\}$ are uniformly non-heavy tailed if there are $C > 0, \alpha \in (0,1)$ so that $p_m^i \le C\alpha^m$ for $m \in \mathbb{N}$, $i \in \mathbb{Z}$.*

We will refer to α in the above definition as the tail-decay *base*. Our interest in non-heavy tailed mass functions is in characterizing solutions to (1.4).

Definition 2.7. *Solution $\{X^i\}$ is sub-explosive if for any $T > 0$, $\{\mathbb{M}^i\}$ has uniformly non-heavy tailed probability mass functions, where*

$$\mathbb{M}^i = \mathbb{M}_T^i = \sup_{t\le T} \mathbb{B}_t^i, \quad \mathbb{B}_t^i = \#\{j : |X_t^j - X_t^i| \le r\}.$$

Note: We will use this notation $\mathbb{B}_t^i$ and $\mathbb{M}^i$ throughout this note.

Remark 1. The particles of sub-explosive solutions to (1.4) have finite moments. Indeed, let $T > 0$ and m be even. Then, Condition (B), Ito's formula, Young's inequality and Gronwall's inequality imply a $C_m = C_m(T) > 0$ such that

$$\begin{aligned} \mathbb{E}|X_t^i - x^i|^m \le\ & m\int_0^t \mathbb{E}\Big[|X_s^i - x^i|^{m-1}|b(X_s^i, N_s)|\Big]ds + 2K^2\binom{m}{2}\int_0^t \mathbb{E}|X_s^i - x^i|^{m-2}ds \\ \le\ & \int_0^t (m-1)\mathbb{E}|X_s^i - x^i|^m + \frac{\mathbb{E}|\mathbb{B}_s^i|^m}{m}ds + 2K^2\int_0^t \binom{m-1}{2}\mathbb{E}|X_s^i - x^i|^m + \frac{2}{m}ds \\ \le\ & C_m,\ \forall t \in [0,T]. \end{aligned}$$

We also consider the Euler approximations to (1.4): If $D_n(s) = \frac{\lfloor sn\rfloor}{n}$, $W_n(s) = W(\frac{\lfloor sn\rfloor}{n})$ and $B_n^i(s) = B^i(\frac{\lfloor sn\rfloor}{n})$, then X_n, defined by the Euler equations

$$\begin{aligned} X_n^i(t) &= x^i + \int_0^t \sigma(X_n^i(s-))dB_n^i(s) \\ &+ \int_0^t b(X_n^i(s-), N_n(s-))dD_n(s) + \int_0^t \alpha(X_k^i(s-))dW_n(s) \end{aligned} \tag{2.4}$$

for $i \in \mathbb{Z}$, where $N_n(s) = \sum_j \delta_{X_n^j(s)}$ is the discretized particle configuration, exist in $D(\mathbb{R}_+, (\mathbb{R}^d)^{\mathbb{Z}})$ for every $n \in \mathbb{N}$. When we say that the Euler approximations are uniformly sub-explosive we mean that the C, α implied from Definitions 2.6 and 2.7 can be chosen independent of n. This implies the C_m in Remark 1 is independent of n.

The principle result of this note is the following, which will be proved in later sections:

Theorem 1. Suppose Conditions (I, B, Lip) hold. If any solution is sub-explosive, then the system of stochastic differential equations (1.4) is pathwise unique. If, in addition, the Euler approximations are uniformly sub-explosive, then there exists a sub-explosive solution.

We give a second example of coefficient b.

Example 2.8. Suppose ζ is a bounded Lipschitz function in both variables, h is a Lipschitz function with compact support $B(0,r)$ and

$$b(x,\nu) = \zeta\left(x, \int h(x-y)\nu(dy)\right).$$

Then, it follows by the support property of h that

$$\begin{aligned}
& \left|b\left(x, \sum \delta_{z^i}\right) - b\left(\tilde{x}, \sum \delta_{\tilde{z}^i}\right)\right| \qquad (2.5)\\
\leq\ & K|x-\tilde{x}| + K\left|\sum_{|\tilde{x}-\tilde{z}^i|\leq r} h(\tilde{x}-\tilde{z}^i) - \sum_{|x-z^i|\leq r} h(x-z^i)\right| \\
\leq\ & K|x-\tilde{x}| + K\sum_{|\tilde{x}-\tilde{z}^i|\leq r} |h(\tilde{x}-\tilde{z}^i) - h(x-z^i)| + K\sum_{|x-z^i|\leq r} |h(x-z^i) - h(\tilde{x}-\tilde{z}^i)| \\
\leq\ & K|x-\tilde{x}| + K^2|x-\tilde{x}|\#\{i : |\tilde{x}-\tilde{z}^i| \leq r\} + K^2\sum_{|\tilde{x}-\tilde{z}^i|\leq r} |\tilde{z}^i - z^i| \\
+\ & K^2|x-\tilde{x}|\#\{i : |x-z^i| \leq r\} + K^2\sum_{|x-z^i|\leq r} |\tilde{z}^i - z^i|.
\end{aligned}$$

(Lip, B) follow if α, σ are bounded and Lipschitz continuous. b is in fact bounded.

The following result, proved in Section 6, shows that our main result is not hollow.

Lemma 2.9. *Suppose Conditions (I,B,Lip) hold, b is bounded, there is a $\kappa \in (0,2)$ such that $\sum_{j\neq i} |x^j - x^i|^{-\kappa} < \infty$ for all i, and $\{X^i\}$ solves (1.4). Then, $\{X^i\}$ is sub-explosive.*

The following directions for future work would like require new or enhanced methods. Interactions that die out quickly rather than cut off immediately would be desirable for some applications. There is hope for building upon our results in this direction by considering multiple radii with different levels of interaction in each. Next, our results are independent of the possible attractive or repulsive nature of b. It would be interesting to understand how the possible initial conditions depend upon choices of b. Finally, it would be interesting to allow the diffusion coefficients to depend upon the configuration.

The model in Fritz (1987) seems more general than (1.4) in the sense that the diffusion term can be interactive. Still, it has a structure mimicking that of (1.1). The state space is $\Omega = \{(\mathbb{R}^d)^{\mathbb{Z}} :\ \bar{H}(\omega) < \infty\}$, where $\bar{H}$ is a functional defined through the potential U. The boundedness and Lipschitz conditions (through assumptions on U) are not comparable (B, Lip), which we think are natural. Finally, instead of the sub-explosive solutions, the concept of tempered solutions,defined through the potential U again, is employed. The results in these two papers appear to supplement each other.

3 Proof of Uniqueness

We use a function from in Mitoma (1985). Let c_ρ be such that ρ is a density, where

$$\rho(x) = c_\rho 1_{|x|<1} \exp\left(-1/(1-x^2)\right), \qquad \phi(x) = \int_{\mathbb{R}} e^{-|a|}\rho(x-a)da, \qquad x \in \mathbb{R}.$$

Then, ϕ is a smooth function, and for each $n \in \mathbb{N}$, there exists constants $C_n > 1$ and $K_n > 1$ such that the n^{th} derivative, $\phi^{(n)}$, of ϕ satisfies

$$\phi^{(n)}(x) \le C_n e^{-|x|} \le K_n \phi(x), \qquad \forall\, x \in \mathbb{R}. \tag{3.1}$$

It is also easy to verify that

$$\phi(x^i) \le e^2 \phi(\tilde{x}^j) \exp\left(|x^i - \tilde{x}^i| + |\tilde{x}^j - \tilde{x}^i|\right). \tag{3.2}$$

For simplicity of notation, we take $r = 1$ in our proofs. Also, fix $T > 0$.

Proposition 1. Suppose Conditions (I, B, Lip) hold. If any solution is sub-explosive, then the system (1.4) satisfies pathwise uniqueness.

Proof: For simplicity of notation, we assume $d = 1$ here. Let $\{\tilde{X}^i_t,\ i \in \mathbb{Z}\}$ be another solution to (1.4) satisfying Condition (I) with $\tilde{N}_t$ being defined as $\tilde{N}_t = \sum_i \delta_{\tilde{X}^i_t}$.

Suppose $t_k = k\delta$ for all $k \in \mathbb{N}_0$ and some fixed $\delta = \frac{T}{k_0}$ with $k_0 \in \mathbb{N}$ to be determined later. Moreover, assume our two systems are indistinguishable on $[0, t_k]$ for some $k < k_0$, which is known to be true for $k = 0$. We will show $\{X_t,\ t \in [0, t_{k+1}]\} = \{\tilde{X}_t,\ t \in [0, t_{k+1}]\}$ a.s. Applying Itô's formula for $s \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} d|X^i_s - \tilde{X}^i_s|^2 \;=\;& 2(X^i_s - \tilde{X}^i_s)(\sigma(X^i_s) - \sigma(\tilde{X}^i_s))dB^i_s \\ &+2(X^i_s - \tilde{X}^i_s)(b(X^i_s, N_s) - b(\tilde{X}^i_s, \tilde{N}_s))ds \\ &+2(X^i_s - \tilde{X}^i_s)(\alpha(X^i_s) - \alpha(\tilde{X}^i_s))dW_s \\ &+\left\{|\sigma(X^i_s) - \sigma(\tilde{X}^i_s)|^2 + |\alpha(X^i_s) - \alpha(\tilde{X}^i_s)|^2\right\} ds. \end{aligned}$$

As $\phi \in C^2(\mathbb{R})$, we have by Itô's formula again that

$$d\phi(X^i_s) = \phi'(X^i_s)\{\sigma(X^i_s)dB^i_s + \alpha(X^i_s)dW_s + b(X^i_s, N_s)ds\} + \frac{1}{2}\phi''(X^i_s)(\sigma^2(X^i_s) + \alpha^2(X^i_s))ds.$$

Next, by integration by parts and the previous two equations, we have

$$\begin{aligned} d|X^i_s - \tilde{X}^i_s|^2\phi(X^i_s) \;=\;& \phi(X^i_s)2(X^i_s - \tilde{X}^i_s)(b(X^i_s, N_s) - b(\tilde{X}^i_s, \tilde{N}_s))ds \\ &+\; \phi(X^i_s)2(X^i_s - \tilde{X}^i_s)[(\sigma(X^i_s) - \sigma(\tilde{X}^i_s))dB^i_s + (\alpha(X^i_s) - \alpha(\tilde{X}^i_s))dW_s] \\ &+\; \phi(X^i_s)\left\{(\sigma(X^i_s) - \sigma(\tilde{X}^i_s))^2 + (\alpha(X^i_s) - \alpha(\tilde{X}^i_s))^2\right\} ds \\ &+\; |X^i_s - \tilde{X}^i_s|^2\phi'(X^i_s)[\sigma(X^i_s)dB^i_s + \alpha(X^i_s)dW_s + b(X^i_s, N_s)ds] \\ &+\; \frac{1}{2}|X^i_s - \tilde{X}^i_s|^2\phi''(X^i_s)(\sigma^2(X^i_s) + \alpha^2(X^i_s))ds \\ &+\; 2(X^i_s - \tilde{X}^i_s)(\sigma(X^i_s) - \sigma(\tilde{X}^i_s))[\phi'(X^i_s)\sigma(X^i_s)]ds \\ &+\; 2(X^i_s - \tilde{X}^i_s)(\alpha(X^i_s) - \alpha(\tilde{X}^i_s))[\phi'(X^i_s)\alpha(X^i_s)]ds. \end{aligned} \tag{3.3}$$

Now, it follows by Conditions (B,Lip) as well as the properties of ϕ that there is a constant $\overline{K} > 0$ (depending only on K, K_1 and K_2) such that

$$\begin{aligned} &\left|\phi(X^i_s)\left\{(\sigma(X^i_s) - \sigma(\tilde{X}^i_s))^2 + (\alpha(X^i_s) - \alpha(\tilde{X}^i_s))^2\right\}\right| \\ &+\frac{1}{2}|X^i_s - \tilde{X}^i_s|^2\phi''(X^i_s)(\sigma^2(X^i_s) + \alpha^2(X^i_s)) \\ &+2(X^i_s - \tilde{X}^i_s)\{(\sigma(X^i_s) - \sigma(\tilde{X}^i_s))[\phi'(X^i_s)\sigma(X^i_s)] + (\alpha(X^i_s) - \alpha(\tilde{X}^i_s))[\phi'(X^i_s)\alpha(X^i_s)]\} \\ &\le \overline{K}|X^i_s - \tilde{X}^i_s|^2\phi(X^i_s). \end{aligned} \tag{3.4}$$

Next, it follows by Condition (Bb) and the properties of ϕ that

$$\phi'(X_s^i)|b(X_s^i, N_s)| \leq K(1 + \mathbb{B}_s^i)\phi(X_s^i) \tag{3.5}$$

and, if $\tilde{\mathbb{B}}_s^i = \#\{j : |\tilde{X}_s^j - \tilde{X}_s^i| \leq 1\}$, that

$$|b(X_s^i, N) - b(\tilde{X}_s^i, \tilde{N})|\phi(X_s^i) \leq K(2 + \mathbb{B}_s^i + \tilde{\mathbb{B}}_s^i)\phi(X_s^i), \tag{3.6}$$

which is used when $|X_s^i - \tilde{X}_s^i| > 1$. But, if $|X_s^i - \tilde{X}_s^i| \leq 1$, then by (2.3) with $N_s = \sum_i \delta_{X_s^i}$,

$$\begin{aligned} & |X_s^i - \tilde{X}_s^i||b(X_s^i, N_s) - b(\tilde{X}_s^i, \tilde{N}_s)|\phi(X_s^i) \\ \leq\ & K|X_s^i - \tilde{X}_s^i|^2 \left(1 + \mathbb{B}_s^i + \tilde{\mathbb{B}}_s^i\right)\phi(X_s^i) \\ +\ & K\left(\sum_{j \in \mathbb{B}_s^i} |X_s^i - \tilde{X}_s^i||X_s^j - \tilde{X}_s^j| + \sum_{j \in \tilde{\mathbb{B}}_s^i} |X_s^i - \tilde{X}_s^i||X_s^j - \tilde{X}_s^j|\right)\phi(X_s^i). \end{aligned} \tag{3.7}$$

However, in this $|X_s^i - \tilde{X}_s^i| \leq 1$ case

$$\begin{aligned} & \sum_i \sum_{j \in \mathbb{B}_s^i} |X_s^i - \tilde{X}_s^i||X_s^j - \tilde{X}_s^j|\phi(X_s^i) \\ \leq\ & \frac{1}{2}\sum_i \sum_{j \in \mathbb{B}_s^i} |X_s^i - \tilde{X}_s^i|^2\phi(X_s^i) + \frac{1}{2}\sum_i \sum_j |X_s^j - \tilde{X}_s^j|^2\phi(X_s^i)1_{|X_s^j - X_s^i| \leq 1} \\ \leq\ & \frac{1}{2}\sum_i \mathbb{B}_s^i|X_s^i - \tilde{X}_s^i|^2\phi(X_s^i) + \frac{e^2}{2}\sum_j \mathbb{B}_s^j|X_s^j - \tilde{X}_s^j|^2\phi(X_s^j), \end{aligned} \tag{3.8}$$

where we used (3.2). Similarly, when $|X_s^i - \tilde{X}_s^i| \leq 1$

$$\begin{aligned} & \sum_i \sum_{j \in \tilde{\mathbb{B}}_s^i} |X_s^i - \tilde{X}_s^i||X_s^j - \tilde{X}_s^j|\phi(X_s^i) \\ \leq\ & e^2 \sum_i \sum_{j \in \tilde{\mathbb{B}}_s^i} |X_s^i - \tilde{X}_s^i||X_s^j - \tilde{X}_s^j|\phi(\tilde{X}_s^i) \\ \leq\ & \frac{e^2}{2}\sum_i \tilde{\mathbb{B}}_s^i|X_s^i - \tilde{X}_s^i|^2\phi(\tilde{X}_s^i) + \frac{e^4}{2}\sum_i \tilde{\mathbb{B}}_s^i|X_s^i - \tilde{X}_s^i|^2\phi(\tilde{X}_s^i) \end{aligned} \tag{3.9}$$

so, combining the cases, we have by (3.6-3.9) a constant $\tilde{K} > 0$ (depending only upon K, K_1, K_2 and e) such that

$$\begin{aligned} & \sum_i |X_s^i - \tilde{X}_s^i||b(X_s^i, N) - b(\tilde{X}_s^i, \tilde{N})|\phi(X_s^i) \\ \leq\ & \tilde{K}\sum_i |X_s^i - \tilde{X}_s^i|^2 \left(\phi(X_s^i) + \mathbb{B}_s^i\phi(X_s^i) + \tilde{\mathbb{B}}_s^i\phi(X_s^i) + \tilde{\mathbb{B}}_s^i\phi(\tilde{X}_s^i)\right). \end{aligned} \tag{3.10}$$

Summing up and taking expectation on both sides of (3.3), we have by (3.4), (3.5) and (3.10) a $\hat{K} > 0$ (depending only upon K, K_1, K_2) such that

$$f(t) \equiv \mathbb{E}\left[\sum_i |X_t^i - \tilde{X}_t^i|^2\phi(X_t^i)\right] \leq \hat{K}\left[\int_{t_k}^t f(s)ds + I_t + \tilde{I}_t + \tilde{J}_t\right], \tag{3.11}$$

where

$$\tilde{f}(t) = \mathbb{E}\left[\sum_i |X_t^i - \tilde{X}_t^i|^2\phi(\tilde{X}_t^i)\right] \leq \hat{K}\left[\int_{t_k}^t \tilde{f}(s)ds + J_t + \tilde{J}_t + I_t\right],$$

$$I_t=\int_{t_k}^t \mathbb{E}\sum_i \mathbb{B}_s^i|X_s^i-\tilde X_s^i|^2\phi(X_s^i)ds,\ \tilde I_t=\int_{t_k}^t \mathbb{E}\sum_i \tilde{\mathbb{B}}_s^i|X_s^i-\tilde X_s^i|^2\phi(X_s^i)ds,$$

and

$$J_t=\int_{t_k}^t \mathbb{E}\sum_i \mathbb{B}_s^i|X_s^i-\tilde X_s^i|^2\phi(\tilde X_s^i)ds,\ \tilde J_t=\int_{t_k}^t \mathbb{E}\sum_i \tilde{\mathbb{B}}_s^i|X_s^i-\tilde X_s^i|^2\phi(\tilde X_s^i)ds.$$

Now, by Holder's inequality and Remark 1 (for $\mathbb{E}|X_s^i-x^i|^6$, $\mathbb{E}|\tilde X_s^i-x^i|^6$) there is a K_+ so

$$\begin{aligned} I_t-n\int_{t_k}^t f(s)ds &\le \int_{t_k}^t \mathbb{E}\left\{\sum_i |X_s^i-\tilde X_s^i|^2\phi(X_s^i)(\mathbb{B}_s^i-n)1_{\mathbb{B}_s^i>n}\right\}ds \\ &\le K_+\int_{t_k}^t\sum_i\left\{\mathbb{E}^{\frac13}[\phi^3(X_s^i)]\mathbb{E}^{\frac13}\left[(\mathbb{B}_s^i-n)^3 1_{\mathbb{B}_s^i>n}\right]\right\}ds. \end{aligned} \tag{3.12}$$

Thus, if $\mathbb{M}^i=\sup_{s\le t}\mathbb{B}_s^i$, then by Lemma 5.1 and Lemma 5.2 (both to follow) as well as (I) there is a $K_\#>0$, a $p\in(0,1)$ and a $c=c(T)$ such that

$$\begin{aligned} I_t &\le n\int_{t_k}^t f(s)ds+K_\#\int_{t_k}^t\sum_i e^{-\frac{p}{3}|x^i|}\alpha^{\frac{n}{3}}ds \\ &\le n\int_{t_k}^t f(s)ds+c\delta\alpha^{\frac{n}{3}}\ . \end{aligned} \tag{3.13}$$

Hence, by symmetry

$$\tilde I_t\le n\int_{t_k}^t f(s)ds+c\delta\alpha^{\frac{n}{3}},\ J_t\vee\tilde J_t\le n\int_{t_k}^t\tilde f(s)ds+c\delta\alpha^{\frac{n}{3}}$$

so

$$f(t)+\tilde f(t)\ \le\ \hat K\left[\int_{t_k}^t(1+3n)(f(s)+\tilde f(s))ds+6c\delta\alpha^{\frac{n}{3}}\right]. \tag{3.14}$$

By Gronwall's inequality, for $t\le(t_k+\delta)$, we have

$$f(t)+\tilde f(t)\le 6c\delta\alpha^{\frac{n}{3}}e^{\hat K(1+3n)(t-t_k)}\to 0,$$

provided $0<\delta<-\frac{\ln\alpha}{9\hat K}$. Hence, $\{X_t,\,t\in[t_k,t_{k+1}]\}$ and $\{\tilde X_t,\,t\in[t_k,t_{k+1}]\}$ are indistinguishable by their continuous paths. Pathwise uniqueness on $[0,T]$ follows by induction. Pathwise uniqueness on $[0,\infty)$ follows since $T>0$ was arbitrary. □

4 Existence

Proposition 2. Suppose Conditions (I, B, Lip) hold. If, in addition, the Euler approximations are uniformly sub-explosive, then there exists a sub-explosive solution.

Proof: Let $\{\mathcal{F}_t^n\}$ be the filtration generated by X_n, defined in (2.4). Now,

$$|b(X_n^i(s-),N_n(s-))|\le K\mathbb{B}_{n,s-}^i$$

by Condition (B), where $\mathbb{B}_{n,s-}^i=\#\{j:|X_n^j(s-)-X_n^i(s-)|\le 1\}$. For $\eta>0$, there is an m_η^i such that $P(\sup_{s\le T}K\mathbb{B}_{n,s-}^i\ge m_\eta^i)\le C\alpha^{m_\eta^i}<\frac{\eta}{6}e^{-|i|}$ by the assumption that the Euler approximations were uniformly sub-explosive. Hence, by Markov's inequality

$$\mathbb{P}\left(X_n(s)\in\prod_{i\in\mathbb{Z}}\left[x^i-m_\eta^i s-\sqrt{\frac{12K^2s}{\eta e^{-|i|}}},x^i+m_\eta^i s+\sqrt{\frac{12K^2s}{\eta e^{-|i|}}}\right]\right)\ge 1-\eta$$

for all $n \in \mathbb{Z}$, so the pointwise compact containment condition is satisfied. Next, moving to the modulus of continuity condition and taking $\delta \in (0,1)$, we have

$$\mathbb{E}[q^2(X_n(t+h),X_n(t))|\mathcal{F}_t^n] = \sum_{i\in\mathbb{Z}} e^{-|x^i|}\mathbb{E}[|X_n^i(t+h)-X_n^i(t)|^2 \big|\mathcal{F}_t^n] \tag{4.1}$$

for all $t, t+h \in [0,T]$ with $h \in (0,\delta)$ but

$$|X_n^i(t+h)-X_n^i(t)|^2 = \Bigg|\sum_{j=\lfloor tn\rfloor+1}^{\lfloor (t+h)n\rfloor}\Big\{\sigma(X_n^i(\frac{j-1}{n}))[B_n^i(\frac{j}{n})-B_n^i(\frac{j-1}{n})] \tag{4.2}$$
$$+\ \alpha(X_n^i(\frac{j-1}{n}))[W_n(\frac{j}{n})-W_n(\frac{j-1}{n})]+b(X_n^i(\frac{j-1}{n}),N_n(\frac{j-1}{n}))\frac{1}{n}\Big\}\Bigg|^2.$$

Hence, it follows by independence and Condition (B) that

$$\mathbb{E}[|X_n^i(t+h)-X_n^i(t)|^2|\mathcal{F}_t^n] \le \frac{K^2}{n^2}\mathbb{E}\Big[\Bigg|\sum_{j=\lfloor tn\rfloor+1}^{\lfloor (t+h)n\rfloor}\mathbb{B}^i_{n,\frac{j-1}{n}}\Bigg|^2\Big|\mathcal{F}_t^n\Big] \tag{4.3}$$
$$+\sum_{j=\lfloor tn\rfloor+1}^{\lfloor (t+h)n\rfloor}\mathbb{E}\Big[\sigma^2(X_n^i(\frac{j-1}{n}))+\alpha^2(X_n^i(\frac{j-1}{n}))|\mathcal{F}_t^n\Big]\frac{1}{n}$$

and so by Condition (B) again

$$\begin{aligned}\mathbb{E}[q(X_n(t+h),X_n(t))^2|\mathcal{F}_t^n] &\le \sum_{i\in\mathbb{Z}} e^{-|x^i|}K^2\mathbb{E}\Big[\Big|\mathbb{M}^i_{n,T}\Big|^2\Big|\mathcal{F}_t^n\Big]\Bigg|\frac{\lfloor (t+h)n\rfloor-\lfloor tn\rfloor}{n}\Bigg|^2 \qquad (4.4)\\ &+\sum_{i\in\mathbb{Z}} e^{-|x^i|}2K^2\frac{\lfloor (t+h)n\rfloor-\lfloor tn\rfloor}{n}\\ &\le \sum_{i\in\mathbb{Z}} e^{-|x^i|}K^2\left(\delta+\frac{1}{n}\right)\Big\{2+\mathbb{E}\Big[\Big|\mathbb{M}^i_{n,T}\Big|^2\Big|\mathcal{F}_t^n\Big]\left(\delta+\frac{1}{n}\right)\Big\}\\ &\doteq \gamma_n(\delta).\end{aligned}$$

However, by Condition (I) and the uniform sub-explosive assumption of the Euler approximations there is $c_\gamma > 0$ (see (5.6) below with $n=0$ and $k=2$) such that

$$\limsup_{n\to\infty}\mathbb{E}[\gamma_n(\delta)] \le c_\gamma\delta. \tag{4.5}$$

Hence, $\{X_n\}$ is tight in $D(\mathbb{R}_+,\mathbb{R}^{\mathbb{Z}})$ and $\{X_n, W_n, B_n\}$ is distributionally relatively compact by (Ethier and Kurtz, 1986, Theorem 3.8.6 and Remark 3.8.7). Now, as in (Kurtz and Protter, 1996, Proposition 7.4) the limit (X,W,B) of any convergent subsequence (X_n,W_n,B_n) is a distribution solution to (1.4), which we know is pathwise hence distributionally unique. Hence, a solution to (1.4) exists weakly and hence strongly. □

5 Auxilliary Lemmas

Lemma 5.1. *Suppose Conditions (I,B,Lip) hold,* $m \ge 1$, $T > 0$ *and* $\{X^i\}$ *is a sub-explosive solution to (1.4). Then, there are constants* $C > 0$, $p \in (0,1)$ *(depending on* T*) such that*

$$\mathbb{E}\left[\phi^m(X_t^i)\right] \le C\exp(-p\,|x^i|),\quad i\in\mathbb{Z}, t\in[0,T]. \tag{5.1}$$

Proof: By ϕ properties (3.2) and (3.1)

$$\phi(X_t^i) \le e^2\phi(x^i)\exp(|X_t^i - x^i|) \tag{5.2}$$

so noting $\phi^m \leq c_{m,p}\phi^p$ for $p \in (0,1)$ and using (B), one finds $K_0, K_1 > 0$ such that

$$\begin{aligned}\phi^m(X_t^i) &\leq K_0 e^{-p\,|x^i|}\exp(p|\int_0^t \sigma(X_s^i)dB_s^i + \int_0^t \alpha(X_s^i)dW_s| + p\int_0^t |b(X_s^i, N_s)|ds) \\ &\leq K_1 e^{-p\,|x^i|}\exp(p \sup_{u_1,\dots,u_d \leq 2K^2 t} |(\beta_{u_1}^{i,1}, \dots, \beta_{u_d}^{i,d})| + ptK \sup_{s\leq t} \mathbb{B}_s^i). \qquad (5.3)\end{aligned}$$

Here, each β^i is a d-dimensional Brownian motion such that

$$\int_0^u \sigma^j(X_s^i)dB_s^i + \int_0^u \alpha^j(X_s^i)dW_s = \beta^{i,j}_{\tau_u^{i,j}}$$

and, by Condition (Bσ, Bα), $\tau_u^{i,j}$ is a stopping time such that

$$\tau_u^{i,j} = \int_0^u \sigma^j(X_s^{i,M})\sigma^j(X_s^{i,M})^T + \alpha^j(X_s^{i,M})\alpha^j(X_s^{i,M})^T ds \leq 2K^2 t$$

for $u \leq t$. But, $\mathbb{P}\left(\sup_{u\leq U}|\beta_u| \geq y\right) \leq 4e^{-\frac{y^2}{2U}}$ is a standard large-deviation-type bound for scalar standard Brownian motion β so

$$\mathbb{P}\left(2p \sup_{u\leq 2K^2 t}|\beta_u^{i,j}| \geq \frac{x-2pKmt}{\sqrt{d}}\right) \leq 4e^{-\frac{(x-2pKmt)^2}{(4pK)^2 dt}}$$

and consequently

$$\mathbb{P}\left(2p \sup_{u_1,\dots,u_d\leq 2K^2 t}|(\beta_{u_1}^{i,1}, \dots, \beta_{u_d}^{i,d})| \geq x-2pKmt\right) \leq 4de^{-\frac{(x-2pKmt)^2}{(4pK)^2 dt}}. \qquad (5.4)$$

If we let $\mathbb{M}^i = \sup_{s\leq T}\#\{j : |X_s^j - X_s^i| \leq 1\}$, then by (5.3), (5.4), Holder's inequality and the Gaussian moment generating function, there are $K_0,\ K_1 > 0$ such that

$$\begin{aligned}&\mathbb{E}\phi^m(X_t^i) \qquad (5.5)\\ \leq\ & K_0 e^{-p\,|x^i|}\sum_{m\in\mathbb{N}}\mathbb{E}\left[\exp(p \sup_{u_1,\dots,u_d\leq 2K^2 t}|(\beta_{u_1}^{i,1}, \dots, \beta_{u_d}^{i,d})| + tpKm)1_{\mathbb{M}^i=m}\right] \\ \leq\ & K_0 e^{-p\,|x^i|}\sum_{m\in\mathbb{N}}\mathbb{E}^{\frac{1}{2}}\left[\exp(2p \sup_{u_1,\dots,u_d\leq 2K^2 t}|(\beta_{u_1}^{i,1}, \dots, \beta_{u_d}^{i,d})| + 2tpKm)\right]\mathbb{P}^{\frac{1}{2}}(\mathbb{M}^i = m) \\ \leq\ & K_0 e^{-p\,|x^i|}\sum_{m\in\mathbb{N}}\sqrt{\int_{2tpm}^\infty \mathbb{P}\left(2p \sup_{u_1,\dots,u_d\leq 2K^2 t}|(\beta_{u_1}^{i,1}, \dots, \beta_{u_d}^{i,d})| \geq x-2pKmt\right)e^x dx}\,\mathbb{P}^{\frac{1}{2}}(\mathbb{M}^i = m) \\ \leq\ & K_0 e^{-p\,|x^i|}\sqrt{\int_0^\infty 4de^{-\frac{y^2}{(4pK)^2 dt}}e^y dy}\sum_{m\in\mathbb{N}} e^{pKmt}\,\mathbb{P}^{\frac{1}{2}}(\mathbb{M}^i = m) \\ \leq\ & K_1 e^{-p\,|x^i|}\sqrt{t}e^{2(pK)^2 dt}\sum_{m\in\mathbb{N}} e^{pKmt}\,\mathbb{P}^{\frac{1}{2}}(\mathbb{M}^i = m).\end{aligned}$$

Since our solutions are sub-explosive there are constants $\alpha \in (0,1)$ and $C > 0$ such that

$$\sum_{m\in\mathbb{N}} e^{pKmt}\,\mathbb{P}^{\frac{1}{2}}(\mathbb{M}^i = m) \leq C\sum_{m\in\mathbb{N}}(e^{pKT}\alpha^{\frac{1}{2}})^m < \infty$$

by choosing $p > 0$ small enough. The lemma follows easily by the last two equations. □

Lemma 5.2. *Suppose Conditions (I,B,Lip) hold and $\{X^i\}$ is a sub-explosive solution to (1.4) with base α. Then, there exists positive constant $C > 0$ (depending on T) such that*

$$\mathbb{E}\Big[(\mathbb{B}_s^i - n)^k 1_{\mathbb{B}_s^i > n}\Big] \leq C\alpha^n \quad \forall i \in \mathbb{Z}, s \in [0,T].$$

Proof: Recalling $\mathbb{M}^i = \sup_{s\leq T} \mathbb{B}^i_s$, one finds by the sub-explosive assumption that

$$\begin{aligned}
\mathbb{E}\Big[(\mathbb{B}^i_s - n)^k 1_{\mathbb{B}^i_s > n}\Big] &\leq \mathbb{E}\Big[(\mathbb{M}^i - n)^k 1_{\mathbb{M}^i > n}\Big] \\
&= \sum_{m=n+1}^{\infty} (m-n)^k P(\mathbb{M}^i = m) \\
&\leq C\sum_{m=1}^{\infty} m^k \alpha^{n+m}.
\end{aligned} \tag{5.6}$$

The lemma follows immediately. □

6 Prevalence of Sub-explosive Solutions

Proof of Lemma 2.9: To ease the notation, we take $i=0$ and abbreviate $\mathbb{M}^i_T$ to $\mathbb{M}$. Particle 0 is within a distance 1 of itself. Testing the others, we have by the boundedness of b (recall $|b|\leq K$ for this lemma) that

$$\begin{aligned}
\mathbb{P}\left(\mathbb{M} > x\right) &\leq \mathbb{P}\left(\#\Big\{|j|>0 : \inf_{u\leq T} |X^j_u - X^0_u| \leq 1\Big\} > x-1\right) \\
&\leq \mathbb{P}\left(\#\Big\{|j|>0 : \sup_{u_1,\dots,u_d\leq K_T} |(\beta^{j,1}_{u_1},\dots,\beta^{j,d}_{u_d})| + 2KT \geq |x^j - x^0| - 1\Big\} > x-1\right)
\end{aligned}$$

since, for $|j|>0$, there exists a d-dimensional Brownian motion β^j such that

$$\int_0^u [\sigma^l(X^j_s)dB^j_s - \sigma^l(X^0_s)dB^0_s + \alpha^l(X^j_s)dW_s - \alpha^l(X^0_s)dW_s] = \beta^{l,j}_{\tau^{l,j}_u},$$

where $\tau^{l,j}_u \leq K_T \doteq 6K^2T$ is a stopping time. Thus, for $|j|>0, l\in\{1,\dots,d\}$

$$\mathbb{P}\left(\sqrt{d}\sup_{u\leq K_T} |\beta^{l,j}_u| \geq |x^j - x^0| - 1 - 2KT\right) \leq c_1 e^{-c_2|x^j-x^0|^2}, \tag{6.1}$$

where $c_1 = 4e^{\frac{(1+2KT)^2}{2dK_T}}$ and $c_2 = \frac{1}{4dK_T}$, by a standard large-deviation-type bound for the Brownian motion and Young's inequality, so

$$\mathbb{P}\left(\sup_{u_1,\dots,u_d\leq K_T} |(\beta^{j,1}_{u_1},\dots,\beta^{j,d}_{u_d})| \geq |x^j - x^0| - 1 - 2KT\right) \leq c_1 d e^{-c_2|x^j-x^0|^2}. \tag{6.2}$$

Now, take $p_j = c_\kappa |x^j - x^0|^\kappa$ where $c_\kappa = \sum_{|j|>0} |x^j - x^0|^{-\kappa}$ so $\sum\limits_{|j|>0} \frac{1}{p_j} = 1$. Then, by Markov's and Holder's inequalities as well as the facts

$$\mathbb{E}\exp\left(p_j 1_A\right) = 1 + (e^{p_j}-1)\mathbb{P}\left(A\right), \quad \log(1+x)\leq x$$

one has from above that:

$$\begin{aligned}
e^x\mathbb{P}\left(\mathbb{M} > x\right) &\leq e\mathbb{E}\exp\left(\sum_{|j|>0} 1_{\sup_{u_1,\dots,u_d\leq K_T}|(\beta^{j,1}_{u_1},\dots,\beta^{j,d}_{u_d})|+2KT\geq |x^j-x^0|-1}\right) \\
&\leq e\Pi_{|j|>0}\left(\mathbb{E}\exp\Big(p_j 1_{\sup_{u_1,\dots,u_d\leq K_T}|(\beta^{j,1}_{u_1},\dots,\beta^{j,d}_{u_d})|+2KT\geq |x^j-x^0|-1}\Big)\right)^{1/p_j} \\
&\leq e\exp\left(\sum_{|j|>0} \frac{e^{p_j}-1}{p_j}\mathbb{P}\left(\sup_{u_1,\dots,u_d\leq K_T} |(\beta^{j,1}_{u_1},\dots,\beta^{j,d}_{u_d})| \geq |x^j - x^0| - 1 - 2KT\right)\right).
\end{aligned} \tag{6.3}$$

Thus, it follows from (6.2,6.3), Taylor's theorem as well as the fact that for any $p > 0$ there is a K_p^κ such that $y^\kappa \le K_p^\kappa + py^2,\ \forall y > 0$ that

$$\begin{aligned}\mathbb{P}\left(\mathbb{M}_t > x\right) &\ll e^{-x}\exp\left(c_1 d\sum_{|j|>0} e^{p_j - c_2|x^j - x^0|^2}\right) \\ &\ll e^{-x}\exp\left(\overline{K}\sum_{|j|>0} e^{-\frac{c_2}{2}|x^j - x^0|^2}\right)\end{aligned} \tag{6.4}$$

for some $\overline{K} > 0$. However, it follows from elementary inequalities that

$$|x^j - x^0|^{-\kappa} = \exp(-\kappa\ln|x^j - x^0|) \ge \exp(-K_{c_2}^2 - \frac{c_2}{2}|x^j - x^0|^2) \tag{6.5}$$

for some $K_{c_2}^2 > 0$. Hence, the sum within the exponential function in (6.4) is finite by the hypothesis $\sum_{|j|>0}|x^j - x^0|^{-\kappa} < \infty$. □

References

Athreya, S. R.; Bass, R.F.; Gordina, M.; Perkins, Edwin A. (2006) Infinite dimensional stochastic differential equations of Ornstein-Uhlenbeck type. *Stochastic Process. Appl.* **116 (3)**, 381-406.

Conrad, F.; Fattler, T.; Grothaus, M. (2013) An Invariance Principle for the Tagged Particle Process in Continuum with Singular Interaction Potential. *Infinite Dimensional Analysis Quantum Probability and Related Topics* **16 (4)**, 1350032, 37 pp.

Ethier, S. N. and Kurtz, T. G. (1986). *Markov processes.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York. Characterization and convergence.

Fradon, M. and Roelly, S. (2007). Infinite system of Brownian balls with interaction: the non-reversible case. *ESAIM Probab. Stat.* **11**, 55-79.

Fritz, J. (1987) Gradient dynamics of infinite point systems. *Ann. Probab.* **15(2)**, 478-514.

Karatzas, I. and Shreve, S. E. (1988). *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York.

Kondratiev, V.N.; Lytvynov, E.; Rockner, M. (2006). Infinite interacting diffusion particles. I. Equilibrium process and its scaling limit. *Forum Math.* **18(1)** 9-43.

Kurtz, T.G. and Protter, Ph. (1996). Weak convergence of stochastic integrals and differential equations. II. Infinite-dimensional case. *Probabilistic Models for Nonlinear Partial Differential Equations, Lecture Notes in Math.*, **1627**, 197-285.

Kurtz, T.G. and Xiong, J. (1999). Particle representations for a class of nonlinear SPDEs *Stochastic Processes and their Applications* **83(1)**, 103-126.

Lanford, O. E. (1968). The classical mechanics of one-dimensional systems of infinitely many particles I. An existence theorem. *Comm. Math. Phys.* **9** 176-191.

Lang, R. (1977). Unendlich-dimensionale Wienerprocesse mit Wechselwirkung I. *Z. Wahrsch. verw. Gebiete* **38** 55-72.

Ma, Z-M. and Röckner, M. (2000). Construction of diffusions on configuration spaces. *Osaka J. Math.* **37 (2)**, 273–314.

Mitoma, I. (1985). An ∞-dimensional inhomogeneous Langevin equation. *J. Func. Anal.* **61** 342-359.

Osada, H. and Tanemura, H. (2020). Infinite-dimensional stochastic differential equations and tail σ-fields. *Probab. Theory Related Fields* **177**, no. 3-4, 1137-1242.

Röckner, M. and Zhang, T. S. (2004). Sample path large deviations for diffusion processes on configuration spaces over a Riemannian manifold. (English summary) *Publ. Res. Inst. Math. Sci.* **40 (2)**, 385–427.

Tsai, L. (2016). Infinite dimensional stochastic differential equations for Dysonś model. *Probab. Theory Related Fields* **166**, no. 3-4, 801-850.

Zambotti, L. (2000). An analytic approach to existence and uniqueness for martingale problems in infinite dimensions. *Probab. Theory Related Fields* **118**, 147-168.