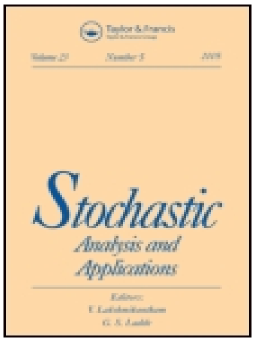

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On Markov Chain Approximations to Semilinear Partial Differential Equations Driven by Poisson Measure Noise

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ABSTRACT

We consider the stochastic model of water pollution, which mathematically can be written with a stochastic partial differential equation driven by Poisson measure noise. We use a stochastic particle Markov chain method to produce an implementable approximate solution. Our main result is the annealed law of large numbers establishing convergence in probability of our Markov chains to the solution of the stochastic reaction-diffusion equation while considering the Poisson source as a random medium for the Markov chains.

Key Words: Stochastic reaction diffusion equations; Markov chains; Poisson processes; Annealed law of large numbers.

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1. INTRODUCTION AND NOTATION

Based upon Kallianpur and Xiong,[7] we consider a stochastic pollution model which characterizes the transport of contaminants (e.g. chemical or bacterial) in a moving sheet of water. Suppose that there are r sources of contamination located at different sites $\kappa_1,\ldots,\kappa_r$ in the water region $E = [0, L_1] \times [0, L_2]$. The r sources disperse contaminants at the jump times of independent Poisson processes $N_1(t),\cdots,N_r(t)$. The magnitude of the j^{th} contaminant released by the i^{th} source is A_i^j, where $\{A_i^j, j = 1, 2, \ldots, i = 1, \ldots, r\}$ are independent random variables that are independent of $\{N_1,\ldots,N_r\}$, and $\{A_i^j, j = 1, 2, \ldots\}$ has common distribution $F_i(da)$. The contaminants are initially distributed in the area $B(\kappa_i, \varepsilon) = \{x : |x - \kappa_i| < \varepsilon\} \subset (0, L_1) \times (0, L_2)$ according to a proportional function $\theta_i(x)$ satisfying

$$\theta_i(x) \geq 0 \ \text{ and } \int_{B(\kappa_i,\varepsilon)} \theta_i(x)dx = 1.$$

Each θ_i is continuous and zero off $B(\kappa_i,\varepsilon)$. Upon release, the contaminants diffuse and drift in the water sheet. Also, there is the possibility of nonlinear reaction of the contaminants due to births and deaths of bacteria or chemical adsorption, which refers to adherence of a substance to the surface of the porous medium in groundwater systems. Reaction is modeled by nonlinear term $R(u)$ below.

The stochastic model described above can be written mathematically as follows (abbreviating $\partial_1 := \frac{\partial}{\partial x_1}$, $\partial_2 := \frac{\partial}{\partial x_2}$, $\Delta := \partial_1^2 + \partial_2^2$, and $\nabla := (\partial_1\, \partial_2)^T$)

$$\begin{aligned}\frac{\partial}{\partial t} u(t, x) =& D\Delta u(t, x) - V \cdot \nabla u(t, x) + R(u(t, x)) \\ &+ \sum_{i=1}^{r} \sum_{j=1}^{\infty} A_i^j(\omega)\theta_i(x) 1_{t=\tau_i^j(\omega)}, \quad x \in [0, L_1] \times [0, L_2], \end{aligned} \tag{1.1}$$

subject to

$$\partial_1 u(t, L_1, x_2) = \partial_1 u(t, 0, x_2) = 0,\ \partial_2 u(t, x_1, L_2) = \partial_2 u(t, x_1, 0) = 0,$$

$$u(0, x) = u_0(x),$$

where $u(t,x)$ denotes the concentration of a dissolved or suspended substance, $D > 0$ denotes the dispersion coefficient, $V = (V_1, V_2)$ with $V_1 > 0$, $V_2 = 0$ denotes the water velocity, $R(\cdot)$ denotes the nonlinear reaction term, $\{\tau_i^j, j \in \mathbb{Z}_+\}$ are the jump times of the independent Poisson processes $N_i(t) (i = 1, 2, \cdots, r)$ with parameters η_i, and $u_0(x)$ denotes

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the initial concentration of the contaminants in the region $[0, L_1] \times [0, L_2]$. All the random variables A_i^j and τ_i^j (or $N_i(t)$) are defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, we assume that $R : [0, \infty) \to \mathbb{R}$ is continuous with

$$R(0) \geq 0 \text{ and } \sup_{u>0} \frac{R(u)}{1+u} < \infty, \tag{1.2}$$

and for some $q \geq 1$ and $K > 0$, we have the local Lipschitz condition

$$|R(u) - R(v)| \leq K|u - v|(1 + |u|^{q-1} + |v|^{q-1}), \ |R(u)| \leq K(1 + u^q). \tag{1.3}$$

Let us define a differential operator $\mathcal{A} = D\Delta - V \cdot \nabla$ with Neumann boundary conditions in both variables. We take the initial domain $\mathcal{D}_0(\mathcal{A})$ of $\mathcal{A}$ to be $\{f \in \mathcal{C}^2(E) : \partial_1 f(0, x_2) = \partial_1 f(L_1, x_2) = \partial_2 f(x_1, 0) = \partial_2 f(x_1, L_2) = 0\}$, where $\mathcal{C}^2(E)$ denotes the twice continuously differentiable functions on E. Letting $\rho(x) = e^{-2cx_1}$ and $c = \frac{V_1}{2D}$, we can rewrite $\mathcal{A}$ as

$$\mathcal{A} = D\left[\frac{1}{\rho(x)} \frac{\partial}{\partial x_1}\left(\rho(x) \frac{\partial}{\partial x_1}\right) + \frac{\partial^2}{\partial x_2^2}\right].$$

In the sequel, $(H, < \cdot, \cdot >)$ is the Hilbert space $L^2(E, \rho(x)dx)$. Then, $(\mathcal{A}, \mathcal{D}_0(\mathcal{A}))$ is symmetric on H and admits a unique self-adjoint extension with domain $\mathcal{D}(\mathcal{A}) = \{f \in H : |\nabla f|, \quad \Delta f \in H$ and $\partial_1 f(0, x_2) = \partial_1 f(L_1, x_2) = 0, \ \partial_2 f(x_1, 0) = \partial_2 f(x_1, L_2) = 0\}$. We define random process $\Theta(t)$ by

$$\Theta(t, x, \omega) = \sum_{i=1}^{r} \theta_i(x) \sum_{j=1}^{N_i(t)} A_i^j(\omega) \text{ for } t \geq 0, x \in [0, L_1] \times [0, L_2], \omega \in \Omega,$$

and find that the equation (1.1) can be rewritten as

$$du(t, x) = [\mathcal{A}u(t, x) + R(u(t, x))]dt + d\Theta(t, x), \quad u(0) = u_0. \tag{1.4}$$

We consider mild solutions

$$u(t) = T(t)u_0 + \int_0^t T(t - s)R(u(s))ds + \int_0^t T(t - s)d\Theta(s) \tag{1.5}$$

of our stochastic partial differential equation (SPDE) (1.4), where $T(t)$ is the C_0-semigroup generated by the operator $\mathcal{A}$.

For any separable Hilbert space V, $C_V[0,T]$ and $D_V[0,T]$ denote respectively the V-valued continuous and càdlàg functions h such that

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$h(t) \in V$ for all $0 \leq t \leq T$. For càdlàg functions h, we define

$$h(\tau_-) := \begin{cases} 0 & \tau = 0, \\ \lim_{s \nearrow \tau} h(s) & 0 < \tau \leq T. \end{cases}$$

We shall use the notations C,$C(N,l)$,$C(T)$ and so on, for finite constants (depending on N, l or T etc.), which may be different at various steps in the proofs of our results.

In this note, we shall discuss unique $D_H[0,T]$-valued solutions and Markov chain approximations to SPDE (1.4). In Ref.[11], Kouritzin and Long established the quenched law of large numbers for the Markov chain approximations to SPDE (1.4) for each fixed path of the Poisson sources and gave an annealed law of large numbers, where the Poisson source is treated as a random medium, as a corollary. It turns out that a more general annealed result is possible if alternative methods are used. In this note, we shall use a different method to establish a general annealed law of large numbers for the Markov chain approximations of the stochastic reaction diffusion model. We remark that our hypotheses (to follow in section 3) are weaker than those given in Ref.[11] In Ref.[11], uniform boundedness was imposed on u_0. Here, we only require that the expectation of u_0 is bounded. Also, there is significant difference between our current proof method and the one used in Ref.[11] In Ref.[11], a relative compactness method and Skorohod representation theorem were crucial in the proof of laws of large numbers. In this note, we directly apply Cauchy criterion (convergence in probability) to our Markov chains and utilize the nice regularity property of Green's function. The current method is clearer and more elegant, especially for people with stronger analysis vis-à-vis probability background.

The contents of this note are organized as follows: In Section 2, we review the Markov chain approximations to our pollution model (1.4) via the stochastic particle method and the random time changes approach. In Section 3, we show that there exists a unique mild solution to (1.4) and prove the annealed law of large numbers.

2. CONSTRUCTION OF MARKOV CHAIN

The Markov chain approximation discussed in this paper is motivated by the stochastic particle simulation method for differential equations studied by Kurtz,[12] Arnold and Theodosopulu,[1] Kotelenez,[8,9] Blount,[3−5] and Kouritzin and Long.[11] They proved that a sequence of Markov chain approximations converges to the unique solution of a reaction-diffusion or

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stochastic reaction diffusion equation in probability. Stochastic limits were considered in Ref.[11] In this case, the Markov chain approximations have two kinds of randomness, which are the external fluctuation coming from the Poisson sources and the internal fluctuation in implementing the reaction and diffusion.

Here, we state some necessary background and results from Ref.[11] for our later development. We denote by $\{(\lambda_p, \phi_p)\}_{p=(p_1,p_2)\in(N_0)^2}$ the eigenvalues and eigenfunctions of $\mathcal{A}$ (see Lemma 2.1 of Kouritzin and Long[11] for their expressions). Now, we introduce the discretized approximation operator $\mathcal{A}^N$ of $\mathcal{A}$. We divide $[0, L_1) \times [0, L_2)$ into $L_1 N \times L_2 N$ cells of size $\frac{1}{N} \times \frac{1}{N}$:

$$I_k := \left[\frac{k_1 - 1}{N}, \frac{k_1}{N}\right) \times \left[\frac{k_2 - 1}{N}, \frac{k_2}{N}\right), k = (k_1, k_2), k_1 = 1, 2, \ldots, L_1 N,$$
$$k_2 = 1, 2, \ldots, L_2 N.$$

We also define the class of cells where the contaminants can enter

$$K_i^N = \{k : I_k \subset B(\kappa_i, \varepsilon)\}, i = 1, 2, \cdots, r.$$

Let $H^N = \{\varphi \in H : \varphi$ is constant on each $I_k\}$. We define the following discrete gradients:

$$\widetilde{\nabla}_{Nx_i} f(x) = N\left[f\left(x + \frac{e_i}{2N}\right) - f\left(x - \frac{e_i}{2N}\right)\right],$$

$$\nabla^+_{Nx_i} f(x) = N\left[f\left(x + \frac{e_i}{N}\right) - f(x)\right]$$

and

$$\nabla^-_{Nx_i} f(x) = N\left[f\left(x - \frac{e_i}{N}\right) - f(x)\right], \quad i = 1, 2,$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. We define $\mathcal{A}^N$ by

$$\mathcal{A}^N f(x) := D\left[\frac{1}{\rho}\tilde{\nabla}_{Nx_1}(\rho\tilde{\nabla}_{Nx_1}) + \Delta_{Nx_2}\right] f(x), \tag{2.1}$$

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where

$$\Delta_{Nx_2} f(x) = -\nabla^{+}_{Nx_2}(\nabla^{-}_{Nx_2} f)(x)$$
$$= N^2\left[f\left(x+\frac{e_2}{N}\right) - 2f(x) + f\left(x-\frac{e_2}{N}\right)\right].$$

In order to take the boundary conditions into account for the discretized approximation scheme, we extend all function $f \in H^N$ to the region $[-\frac{1}{N}, L_1+\frac{1}{N}]\times[-\frac{1}{N}, L_2+\frac{1}{N}]$ by letting

$$f(x_1,x_2) = f\left(x_1+\frac{1}{N}, x_2\right), x_1 \in \left[-\frac{1}{N}, 0\right), x_2 \in [0, L_2];$$

$$f(x_1,x_2) = f\left(x_1-\frac{1}{N}, x_2\right), x_1 \in \left[L_1, L_1+\frac{1}{N}\right), x_2 \in [0, L_2];$$

$$f(x_1,x_2) = f\left(x_1, x_2+\frac{1}{N}\right), x_1 \in [0, L_1], x_2 \in \left[-\frac{1}{N}, 0\right);$$

$$f(x_1,x_2) = f\left(x_1, x_2-\frac{1}{N}\right), x_1 \in [0, L_1], x_2 \in \left[L_2, L_2+\frac{1}{N}\right)$$

and denote this class of functions by H^N_{bc}. Then, H^N_{bc} is the domain of $\mathcal{A}^N$. We denote by $\{\lambda^N_p, \phi^N_p\}^{(L_1N-1, L_2N-1)}_{p=(p_1,p_2)=0}$ the eigenvalues and eigenfunctions of $\mathcal{A}^N$. For their precise expressions, we refer to Lemma 2.2 of Kouritzin and Long.[11] For $p=(p_1,p_2)\in\mathbb{N}_0^2$, let $|p| = \sqrt{p_1^2+p_2^2}$.

Let $T^N(t) = \exp(\mathcal{A}^N t)$. Then, ϕ^N_p are eigenfunctions of $T^N(t)$ with eigenvalues $\exp\{\lambda^N_p t\}$. Now, we describe the stochastic particle systems. Let $l = l(N)$ be a function such that $l(N)\to\infty$ as $N\to\infty$. l^{-1} can loosely be thought of as the "mass" or the "amount of concentration" of one particle. We let $n_k(t)$ denote the number of particles in cell k at time t for $k=(k_1,k_2)\in\{1,\ldots,L_1N\}\times\{1,\ldots,L_2N\}$ and also, to account for our Neumann boundary conditions, we set

$$n_{0,k_2}(t) = n_{1,k_2}(t), \quad n_{L_1N+1,k_2}(t) = n_{L_1N,k_2}(t), \quad k_2 = 1,\ldots,L_2N,$$

$$n_{k_1,0}(t) = n_{k_1,1}(t), \quad n_{k_1,L_2N+1}(t) = n_{k_1,L_2N}(t), \quad k_1 = 1,\ldots,L_1N.$$

Particles undergo diffusion between adjacent cells, and give births or die in each cell due to reaction. For the transition rates of particles evolving in cells, we refer to Kouritzin and Long.[11] Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ be another

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probability space on which is defined independent standard Poisson processes $\{X^{k,R}_{+,N}, X^{k,R}_{-,N}, X^{k,1}_{+,N}, X^{k,1}_{-,N}, X^{k,2}_{+,N}, X^{k,2}_{-,N}\}^{(L_1N,L_2N)}_{k=(1,1)}$, $\{X^{k,1}_{+,N}, X^{k,1}_{-,N}, k = (0,k_2)\}^{L_2N}_{k_2=1}$ and $\{X^{k,2}_{+,N}, X^{k,2}_{-,N}, k=(k_1,0)\}^{L_1N}_{k_1=1}$. We define the product probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0) = (\Omega\times\bar{\Omega}, \mathcal{F}\otimes\bar{\mathcal{F}}, \mathbb{P}\times\bar{\mathbb{P}})$. In the sequel, $\lfloor r \rfloor$ denotes the greatest integer not more than a real number r. For any real a, $a^+ := a\vee 0$ and $a^- := -(a\wedge 0)$. Then, from Ethier and Kurtz[6] and Kouritzin and Long,[11] we have for $k\in\{(1,1),\dots,(L_1N,L_2N)\}$

$$\begin{aligned} n^N_k(t) = {} & n^N_k(0) + X^{k,R}_{+,N}\left(l\int_0^t R^+(n^N_k(s)l^{-1})ds\right) \\ & - X^{k,R}_{-,N}\left(l\int_0^t R^-(n^N_k(s)l^{-1})ds\right) + \sum_{i=1}^{2}\left[X^{k,i}_{+,N}\left(\int_0^t \delta^+_{i,N}(n^N_k(s))ds\right)\right. \\ & \left. - X^{k,i}_{-,N}\left(\int_0^t \delta^-_{i,N}(n^N_k(s))ds\right)\right] - \sum_{i=1}^{2}\left[X^{k-e_i,i}_{+,N}\left(\int_0^t \delta^+_{i,N}(n^N_{k-e_i}(s))ds\right)\right. \\ & \left. - X^{k-e_i,i}_{-,N}\left(\int_0^t \delta^-_{i,N}(n^N_{k-e_i}(s))ds\right)\right] \\ & + \sum_{i=1}^{r}\sum_{j=1}^{\infty}\lfloor l\theta_i(k)A^j_i + 0.5\rfloor 1_{t\ge\tau^j_i} 1_{k\in K^N_i}, \end{aligned} \tag{2.2}$$

where $\delta_{1,N}(n_k) = DN^2e^{-\frac{c}{N}}n_{k+e_1} - DN^2e^{\frac{c}{N}}n_k$ and $\delta_{2,N}(n_k) = DN^2n_{k+e_2} - DN^2n_k$.

Equation (2.2) is our Markov chain approximation to equation (1.4) that can be implemented directly on a computer. We let $\{\mathcal{G}^N_t\}_{t\ge 0}$ be the smallest right continuous standard filtration such that $\{X^{k,R}_{\sigma,N}(l\int_0^t R^\sigma(n^N_k(s)l^{-1})ds), X^{k,i}_{\sigma,N}(\int_0^t \delta^\sigma_{i,N}(n^N_k(s))ds), \sigma = +,-, i=1,2\}^{(L_1N,L_2N)}_{k=(1,1)}$, $\{X^{k,1}_{\sigma,N}(\int_0^t \delta^\sigma_{1,N}(n^N_k(s))ds), \sigma = +,-, k=(0,k_2)\}^{L_2N}_{k_2=1}$, and $\{X^{k,2}_{\sigma,N}(\int_0^t \delta^\sigma_{2,N}(n^N_k(s))ds), \sigma=+,-, k=(k_1,0)\}^{L_1N}_{k_1=1}$ as well as $\{N_i, i=1,2,\dots,r\}$ are adapted to $\{\mathcal{G}^N_t\}$.

To get the concentration in each cell, we divide $n^N_k(t)$ by l

$$u^N(t,x) = \sum_{k_1=1}^{L_1N}\sum_{k_2=1}^{L_2N}\frac{n^N_k(t)}{l}1_k(x), \tag{2.3}$$

where $1_k(\cdot)$ denotes the indicator function on I_k. Then, from, (2.2) it follows

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that

$$u^N(t) = u^N(0) + \int_0^t \mathcal{A}^N u^N(s)ds + \int_0^t R(u^N(s))ds + Z^N(t) + \Theta^N(t), \tag{2.4}$$

where

$$\Theta^N(t,\cdot) = \sum_{i=1}^{r} \sum_{j=1}^{N_i(t)} \sum_{k \in K_i^N} l^{-1} \lfloor l\theta_i(k) A_i^j(\omega) + 0.5 \rfloor 1_k(\cdot),$$

$$Z^N(t) := \sum_{k=(1,1)}^{(L_1N, L_2N)} l^{-1} \left[Z_{k,R,+}^N(t) + Z_{k,R,-}^N(t) + \sum_{i=1}^{2} \left(Z_{k,i}^N(t) - Z_{k-e_i,i}^N(t) \right) \right] 1_k,$$

and

$$Z_{k,R,+}^N(t) = X_{+,N}^{k,R}\left(l \int_0^t R^+(n_k^N(s) l^{-1}) ds \right) - l \int_0^t R^+(n_k^N(s) l^{-1}) ds,$$

$$Z_{k,R,-}^N(t) = -X_{-,N}^{k,R}\left(l \int_0^t R^-(n_k^N(s) l^{-1}) ds \right) + l \int_0^t R^-(n_k^N(s) l^{-1}) ds,$$

$$Z_{k,i}^N(t) = X_{+,N}^{k,i}\left(\int_0^t \delta_{i,N}^+(n_k^N(s)) ds \right) - X_{-,N}^{k,i}\left(\int_0^t \delta_{i,N}^-(n_k^N(s)) ds \right)$$
$$- \int_0^t \delta_{i,N}(n_k^N(s)) ds, \quad i = 1, 2$$

are $\mathcal{L}^2$- martingales with respect to $\{\mathcal{G}_t^N\}$ under probability measure $\mathbb{P}_0$ (see Lemma 2.5 of Kouritzin and Long[11]). By variation of constants and (2.4), it follows that $u^N(t) = u^N(t, \omega_0)$ satisfies

$$u^N(t) = T^N(t) u^N(0) + \int_0^t T^N(t-s) R(u^N(s)) ds$$
$$+ Y^N(t) + \int_0^t T^N(t-s) d\Theta^N(s), \tag{2.5}$$

where

$$Y^N(t) = \int_0^t T^N(t-s) dZ^N(s).$$

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3. LAW OF LARGE NUMBERS: ANNEALED APPROACH

In this section, we shall prove the law of large numbers for u^N via annealed approach, which means that $N_i(t)$ and A_i^j are considered to be random variables, and the Markov chain u^N evolves in this random medium. As already mentioned in Section 2, u^N is defined on the product probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0) = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) \times (\Omega, \mathcal{F}, \mathbb{P})$. For $f : E \to \mathbb{R}$, let $\|f\|_\infty = \sup_{x\in E}|f(x)|$. In the sequel, we always consider the Skorohod metric d on $D_H[0,T]$ so that $(D_H[0,T],d)$ is a complete separable metric space. We introduce the following hypotheses (abbreviating $\mathbb{E}_0 := \mathbb{E}^{\mathbb{P}_0}$) :

Hypotheses

(i) $\|\mathbb{E}_0(u^N(0)^{2q}\|_\infty \le C < \infty$.
(ii) $\{l(N)\}$ is any sequence satisfying $l(N) \to \infty$ as $N \to \infty$.
(iii) $\|u^N(0) - u_0\| \to 0$ in probability $\mathbb{P}_0$.
(iv) $\|u^N(0)\|_\infty \le C(N, l) < \infty$.
(v) The distribution of the deposit magnitudes satisfies $\int_{\mathbb{R}^+} a^{2q} F_i(da) < \infty$.
(vi) $\mathbb{E}_0\|u_0\|_\infty \le c_0 < \infty$.

We have the following *annealed* law of large numbers:

Theorem 3.1. *Let the Hypotheses be fulfilled. Then, there exists a pathwise unique mild solution u to* (1.4) *and*

$$\sup_{t\le T}\|u^N(t) - u(t)\| \to 0 \tag{3.1}$$

in probability $\mathbb{P}_0$ *as* $N \to \infty$.

Remark 3.2. Let $\{N_k\}_{k=0}^\infty$ be an increasing sequence in $\mathbb{N}$ such that $N_k \to \infty$ as $k \to \infty$. We define $\tilde{\Omega} = \prod_{m=0}^\infty \tilde{\Omega}_m$, where $\tilde{\Omega}_m = D_{\mathbb{R}^{L_1 N_m \times L_2 N_m}}[0, \infty)$. Set $\tilde{\mathcal{F}} = \otimes_{m=0}^\infty \mathcal{B}(\tilde{\Omega}_m)$, which is the σ-algebra generated by open sets under Skorohod topology and countable products. From Lemma 2.5 of Kouritzin and Long,[11] we know that $n^N(t) = \{n_k^N(t)\}_{k=(1,1)}^{(L_1N,L_2N)}$ is well defined, and for each $\omega \in \Omega$ there exists a unique probability measure $\tilde{\mathbb{P}}^\omega$ on $(\tilde{\Omega},\tilde{\mathcal{F}})$ such that $\bar{\mathbb{P}}(\bar{\omega} \in \bar{\Omega} : n^{N_{m_1}}(\bar{\omega}, \omega) \in A_1, \ldots n^{N_{m_j}}(\bar{\omega}, \omega) \in A_j) = \tilde{\mathbb{P}}^\omega(\tilde{\omega} \in \tilde{\Omega} : \tilde{\omega}_{m_1} \in A_1, \ldots, \tilde{\omega}_{m_j} \in A_j)$ for all $A_i \in \mathcal{B}(D_{\mathbb{R}^{L_1 N_{m_i} \times L_2 N_{m_i}}}[0, \infty)), i = 1, \ldots, j;$ $j = 1, 2, \ldots$. Moreover, we have that for each $B \in \tilde{\mathcal{F}}$, $\omega \to \tilde{P}^\omega(B)$ is $(\Omega,\mathcal{F})$-measurable, and

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$\omega \to \int_{\tilde{\Omega}} f(\omega, \tilde{\omega}) \tilde{P}^{\omega}(d\tilde{\omega})$ is $\mathcal{F}$-measurable for each bounded measurable function f. We can write

$$\mathbb{P}_0(d\omega_0) = \tilde{\mathbb{P}}^{\omega}(d\tilde{\omega})\mathbb{P}(d\omega), \quad \omega_0 = (\omega, \tilde{\omega}).$$

The statements of Lemma 3.3 and Lemma 3.4 of Ref.[11] are still valid if we replace $\tilde{\mathbb{E}}^{\omega}$ by $\mathbb{E}_0$. We may just take expectations with respect to $\mathbb{P}$.

Lemma 3.3. *For* $2\beta \in [1, 2q]$,

$$\sup_{s \le t} \|\mathbb{E}_0(u^N(s))^{2\beta}\|_{\infty} \le C(t, l, \|\mathbb{E}_0(u^N(0))^{2\beta}\|_{\infty}) < \infty,$$

where C is decreasing in l.

Proof. Our proof follows the proof of Lemma 3.5 of Ref.[11]. The only significant modification is to estimate the expectation (under $\mathbb{P}_0$) of the last term on the right-hand side of (3.12) in the proof of Lemma 3.5 of Ref.[11]. In fact, letting $\sigma_N(k) = \int_{I_k} \rho(x)dx$ and using (3.22) in the proof of Lemma 3.5 of Ref.[11], we have

$$\mathbb{E}_0 \left| \left\langle \int_0^t T^N(t-s) d\Theta^N(s), (\sigma_N(k))^{-1} 1_k \right\rangle \right|^{2\beta}$$
$$\le 2^{2\beta} \left(\mathbb{E}_0 \left| \sum_{i=1}^{r} \|\theta_i\|_{\infty} \sum_{j=1}^{N_i(t)} A_i^j \right|^{2\beta} + l^{-2\beta} \mathbb{E}_0 |N_1(t) + \cdots + N_r(t)|^{2\beta} \right). \tag{3.2}$$

Recalling that $\{N_i, i = 1,2,\ldots,r\}$ are independent Poisson processes with parameters η_i, we set

$$\mathcal{N}_i([0,t] \times B) = \sum_{j=1}^{N_i(t)} 1_B(A_i^j),$$

which is a Poisson measure with characteristic measure $\mu_i(B) = \eta_i F_i(B)$. Denote by $\tilde{\mathcal{N}}_i(dsda)$ the compensated martingale measure of $\mathcal{N}_i(dsda)$. Wald's equation gives

$$\mathbb{E}_0 \left[\sum_{j=1}^{N_i(t)} A_i^j \right] = t\eta_i \int_{\mathbb{R}_+} aF_i(da) = t\eta_i a_i, \quad a_i := \int_{\mathbb{R}_+} aF_i(da). \tag{3.3}$$

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Applying Jensen's inequality and Burkholder's inequality, we find that

$$
\begin{aligned}
&\mathbb{E}_0\left|\sum_{i=1}^{r}\|\theta_i\|_\infty\sum_{j=1}^{N_i(t)}A_i^j\right|^{2\beta}\\
&\quad\le r^{2\beta-1}\sum_{i=1}^{r}\|\theta_i\|_\infty^{2\beta}\cdot\mathbb{E}_0\left|\sum_{j=1}^{N_i(t)}A_i^j\right|^{2\beta}\\
&\quad\le r^{2\beta-1}\sum_{i=1}^{r}\|\theta_i\|_\infty^{2\beta}\cdot\mathbb{E}_0\left|t\eta_i a_i+\int_0^t\int_{\mathbb{R}_+}a\tilde{\mathcal{N}}_i(dsda)\right|^{2\beta}\\
&\quad\le r^{2\beta-1}\sum_{i=1}^{r}\|\theta_i\|_\infty^{2\beta}\cdot 2^{2\beta-1}\left[(t\eta_i a_i)^{2\beta}+\mathbb{E}_0\left|\int_0^t\int_{\mathbb{R}_+}a\tilde{\mathcal{N}}_i(dsda)\right|^{2\beta}\right]\\
&\quad\le (2r)^{2\beta-1}\sum_{i=1}^{r}\|\theta_i\|_\infty^{2\beta}\left\{t^{2\beta}\eta_i^{2\beta}a_i^{2\beta}+C(\beta)\mathbb{E}_0\left[\int_0^\cdot\int_{\mathbb{R}_+}a\tilde{\mathcal{N}}_i(dsda)\right]_t^{\beta}\right\}.
\end{aligned}
\tag{3.4}
$$

When $\beta\le 1$, we have by Jensen's inequality

$$
\mathbb{E}_0\left[\int_0^\cdot\int_{\mathbb{R}_+}a\tilde{\mathcal{N}}_i(dsda)\right]_t^{\beta}\le\left(t\eta_i\int_{\mathbb{R}_+}a^2F_i(da)\right)^{\beta},
\tag{3.5}
$$

and otherwise (when $\beta>1$) we have by Jensen's inequality that

$$
\begin{aligned}
\mathbb{E}_0\left[\int_0^\cdot\int_{\mathbb{R}_+}a\tilde{\mathcal{N}}_i(dsda)\right]_t^{\beta}&=\mathbb{E}_0\left[\sum_{j=1}^{N_i(t)}(A_i^j)^2\right]^{\beta}\\
&=\sum_{n=0}^{\infty}\mathbb{E}_0\left[\left(\sum_{j=1}^{N_i(t)}(A_i^j)^2\right)^{\beta}\middle|N_i(t)=n\right]\mathbb{P}_0(N_i(t)=n)\\
&\le\sum_{n=0}^{\infty}n^{\beta-1}\mathbb{E}_0\left[\sum_{j=1}^{n}(A_i^j)^{2\beta}\middle|N_i(t)=n\right]\mathbb{P}_0(N_i(t)=n)\\
&=\mathbb{E}_0\left[(N_i(t))^{\beta}\right]\int_{\mathbb{R}_+}a^{2\beta}F_i(da).
\end{aligned}
\tag{3.6}
$$

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Thus, by (3.4), (3.5), (3.6) and Hypothesis (v), one has that

$$\mathbb{E}_0\left|\sum_{i=1}^{r}\|\theta_i\|_\infty\sum_{j=1}^{N_i(t)}A_i^j\right|^{2\beta}\leq C(t).$$

Therefore, noting $\mathbb{E}_0|N_1(t)+\cdots+N_r(t)|^{2\beta}\leq C(t)$, we have by (3.2) that

$$\mathbb{E}_0\left|\left\langle\int_0^t T^N(t-s)d\Theta^N(s),(\sigma_N(k))^{-1}1_k\right\rangle\right|^{2\beta}\leq C(t,l)<\infty.$$

Now, the lemma follows from the proof of Lemma 3.5 of Ref.[11] □

Lemma 3.4. $\sup_{t\leq T}\|Y^N(t)\|\to 0$ *in probability* $\mathbb{P}_0$ *as* $N\to\infty$.

Proof. Note that Lemma 3.6 of Kouritzin and Long[11] is still valid under our new Hypotheses if we replace $\tilde{\mathbb{P}}^\omega$ by $\mathbb{P}_0$ and $\tilde{\mathbb{E}}^\omega$ by $\mathbb{E}_0$. Since the proof is almost the same as that of Lemma 3.6 (iv) of Ref.[11], we omit the details here.

Immediately from Lemma 3.7 of Ref.[11], we have the following lemma.

Lemma 3.5. *For almost all* $\omega_0\in\Omega_0$

$$\sup_{t\leq T}\left\|\int_0^t T^N(t-s)d\Theta^N(s)-\int_0^t T(t-s)d\Theta(s)\right\|\to 0 \ \text{ as } \ \mathrm{N}\to\infty.$$

The remainder of the development differs substantially from Ref.[11]

Lemma 3.6. *There exists a càdlàg H-valued process* φ *such that* $\sup_{t\leq T}\|u^N(t)-\varphi(t)\|\to 0$ *in probability* $\mathbb{P}_0$ *as* $\mathrm{N}\to\infty$.

Proof. We apply Cauchy criterion to prove our statement. It follows from (2.5) that for $N,M\in\mathbb{N}$

$$\begin{aligned}\|u^N(t)-u^M(t)\|\leq&\|T^N(t)u^N(0)-T^M(t)u^M(0)\|+\|Y^N(t)\|+\|Y^M(t)\|\\&+\left\|\int_0^t T^N(t-s)d\Theta^N(s)-\int_0^t T^M(t-s)d\Theta^M(s)\right\|\\&+\left\|\int_0^t T^N(t-s)R(u^N(s))ds-\int_0^t T(t-s)R(u^N(s))ds\right\|\\&+\left\|\int_0^t T^M(t-s)R(u^M(s))ds-\int_0^t T(t-s)R(u^M(s))ds\right\|\end{aligned}$$

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$$+\int_0^t \|T(t-s)\big(R(u^N(s))-R(u^M(s))\big)\|ds$$

$$:= V_t^{N,M}+\int_0^t \|T(t-s)(R(u^N(s))-R(u^M(s)))\|ds. \tag{3.7}$$

We claim that $\sup_{t\le T} V_t^{N,M}\to 0$ in probability as $N, M\to\infty$ (in any way). This is true by Hypothesis (iii), Trotter-Kato theorem, Lemma 3.4, and Lemma 3.5 if we can prove that

$$\sup_{t\le T}\left\|\int_0^t T^N(t-s)R(u^N(s))ds-\int_0^t T(t-s)R(u^N(s))ds\right\|$$

converges to zero in probability when $N\to\infty$. Note that

$$\int_0^t T^N(t-s)R(u^N(s))ds-\int_0^t T(t-s)R(u^N(s))ds$$

$$=\sum_{|p|\le n}\Big[\int_0^t \exp(\lambda_p^N(t-s))\langle\phi_p^N, R(u^N(s))\rangle ds\phi_p^N-\int_0^t \exp(\lambda_p(t-s))$$

$$\times\langle\phi_p, R(u^N(s))\rangle ds\phi_p\Big]+\sum_{|p|>n}\int_0^t \exp(\lambda_p^N(t-s))\langle\phi_p^N, R(u^N(s))\rangle ds\phi_p^N$$

$$-\sum_{|p|>n}\int_0^t \exp(\lambda_p(t-s))\langle\phi_p, R(u^N(s))\rangle ds\phi_p. \tag{3.8}$$

For $|p|\neq 0$, $c_1|\lambda_p|\le|\lambda_p^N|\le c_2|\lambda_p|$ and $|\lambda_p|\ge c_3|p|^2$ (see Lemma 2.1 and Remark 2.3 of Kouritzin and Long[11]). So, by Cauchy-Schwarz inequality, we have

$$\left(\int_0^t \exp(\lambda_p^N(t-s))\langle\phi_p^N, R(u^N(s))\rangle ds\right)^2$$

$$\le\left(\int_0^t \exp(2\lambda_p^N(t-s))ds\right)\cdot\left(\int_0^t \langle\phi_p^N, R(u^N(s))\rangle^2 ds\right)$$

$$\le C|p|^{-2}\int_0^t \langle\phi_p^N, R(u^N(s))\rangle^2 ds.$$

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Thus,

$$\sum_{|p|>n}\left(\int_0^t \exp(\lambda_p^N(t-s))\langle \phi_p^N, R(u^N(s))\rangle ds\right)^2$$

$$\leq Cn^{-2}\int_0^t \sum_p \langle \phi_p^N, R(u^N(s))\rangle^2 ds \leq Cn^{-2}\int_0^T \langle 1, R^2(u^N(s))\rangle ds.$$

It follows by (1.3), Hypothesis (i) and Lemma 3.3 that

$$\mathbb{E}_0\left[\sup_{t\leq T}\left\|\sum_{|p|>n}\int_0^t \exp(\lambda_p^N(t-s))\langle \phi_p^N, R(u^N(s))\rangle ds\phi_p^N\right\|^2\right] \leq \frac{C(T)}{n^2}.$$

Similarly, we can show that

$$\mathbb{E}_0\left[\sup_{t\leq T}\left\|\sum_{|p|>n}\int_0^t \exp(\lambda_p(t-s))\langle \phi_p, R(u^N(s))\rangle ds\phi_p\right\|^2\right] \leq \frac{C(T)}{n^2}.$$

Note that for fixed p, $|\lambda_p^N - \lambda_p| + \|\phi_p^N - \phi_p\|_\infty \to 0$ as $N \to \infty$ and $\sup_{p,N}(\|\phi_p^N\|_\infty + \|\phi_p\|_\infty) < \infty$ (see Lemma 2.1, Lemma 2.2 and Remark 2.3 of Ref.[11]) Therefore, by Lemma 3.3, it follows that for p fixed,

$$\sup_{t\leq T}\left\|\int_0^t \exp(\lambda_p^N(t-s))\langle \phi_p^N, R(u^N(s))\rangle ds\phi_p^N\right.$$

$$\left. - \int_0^t \exp(\lambda_p(t-s))\langle \phi_p, R(u^N(s))\rangle ds\phi_p\right\|_\infty$$

converges to zero in probability as $N \to \infty$. Thus,

$$\sup_{t\leq T}\left\|\int_0^t T^N(t-s)R(u^N(s))ds - \int_0^t T(t-s)R(u^N(s))ds\right\| \to 0 \tag{3.9}$$

in probability as $N \to \infty$.

Next, we come back to the inequality (3.7). We remark that the semigroup $T(t)$ is defined by $(T(t)f)(x) = \int_E G(t;x,y)f(y)\rho(y)dy, x \in E$, where $G(t;x,y)$ is the Green's function corresponding to $\mathcal{A}$ and is given by

$$G(t;x,y) = \sum_p \exp(\lambda_p t)\phi_p(x)\phi_p(y), \quad t \geq 0.$$

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Moreover, using Lemma 2.1 of Ref.[11], we find that for any $t \in [0,T]$, there exists a constant $c(T) > 0$ such that

$$\begin{aligned}
\int_E |G(t;x,y)|^2 \rho(x)dx &= \sum_{p=(0,0)}^{(\infty,\infty)} e^{2\lambda_p t} \phi_p^2(y) \\
&\leq C \sum_{p_1=0}^{\infty} e^{2\lambda_{p_1}^1 t} \sum_{p_2=0}^{\infty} e^{2\lambda_{p_2}^2 t} \\
&= C\left(1 + \sum_{p_1=1}^{\infty} \exp\left\{-2\left(\frac{D\pi^2}{L_1^2} p_1^2 + Dc^2\right)t\right\}\right) \\
&\quad \times \left(1 + \sum_{p_2=1}^{\infty} \exp\left\{-2\frac{D\pi^2}{L_2^2} p_2^2 t\right\}\right) \\
&\leq C\left(1 + \int_0^{\infty} \exp\left\{-2\frac{D\pi^2}{L_1^2} tx^2\right\}dx\right) \\
&\quad \times \left(1 + \int_0^{\infty} \exp\left\{-2\frac{D\pi^2}{L_2^2} tx^2\right\}dx\right) \\
&= C\left(1 + \frac{L_1}{2\sqrt{2D\pi t}}\right)\left(1 + \frac{L_2}{2\sqrt{2D\pi t}}\right) \leq c(T) \cdot t^{-1}.
\end{aligned}$$

By using Minkowski's integral inequality, (1.3), and Cauchy-Schwarz inequality, we find that

$$\begin{aligned}
&\left\| T(t-s)(R(u^N(s)) - R(u^M(s))) \right\| \\
&\quad = \left\{ \int_E \left[\int_E G(t-s;x,y)(R(u^N(s,y)) - R(u^M(s,y)))\rho(y)dy \right]^2 \rho(x)dx \right\}^{\frac{1}{2}} \\
&\quad \leq \int_E \left\{ \int_E |G(t-s;x,y)|^2 \rho(x)dx \right\}^{\frac{1}{2}} |R(u^N(s,y)) - R(u^M(s,y))| \rho(y)dy \\
&\quad \leq C(T)(t-s)^{-\frac{1}{2}} \int_E |u^N(s,y) - u^M(s,y)|(1 + |u^N(s,y)|^{q-1}
\end{aligned}$$

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$$
\begin{aligned}
&\quad + |u^M(s,y)|^{q-1})\rho(y)dy \\
&\le C(T)(t-s)^{-\frac{1}{2}}\|u^N(s)-u^M(s)\|\left\{\int_E (1+|u^N(s,y)|^{q-1}\right. \\
&\quad \left. +|u^M(s,y)|^{q-1})^2\rho(y)dy\right\}^{\frac{1}{2}}. \qquad (3.10)
\end{aligned}
$$

For convenience, let

$$
f(s) = \left\{\int_E (1+|u^N(s,y)|^{q-1}+|u^M(s,y)|^{q-1})^2\rho(y)dy\right\}^{\frac{1}{2}}.
$$

The following method is motivated in part by Kouritzin.[10] Substituting (3.10) into (3.7), we get

$$
\|u^N(t)-u^M(t)\| \le V_t^{N,M} + C(T)\int_0^t (t-s)^{-\frac{1}{2}} f(s)\|u^N(s)-u^M(s)\|ds \quad a.s. \qquad (3.11)
$$

By Lemma 3.3, it is easy to see that for $t \in [0,T]$

$$
\mathbb{E}_0 \int_0^t (t-s)^{-\frac{1}{2}} f(s)ds \le C(T) < \infty
$$

and consequently

$$
\int_0^t (t-s)^{-\frac{1}{2}} f(s)ds < \infty, a.s.
$$

By iterating (3.11) n times, we get

$$
\begin{aligned}
&\|u^N(t)-u^M(t)\| \\
&\le \sup_{s\le t} V_s^{N,M}\left(1 + C(T)\int_0^t \frac{f(s)}{\sqrt{t-s}}ds + \cdots + (C(T))^n\right. \\
&\quad \left.\times \int_0^t\int_0^s\cdots\int_0^{s_{n-2}} \frac{f(s)}{\sqrt{t-s}}\cdots\frac{f(s_{n-1})}{\sqrt{s_{n-2}-s_{n-1}}}ds_{n-1}\cdots ds_1 ds\right) + (C(T))^{n+1} \\
&\quad \times \int_0^t\int_0^s\cdots\int_0^{s_{n-1}} \frac{f(s)}{\sqrt{t-s}}\cdots\frac{f(s_n)}{\sqrt{s_{n-1}-s_n}}\|u^N(s_n)-u^M(s_n)\|ds_n\cdots ds_1 ds, \qquad (3.12)
\end{aligned}
$$

If $q=1$, then $f(s) \le C < \infty$. This case is much simpler than the case of

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$q > 1$. So, we only deal with the later case here. For $q > 1$, by applying Hölder's inequality and Young's inequality, we find

$$\begin{aligned}\int_0^t \frac{f(s)}{\sqrt{t-s}}ds &\le \left(\int_0^t \left[(t-s)^{-\frac{1}{2}}\right]^{\frac{2q}{q+1}} ds\right)^{\frac{q+1}{2q}} \cdot \left(\int_0^t f^{\frac{2q}{q-1}}(s)ds\right)^{\frac{q-1}{2q}} \\ &\le \frac{q+1}{2q}\int_0^t (t-s)^{-\frac{q}{q+1}}ds + \frac{q-1}{2q}\int_0^t f^{\frac{2q}{q-1}}(s)ds \\ &= \frac{q+1}{2q}\cdot(q+1)t^{\frac{1}{q+1}} + \frac{q-1}{2q}\int_0^t f^{\frac{2q}{q-1}}(s)ds. \end{aligned} \tag{3.13}$$

Similarly, for $k \ge 1$ with $s_0 = s$, we have

$$\begin{aligned}&\int_0^t \int_0^s \cdots \int_0^{s_{k-1}} \frac{f(s)}{\sqrt{t-s}} \cdots \frac{f(s_k)}{\sqrt{s_{k-1}-s_k}} ds_k \cdots ds \\ &\le \left\{\int_0^t \int_0^s \cdots \int_0^{s_{k-1}} (t-s)^{-\frac{q}{q+1}} \cdots (s_{k-1}-s_k)^{-\frac{q}{q+1}} ds_k \cdots ds\right\}^{\frac{q+1}{2q}} \\ &\quad \times \left\{\int_0^t \int_0^s \cdots \int_0^{s_{k-1}} f^{\frac{2q}{q-1}}(s) \cdots f^{\frac{2q}{q-1}}(s_k) ds_k \cdots ds\right\}^{\frac{q-1}{2q}} \\ &\le \frac{q+1}{2q}\cdot(q+1)\frac{\left(\Gamma(\frac{1}{q+1})\right)^k \Gamma(1+\frac{1}{q+1}) t^{\frac{k+1}{q+1}}}{\Gamma(1+\frac{k+1}{q+1})} + \frac{q-1}{2q}\cdot\frac{\left(\int_0^t f^{\frac{2q}{q-1}}(s)ds\right)^{k+1}}{(k+1)!}.\end{aligned}$$

Therefore, it follows from (3.12) that

$$\begin{aligned}&\sup_{t\le T}\|u^N(t)-u^M(t)\| \\ &\le \sup_{t\le T} V_t^{N,M}\left[1 + \frac{q+1}{2q}\sum_{k=1}^n (q+1)\frac{\left(\Gamma(\frac{1}{q+1})\right)^{k-1}\Gamma(1+\frac{1}{q+1})T^{\frac{k}{q+1}}(C(T))^k}{\Gamma(1+\frac{k}{q+1})}\right. \\ &\quad \left. + \frac{q-1}{2q}\sum_{k=1}^n \frac{(C(T))^k}{k!}\left(\int_0^T f^{\frac{2q}{q-1}}(s)ds\right)^k\right]\end{aligned}$$

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$$+ \sup_{t\leq T}\|u^N(t) - u^M(t)\|\cdot(C(T))^{n+1}\left[\frac{q+1}{2q}\cdot(q+1)\frac{\left(\Gamma(\frac{1}{q+1})\right)^n\Gamma(1+\frac{1}{q+1})T^{\frac{n+1}{q+1}}}{\Gamma(1+\frac{n+1}{q+1})}\right.$$

$$\left. + \frac{q-1}{2q}\cdot\frac{1}{(n+1)!}\left(\int_0^T f^{\frac{2q}{q-1}}(s)ds\right)^{n+1}\right]. \tag{3.14}$$

Let

$$h(T) = \int_0^T f^{\frac{2q}{q-1}}(s)ds.$$

By applying Hölder's inequality, Hypothesis (i) and Lemma 3.3, we have

$$\begin{aligned}\mathbb{E}_0[h(T)] &= \int_0^T \mathbb{E}_0\left[f^{\frac{2q}{q-1}}(s)\right]ds \\ &= \int_0^T \mathbb{E}_0\left[\int_E (1+|u^N(s,y)|^{q-1} + |u^M(s,y)|^{q-1})^2\rho(y)dy\right]^{\frac{q}{q-1}} ds \\ &\leq C\int_0^T \mathbb{E}_0\left[\int_E (1+|u^N(s,y)|^{2q} + |u^M(s,y)|^{2q})dy\right]ds \\ &\leq C_1(T) < \infty. \end{aligned} \tag{3.15}$$

$\sup_{t\leq T}\|u^N(t) - u^M(t)\|$ is almost surely finite for fixed N and M by the fact that $t \to \|u^N(t)\|$ is càdlàg and the argument on page 110 of Billingsley.[2] Thus, it is easy to see that the last term on the right hand side of (3.14) tends to zero almost surely as $n\to\infty$. So, by letting $n\to\infty$ in (3.14), we find that

$$\begin{aligned}&\sup_{t\leq T}\|u^N(t) - u^M(t)\| \\ &\quad\leq \sup_{t\leq T}V_t^{N,M}\left(\frac{q+1}{2q} + \frac{(q+1)^2}{2q}\sum_{k=1}^{\infty}\frac{\left(\Gamma(\frac{1}{q+1})\right)^{k-1}\Gamma(1+\frac{1}{q+1})T^{\frac{k}{q+1}}(C(T))^k}{\Gamma(1+\frac{k}{q+1})}\right. \\ &\quad\quad\left. + \frac{q-1}{2q}\cdot\exp\{C(T)h(T)\}\right),\end{aligned} \tag{3.16}$$

which tends to zero in probability as $N, M\to\infty$ by (3.15) and the fact that

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$\sup_{t\le T} V_t^{N,M}$ converges to zero in probability. It follows that for each positive integer n there exists a number N_n such that

$$\mathbb{P}_0\left(\sup_{t\le T}\|u^N(t)-u^M(t)\|\ge\frac{1}{2^n}\right)<\frac{1}{2^n}$$

if $N, M \ge N_n$. Without loss of generality, we may assume that $\{N_n\}$ is strictly increasing. Let

$$A_n=\left\{\omega_0\in\Omega_0:\sup_{t\le T}\|u^{N_i}(t)-u^{N_j}(t)\|<\frac{1}{2^n}\ \text{ for all }\ i,j\ge n\right\}$$

and $A=\limsup A_n$. It is easy to find that $\mathbb{P}_0(A)=1$. Then, for each $\omega_0\in A$,

$$\lim_{i,j\to\infty}\sup_{t\le T}\|u^{N_i}(t,\omega_0)-u^{N_j}(t,\omega_0)\|=0.$$

Note that $(D_H[0,T],\sup_{t\le T}\|\cdot\|)$ is a complete non-separable metric space. Thus, for each $\omega_0\in A$, there exists $\varphi(\omega_0)\in D_H[0,T]$ such that

$$\lim_{j\to\infty}\sup_{t\le T}\|u^{N_j}(t,\omega_0)-\varphi(t,\omega_0)\|=0. \tag{3.17}$$

For ω_0 not in A, we define $\varphi(\omega_0)$ as any fixed point of $D_H[0,T]$. By the measurability of $\varphi(t,\cdot)$ for each $t\in[0,T]$ and the càdlàg property, we have that $\varphi:\Omega_0\to(D_H[0,T],d)$ is measurable. Finally, it is easy to see that

$$\mathbb{P}_0\left(\sup_{t\le T}\|u^N(t)-\varphi(t)\|>\varepsilon\right)\le\mathbb{P}_0\left(\sup_{t\le T}\|u^N(t)-u^{N_j}(t)\|>\frac{\varepsilon}{2}\right)+\mathbb{P}_0\left(\sup_{t\le T}\|u^{N_j}(t)-\varphi(t)\|>\frac{\varepsilon}{2}\right). \tag{3.18}$$

If we choose $N_j>N$ and N sufficiently large, then the right hand side of (3.18) can be made arbitrarily small. This means that $\sup_{t\le T}\|u^N(t)-\varphi(t)\|\to 0$ in probability as $N\to\infty$. □

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Lemma 3.7. *Let* φ *be the limit process from Lemma* 3.6. *Then for* $1 \le \beta \le 2q$

$$\sup_{t\le T}\mathbb{E}_0\langle\varphi^\beta(t),1\rangle \le C(T) < \infty. \tag{3.19}$$

Proof. By Lemma 3.6, there is a subsequence $\{N_j\}$ such that $u^{N_j}(t)\longrightarrow\varphi(t)$ in H almost surely for each $t\in[0,T]$. It follows that $(u^{N_j}(t,x))^\beta\longrightarrow\varphi^\beta(t,x)$ a.e. $x\in E$ almost surely. Therefore, by Tonelli's theorem and Fatou's lemma, we find that

$$\begin{aligned}\mathbb{E}_0\langle\varphi^\beta(t),1\rangle &= \mathbb{E}_0\int_E\varphi^\beta(t,x)\rho(x)dx \le \int_E \liminf_{j\to\infty}\mathbb{E}_0\left[(u^{N_j}(t,x))^\beta\right]\rho(x)dx\\ &\le L_1L_2\sup_N\sup_{t\le T}\|\mathbb{E}_0(u^N(t))^\beta\|_\infty \le C(T)<\infty. \qquad\square\end{aligned}$$

Finally, we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. We have already proved that $\sup_{t\le T}\|u^N(t)-\varphi(t)\|\longrightarrow 0$ in probability in Lemma 3.6. Thus, we only need to show that $\varphi(t)$ is a unique mild solution to (1.4). By (2.5), we have with $\varphi(0)=u_0$,

$$\begin{aligned}\varphi(t) &= T(t)\varphi(0)+\int_0^t T(t-s)R(\varphi(s))ds+\int_0^t T(t-s)d\Theta(s)\\ &\quad+\sum_{i=1}^4\varepsilon_N^i(t),\end{aligned}\tag{3.20}$$

where

$$\varepsilon_N^1(t)=\varphi(t)-\int_0^t T(t-s)d\Theta(s)-\left(u^N(t)-\int_0^t T^N(t-s)d\Theta^N(s)\right),$$

$$\varepsilon_N^2(t)=(T^N(t)u^N(0)-T(t)\varphi(0))+Y^N(t),$$

$$\varepsilon_N^3(t)=\int_0^t T^N(t-s)R(u^N(s))ds-\int_0^t T(t-s)R(u^N(s))ds,$$

and

$$\varepsilon_N^4(t)=\int_0^t T(t-s)\left[R(u^N(s))-R(\varphi(s))\right]ds.$$

Whereas $\sup_{t\le T}\|\varepsilon_N^1(t)\|\longrightarrow 0$ in probability $\mathbb{P}_0$ by Lemmas 3.5 and 3.6,

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$\sup_{t\le T}\|\varepsilon_N^3(t)\|\to 0$ in probability by (3.9). By Trotter-Kato theorem and Lemma 3.4, we have that $\sup_{t\le T}\|\varepsilon_N^2(t)\|\to 0$ in probability $\mathbb{P}_0$. For $\varepsilon_N^4(t)$, we have by the argument in (3.10) and (3.13) that

$$\begin{aligned}\|\varepsilon_N^4(t)\| &\le \int_0^t \|T(t-s)\big[R(u^N(s)) - R(\varphi(s))\big]\|ds\\ &\le C(T)\sup_{t\le T}\|u^N(s)-\varphi(s)\|\cdot\int_0^t (t-s)^{-\frac{1}{2}}g(s)ds\\ &\le C(T)\sup_{t\le T}\|u^N(s)-\varphi(s)\|\left[\frac{(q+1)^2}{2q}\cdot T^{\frac{1}{q+1}} + \frac{q-1}{2q}\int_0^T g^{\frac{2q}{q-1}}(s)ds\right],\end{aligned}$$

where

$$g(s) = \left\{\int_E (1+|u^N(s,y)|^{q-1} + |\varphi(s,y)|^{q-1})^2\rho(y)dy\right\}^{\frac{1}{2}}.$$

By Lemmas 3.3 and 3.7, we easily find that $\mathbb{E}_0\int_0^T g^{\frac{2q}{q-1}}(s)ds < \infty$. Thus, by Lemma 3.6, it follows that $\sup_{t\le T}\|\varepsilon_N^4(t)\|\to 0$ in probability $\mathbb{P}_0$. Now, from (3.20), we find that $\varphi(t)$ is a mild solution of (1.4). The last task is to prove the uniqueness. If R is Lipschitz, then the uniqueness follows from standard arguments. For the non-Lipschitz case, we define for each $n\in\mathbb{N}$

$$R_n(x) = \begin{cases} R(x) & \text{if } |x|\le n,\\ R\left(\frac{nx}{|x|}\right) & \text{otherwise.}\end{cases}$$

Then, R_n is Lipschitz by (1.3). Let us denote by u^n the solution of (1.4) with R_n instead of R. Let $\tau_n = \inf\{t : \|u^n(t)\|_\infty \le n\}$. Then, $\{\tau_n\}$ is a non-decreasing sequence of stopping times and $u^{n+1}(t) = u^n(t), \forall t\le\tau_n$. Let $\tau = \sup_n\tau_n$ and $u(t) = u^n(t), \forall t\le\tau_n$. Then, $u(t)$ is a unique solution to (1.4) up to time τ. We shall prove that $\tau = \infty\, a.s.$ For any $t>0$, we have by (1.2) some constant C independent of n such that

$$\begin{aligned}u^n(t,x) &= T(t)u(0,x) + \int_0^t T(t-s)R_n(u^n(s,x))ds + \int_0^t T(t-s)d\Theta(s,x)\\ &\le T(t)u(0,x) + \int_0^t T(t-s)R_n^+(u^n(s,x))ds + \int_0^t T(t-s)d\Theta(s,x)\\ &\le \|u(0)\|_\infty + Ct + C\int_0^t \|u^n(s)\|_\infty ds + \sum_{i=1}^r\sum_{j=1}^{N_i(t)}\|\theta_i\|_\infty A_i^j.\end{aligned}$$

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It follows that

$$\|u^n(t\wedge\tau_n)\|_\infty \le \|u(0)\|_\infty + Ct\wedge\tau_n + C\int_0^{t\wedge\tau_n}\|u^n(s)\|_\infty ds + \sum_{i=1}^{r}\sum_{j=1}^{N_i(t\wedge\tau_n)}\|\theta_i\|_\infty A_i^j,$$

and consequently,

$$\mathbb{E}_0\|u^n(t\wedge\tau_n)\|_\infty \le \mathbb{E}_0\|u(0)\|_\infty + Ct + C\int_0^t \mathbb{E}_0\|u^n(s\wedge\tau_n)\|_\infty ds + \sum\nolimits_{i=1}^{r}\|\theta_i\|_\infty \eta_i t\int_{\mathbb{R}_+} aF_i(da).$$

By Gronwall's inequality, we find

$$\mathbb{E}_0\|u^n(t\wedge\tau_n)\|_\infty \le C(t) < \infty.$$

It is easy to see that

$$\mathbb{P}_0(\tau\le t) \le \mathbb{P}_0(\tau_n \le t) \le \mathbb{P}_0\big(\|u^n(t\wedge\tau_n)\|_\infty \ge n\big) \le \frac{\mathbb{E}_0\|u^n(t\wedge\tau_n)\|_\infty}{n} \le \frac{C(t)}{n}$$

which tends to zero as $n\to\infty$. So, $\mathbb{P}_0(\tau\le t)=0, \forall t>0$, i.e. $\tau=\infty$, a.s. □

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