LONG-TIME LIMITS AND OCCUPATION TIMES FOR STABLE FLEMING-VIOT PROCESSES WITH DECAYING SAMPLING RATES

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Abstract. A class of Fleming-Viot processes with decaying sampling rates and α-stable motions that correspond to distributions with growing populations are introduced and analyzed. Almost sure long-time scaling limits for these processes are developed, addressing the question of long-time population distribution for growing populations. Asymptotics in higher orders are investigated. Convergence of particle location occupation and inhabitation time processes are also addressed and related by way of the historical process. The basic results and techniques allow general Feller motion/mutation and may apply to other measure-valued Markov processes.

1. Introduction

We consider an $M_1(\mathbb{R}^d)$-valued Fleming-Viot process $X = (X_t, t \geq 0)$ with mutation generator $-(-\Delta)^{\frac{\alpha}{2}}$ ($\alpha \in (0, 2]$) and sampling rate $1/\phi(t)$ at time t for some positive function ϕ defined on $\mathbb{R}_+ = [0, \infty)$. Such Fleming-Viot processes can be obtained by normalizing and conditioning the total mass of (possibly non-critical) Dawson-Watanabe processes to have total mass $\phi(t)$ for all time t. This was established by Etheridge and March [EM91] for $\phi \equiv 1$ and by Perkins [Per92] for general nonnegative function ϕ. Herein, we investigate the long-time asymptotic of such Fleming-Viot processes when $1/\phi$ satisfies an integrability condition at infinity. Examples of ϕ satisfying this integrability condition include $t \to e^{\beta t}$ for $\beta > 0$ and $t \to 1 + t^N$ for N sufficiently large.

To be more precise, we let $W = (W_t, t \geq 0; \mathbb{P}_m)$ be a Dawson-Watanabe process with motion generator $-(-\Delta)^{\frac{\alpha}{2}}$ on $\mathbb{R}^d$, linear growth β and critical branching rate $\eta > 0$ corresponding to the operator $-(-\Delta)^{\frac{\alpha}{2}} u + \beta u - \frac{\eta}{2} u^2$ on $\mathbb{R}^d$. Then, W is a measure-valued Markov process starting at a finite measure m such that

$$M_t^W(f) := W_t(f) - m(f) - \int_0^t W_s\left((-(-\Delta)^{\frac{\alpha}{2}} + \beta)f\right) \mathrm{d}s$$

is a continuous martingale with quadratic variation $\langle M^W(f)\rangle_t = \int_0^t W_s(\eta f^2)\,\mathrm{d}s$ for all $f \in C_b^2(\mathbb{R}^d)$. (The reader is referred to [Kyp14] for information about stable processes and to [Per95] as well as [BP01] for describing measure-valued processes as martingale problems.) W's mass growth is subcritical, critical or supercritical if $\beta < 0$, $\beta = 0$ or $\beta > 0$ respectively. The expected total mass is found to be $m(1)\,e^{\beta t}$ by substituting $f = 1$ into the above equation and taking expectations. Now, suppose a population is projected to grow according to a positive continuous function ϕ. Then, following Perkins [Per92], one finds that the corresponding Fleming-Viot process attained by taking the angular part of W and conditioning $W_t(1)$ to have total mass $\phi(t)$ at every time t yields an $M_1(\mathbb{R}^d)$-valued process

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$X = (X_t, t \geq 0)$, the (α, ϕ) Fleming-Viot superprocess, starting at $\mu = \frac{m}{m(E)} \in M_1(E)$ and satisfying the martingale problem: For each $f \in \mathcal{F}\left((-\Delta)^{\frac{\alpha}{2}}\right)$ (the domain of $-(-\Delta)^{\frac{\alpha}{2}}$), the process

$$M_t^X(f) := X_t(f) - \mu(f) - \int_0^t X_s(-(-\Delta)^{\frac{\alpha}{2}} f)\mathrm{d}s \tag{1.1}$$

is a continuous martingale with quadratic variation

$$\langle M^X(f)\rangle_t = \int_0^t \eta\phi(s)^{-1}\left[X_s\left(f^2\right) - X_s^2(f)\right]\mathrm{d}s. \tag{1.2}$$

The law of X is denoted by $\mathbb{P}_\mu^\phi$. Technically, to get X from W, we condition on $W_t(1)$ staying within ε of $\phi(t)$ up to T and then let $\varepsilon \to 0$ and $T \to \infty$. Also, it does not matter whether the original Dawson-Watanabe process is supercritical, critical or even subcritical as the resulting Fleming-Viot after normalizing by $W_\cdot(1)$ and conditioning so that $W_\cdot(1) = \phi$ (that is, after considering the angular part) satisfies the same martingale problem. If ϕ is increasing, X can be considered as a Fleming-Viot process that gives the population distribution for growing populations.

Properly normalized supercritical superprocesses have recently been shown (see e.g. Wang [Wan10], Kouritzin and Ren [KR14], Liu et. al. [LRS13], Eckhoff et. al. [EKW15] as well as the more detailed review in Section 2.7 of [Eng15]) to have almost sure long-time scaling limits (often called strong laws of large numbers), generalizing the pioneering branching Markov process work of Watanabe [Wat67] and Asmussen and Hering [AH76]. Traditionally, superprocesses with ergodic and transient motion models have been handled separately with different scalings in laws of large number results. However, while considering strong laws of large numbers for supercritical, (possibly) non-Markov Gaussian branching processes, Kouritzin et. al. [KLS18] showed that these two cases can be considered together. For α-stable Dawson-Watanabe processes, the result of [KR14] states that with probability one, as $t \to \infty$,

$$t^{\frac{d}{\alpha}}\frac{W_t}{W_t(1)} \hookrightarrow \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|\theta|^\alpha}\mathrm{d}\theta\, \lambda_d \quad \text{on } \left\{\lim_{t\to\infty} e^{-\beta t}W_t(1) > 0\right\},$$

where $\hookrightarrow$ denotes shallow convergence of measures and λ_d is the Lebesgue measure on $\mathbb{R}^d$. Here and in the sequel, we ease notation by reducing $\lambda_d(\mathrm{d}\theta)$ to just $\mathrm{d}\theta$ when appropriate. *Shallow* convergence is stronger than vague convergence yet still allows convergence to non-finite measures like Lebesgue measure. It is defined in [KR14] as

$$\nu_t \hookrightarrow \nu \quad \Longleftrightarrow \quad \nu_t(f) \to \nu(f),\ \forall \text{ continuous } f\ :\ \sup_{x\in\mathbb{R}^d}\left|e^{\epsilon|x|^2} f(x)\right| < \infty \text{ for some } \epsilon > 0.$$

1.1. **Statement of Main Results.** For notational simplicity, we will simply call an α-stable Fleming-Viot process with with sampling rate $1/\phi(t)$ an (α,ϕ)-FV process. As explained previously, (α,ϕ)-FV processes corresponds to $W_t/W_t(1)$ conditioned so that $W_t(1) = \phi(t)$ for all t. For supercritical Dawson-Watanabe processes, the total mass $W_t(1)$ has expected mean $m(1)e^{\beta t}$ for some $\beta > 0$. This suggests that if $\phi(t) = e^{\beta t}$, then we should have the almost-sure, shallow-topology, long-time limit

$$t^{\frac{d}{\alpha}} X_t \xrightarrow{t\to\infty} \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|\theta|^\alpha}\mathrm{d}\theta\, \lambda_d\,. \tag{1.3}$$

In fact, our first main result shows the above almost sure limit for a larger class of sampling functions ϕ.

Theorem 1.1. *Assume that μ satisfies*

$$\int_{\mathbb{R}^d} |x|^a \mu(\mathrm{d}x) < \infty \quad \textit{for some} \quad a > 0 \tag{1.4}$$

and ϕ is a positive function on $\mathbb{R}_+$ such that

$$\int_0^\infty s^{\frac{d}{\alpha}+1+\varepsilon_0} \frac{\mathrm{d}s}{\phi(s)} < \infty \quad \textit{for some} \quad \varepsilon_0 > 0\,. \tag{1.5}$$

Then, with $\mathbb{P}^\phi_\mu$-probability one, the shallow limit (1.3) holds.

In addition, if a test function $f \in C^2_c(\mathbb{R}^d)$ is fixed, all higher order asymptotics of $X_t(f)$ can be identified. To state our second main result, we prepare some notation. For each multi-index $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ and $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we let

$$|k| = k_1 + k_2 + \cdots + k_d\,, \quad k! = k_1!k_2!\cdots k_d!\,, \quad x^k = x_1^{k_1}x_2^{k_2}\cdots x_d^{k_d}\,,$$

and define the constant $\vartheta^k_{d,\alpha}$ and the σ-finite signed measure λ^k_d on $\mathbb{R}^d$ respectively by

$$\vartheta^k_{d,\alpha} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|\theta|^\alpha}\theta^k \mathrm{d}\theta \quad \text{and} \quad \lambda^k_d(\mathrm{d}y) = \frac{1}{k!}y^k \mathrm{d}y\,. \tag{1.6}$$

Obviously, λ^0_d is the Lebesgue measure λ_d.

Theorem 1.2. *Let N be a non-negative integer. Assume that μ satisfies (1.4) and ϕ satisfies*

$$\int_0^\infty s^{\frac{2N+d}{\alpha}+1+\varepsilon_0} \frac{\mathrm{d}s}{\phi(s)} < \infty \quad \textit{for some} \quad \varepsilon_0 > 0\,. \tag{1.7}$$

Let f be a function in $b\mathcal{E}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} |f(x)||x|^N \mathrm{d}x < \infty \tag{1.8}$$

with its Fourier transform $\hat{f}$ satisfying

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)||\xi|^\alpha \mathrm{d}\xi < \infty\,. \tag{1.9}$$

Then, $\mathbb{P}^\phi_\mu$-almost surely

$$\lim_{t\to\infty} t^{\frac{N+d}{\alpha}} \left| X_t(f) - \sum_{\substack{k\in\mathbb{N}^d:|k|\le N \\ |k| \text{ is even}}} \frac{(-1)^{\frac{|k|}{2}} t^{-\frac{d+|k|}{\alpha}}}{(2\pi)^d k!} \int_{\mathbb{R}^d} f(y) y^k \mathrm{d}y \int_{\mathbb{R}^d} e^{-|\theta|^\alpha}\theta^k \mathrm{d}\theta \right| = 0\,. \tag{1.10}$$

Written another way, we have $\mathbb{P}^\phi_\mu$-almost surely

$$t^{\frac{d}{\alpha}} X_t(f) = \sum_{\substack{k\in\mathbb{N}^d:|k|\le N \\ |k| \text{ is even}}} (-1)^{\frac{|k|}{2}} t^{-\frac{|k|}{\alpha}} \vartheta^k_{d,\alpha}\lambda^k_d(f) + o(t^{-\frac{N}{\alpha}}) \quad \textit{as} \quad t\to\infty\,. \tag{1.11}$$

In the above statement, $b\mathcal{E}(\mathbb{R}^d)$ denotes the space of real-valued bounded measurable functions on $\mathbb{R}^d$. Theorem 1.2 gives flexible rate-of-convergence information on the convergence of $t^{\frac{d}{\alpha}}X_t$ to scaled Lebesgue measure, depending upon the conditions assumed and the test function used. Some of our results herein were later obtained for Dawson-Watanabe processes [Lê19].

The long-time behaviour of constant rate Fleming-Viot processes has been well studied. When $\phi \equiv 1$ and $\alpha = 2$, the long-time behavior of X_t is discussed in Dawson and Hochberg [DH82]. They show that as time gets large, the measure-valued process $(X_t, t \geq 0)$ concentrates within a random (but stationary) distance from a Brownian motion. The possible long-time distributional limits of even multiple (critical) interacting Fleming-Viot processes are well-known. For example, the stationary distributions were obtained in Shiga [Shi80a, Shi80b], Shiga and Uchiyama [SU86] for the two allele case and in Dawson et. al. [DGV95], Dawson and Greven [DG99] for the general case. These results characterize the possible distributional limits of interacting Fleming-Viot processes. However, Fleming-Viot processes that admit almost sure scaling limits do not appear to have been considered.

The proofs of Theorems 1.1 and 1.2 are presented in Subsection 3.2. Our method, discussed in Subsection 2.2, improves Asmussen and Hering's technique in [AH76] and converts it to the language of martingales and stochastic integration. This formulation provides a clear picture, which may be applicable for superprocesses of both Dawson-Watanabe type and Fleming-Viot type with general Feller motions or mutation processes.

To illustrate our method further, we consider long-time limits of the occupation time $Y_t = \int_0^t X_s \, \mathrm{d}s$ and the inhabitation time, defined for bounded f as $Z_t(f) = \mathbb{X}_t(\ell_f)$. Here,

$$\ell_f(r, y) = \int_0^r f(y_s)\mathrm{d}s \quad \forall r \geq 0,\ y \in D(\mathbb{R}^d),\ f \in b\mathcal{E}(\mathbb{R}^d), \tag{1.12}$$

$D(\mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued cadlag paths equipped with Skorohod J_1 topology and $\mathbb{X}$ is the (α, ϕ) Fleming-Viot historical process satisfying the martingale problem:

$$\mathbb{M}_t(h) = \mathbb{X}_t(h) - \delta_0 \times \mu^*(h) - \int_0^t \mathbb{X}_s(\mathbb{A}h)\mathrm{d}s \tag{1.13}$$

is a continuous martingale starting at 0 such that

$$\langle \mathbb{M}(h) \rangle_t = \int_0^t (\mathbb{X}_s(h^2) - \mathbb{X}_s(h)^2)\frac{\mathrm{d}s}{\phi(s)} \tag{1.14}$$

for all h in the domain of bounded functions $\mathcal{D}(\mathbb{A})$ for the historical generator $\mathbb{A}$. (We define $\mathbb{X}$ precisely and relate it to our (α, ϕ)-FV process below. μ^* will be a variant of μ, defined on the historical path space.) However, ℓ_f with $f = 1_{\mathcal{O}}$ (for an open $\mathcal{O}$) is not bounded, hence $\ell_{1_{\mathcal{O}}} \notin \mathcal{D}(\mathbb{A})$. Still, the martingale problem (1.13,1.14) does hold for such *natural* $\ell = \ell_f$ since

$$\mathbb{A}\ell_f = \text{bp-}\lim_{t\to\infty} \mathbb{A}\ell_f^t \tag{1.15}$$

exists and

$$\mathbb{X}_t(\ell_f) = \mathbb{X}_t(\ell_f^u) \quad \text{and} \quad \mathbb{X}_t(\mathbb{A}\ell_f) = \mathbb{X}_t(\mathbb{A}\ell_f^u)\ \forall u \geq t, \tag{1.16}$$

where ℓ_f^u is the bounded variant of ℓ_f. Namely,

$$\ell_f^u(r, y) = \int_0^{r\wedge u} f(y_s)\mathrm{d}s \quad \forall r \geq 0,\ y \in D(\mathbb{R}^d),\ f \in b\mathcal{E}(\mathbb{R}^d),$$

for each fixed u that we will show is in $\mathcal{D}(\mathbb{A})$. (Herein, bp-lim denotes the bounded, pointwise limit.) The definition of $\mathbb{A}\ell_f$ through the limit (1.15) is established in Lemma 2.13 (to follow). To show (1.16), we let $y^t = y(\cdot \wedge t) \in D(\mathbb{R}^d)$ for $t > 0$ and $y \in D(\mathbb{R}^d)$, consider $\mathbb{E} = \{(r, y^r) : r \geq 0, y \in D(\mathbb{R}^d)\}$ as a topological subspace of $\mathbb{R}_+ \times D(\mathbb{R}^d)$, and show that $\mathbb{X}_t$ is supported on

$$\mathbb{E}^t = \{(r, y) \in \mathbb{E} : r = t\}, \tag{1.17}$$

which we do below (see Proposition 2.10 and Remark 2.11). Having observed (1.15) and (1.16), we can define $\mathbb{M}_t(\ell_f)$ by substituting $h = \ell_f^u$ in (1.13)

$$\mathbb{M}_t(\ell_f) := \mathbb{X}_t(\ell_f^u) - \delta_0 \times \mu^*(\ell_f^u) - \int_0^t \mathbb{X}_s(\mathbb{A}\ell_f^u)\mathrm{d}s \quad \forall u \geq t. \tag{1.18}$$

It also follows from (1.16) that

$$\langle \mathbb{M}(\ell_f)\rangle_t = \langle \mathbb{M}(\ell_f^u)\rangle_t \quad \forall u \geq t\,. \tag{1.19}$$

Remark 1.3. The occupation time $Y_t(1_{\mathcal{O}})$ counts the time in $\mathcal{O}$ of dead lineages i.e. times of particles that are not ancestors of living particles, while the inhabitation time $Z_t(1_{\mathcal{O}})$ counts the time of common ancestors multiple times.

Our third main result, connects these two time processes.

Theorem 1.4. *Defining $\ell_f, \mathbb{M}(\ell_f)$ respectively as in (1.12), (1.18) and for all $t > 0$, $f \in b\mathcal{E}(\mathbb{R}^d)$,*

$$\gamma_d(t) = \begin{cases} t^{1-\frac{d}{\alpha}} & \text{if } d < \alpha \\ \ln(t \vee 1) & \text{if } d = \alpha \\ 1 & \text{if } d > \alpha \end{cases} \quad \text{and} \quad \mathcal{N}_d(f) = \begin{cases} \|f\|_{L^1(\mathbb{R}^d)} & \text{if } d < \alpha \\ \|f\|_{L^1(\mathbb{R}^d)} + \int_{\mathbb{R}^d} |\hat{f}(\theta)||\theta|^{-\alpha}\mathrm{d}\theta & \text{if } d = \alpha \\ \int_{\mathbb{R}^d} |\hat{f}(\theta)||\theta|^{-\alpha}\mathrm{d}\theta & \text{if } d > \alpha \end{cases}. \tag{1.20}$$

One has that:

a: $\mathbb{M}_t(\ell_f) = Z_t(f) - Y_t(f)$ *is a* $\mathcal{F}^X_{t+}$*-martingale for* $f \in b\mathcal{E}(\mathbb{R}^d)$.

b: $\mathbb{M}_\infty(\ell_f) := \lim_{t\to\infty} \mathbb{M}_t(\ell_f)$ *exists a.s. provided that* $\mathcal{N}_d(f) < \infty$ *and* $\int_0^\infty \gamma_d^2(s)\frac{\mathrm{d}s}{\phi(s)} < \infty$.

c: $Z_\infty(f) := \lim_{t\to\infty} Z_t(f), Y_\infty(f) := \lim_{t\to\infty} Y_t(f)$ *both exist a.s. in* $\mathbb{R}$ *and* $\mathbb{M}_\infty(\ell_f) = Z_\infty(f) - Y_\infty(f)$ *a.s. provided that* $\mathbb{M}_\infty(\ell_f)$ *exists,* $d > \alpha$ *and* $\mathcal{N}_d(f) < \infty$.

The proof of parts a, b, and c follow respectively from Proposition 2.14, Proposition 4.4 (i), and the proof of the high dimensional case of Theorem 1.5 as well as Proposition 4.4.

Notice that the martingale $\{\mathbb{M}_t(\ell_f),\, t \geq 0\}$ in Theorem 1.4 is with respect to the right continuous filtration of the the (non-historical) (α, ϕ)-FV process, which is possible when the (α, ϕ)-FV process is defined from the (α, ϕ)-historical process through (2.37) below.

Hereafter, the cases $d < \alpha$, $d = \alpha$ and $d > \alpha$ are respectively called *low dimension, critical dimension* and *high dimension*. One can see from Theorem 1.4 that particle time behaviour is dimensionally dependent as one might expect from the transition from recurrent to transient particle motion.

While in the context of Dawson-Watanabe processes, long-time asymptotics of occupation time processes have been studied extensively starting from the work of Iscoe [Isc86], the corresponding problem for Fleming-Viot processes seems sparse in the literature. As is known in the context of Dawson-Watanabe processes, the limiting behavior of occupation times depends on the relation between d and α. Our fourth main result shows the limiting behaviour of occupation and inhabitation times for (α,ϕ)-Fleming-Viot processes will also be dimensionally dependent. Define

$$\varkappa_d(\alpha) = \begin{cases} (2\pi)^{-d}\frac{\alpha}{\alpha-d}\int_{\mathbb{R}^d} e^{-|\theta|^\alpha}\mathrm{d}\theta & \text{if } d<\alpha \\ (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-|\theta|^\alpha}\mathrm{d}\theta & \text{if } d=\alpha \end{cases} . \tag{1.21}$$

Theorem 1.5. *Suppose ϕ satisfies*

$$\int_1^\infty \frac{\mathrm{d}s}{\phi(s)} < \infty . \tag{1.22}$$

Then, in low and critical dimensions ($d \le \alpha$), with $\mathbb{P}^\phi_\mu$-probability one, scaled occupation and inhabitation times $\frac{Y_t}{\gamma_d(t)}$ and $\frac{Z_t}{\gamma_d(t)}$ both converge shallowly to $\varkappa_d(\alpha)\lambda_d$. While in high dimensions ($d>\alpha$), with $\mathbb{P}^\phi_\mu$-probability one, Y_t and Z_t converge shallowly to some random measures.

This result relies on Proposition 4.1 (to follow) and is proved at the end of Subsection 4.2.

1.2. **Explanation of Sampling Rate Assumptions.** We would like to thank an anonymous referee for inviting us to speculate around our sampling rate assumptions. Condition (1.5) can be considered heuristically in two ways: Kouritzin and Ren [KR14] showed the (shallow) a.s. convergence of $t^{\frac{d}{\alpha}}\frac{W_t}{e^{\beta t}}$ when W was a superstable process with growth factor $\beta>0$. However, it is well known (see [AN04]) that $e^{-\beta t}W_t(1)\to F$ a.s. for some non-trivial random variable F in this case and the limits of $t^{\frac{d}{\alpha}}\frac{W_t}{e^{\beta t}}$ and $t^{\frac{d}{\alpha}}\frac{W_t}{W_t(1)}$ will only differ by this factor F (on the set where $F>0$). Next, conditioning on $W_t(1)$ to be close to $e^{\beta t}$ might not have a huge effect since their ratio converges. Finally, Perkins [Per92]'s argument on $\frac{W_t}{W_t(1)}$ conditioned so $W_t(1)$ is $e^{\beta t}$ has martingale problem (1.1,1.2) with $\phi(t)=e^{\beta t}$. In this way, Theorem 1.1 loosely generalizes Kouritzin and Ren [KR14] from $\phi(t)=e^{\beta t}$ to any ϕ satisfying (1.5). Secondly, the factor $t^{\frac{d}{\alpha}}$ on the left of (1.3) is what is needed for a non-trivial limit in Theorem 1.1 but this factor blows up X_t and its noise M_t. To have an almost sure limit the noise has to die out fast enough through the $\phi(s)^{-1}$ factor in (1.2). We can think of the $s^{\frac{d}{\alpha}}$ factor within the integral of (1.5) as compensation for blowing X_t up by $t^{\frac{d}{\alpha}}$ and the integral without this factor as a condition on the noise of X itself. The full force of (1.5) only comes to bear in Proposition 2.8 through conditions (2.18,2.23). In the proofs of Theorems 1.1 and 1.2, we will decompose $X_t(f)$ as

$$X_{\rho(t_n)}(L_{t_n-\rho(t_n)}f) + [X_{t_n}(T_{t_{n+1}-t_n}f) - X_{\rho(t_n)}(L_{t_n-\rho(t_n)}f)] + [X_t(f) - X_{t_n}(T_{t_{n+1}-t_n}f)],$$

where T_t is the α-stable semigroup, L_t is an N^{th} order approximation of T_t and ρ is a sublinear function. It follows from Proposition 3.2 that $X_{\rho(t_n)}(L_{t_n-\rho(t_n)}f)$ satisfies the stated scaling limits using Fourier analysis under a lesser condition on ϕ so the other two terms can be thought of as errors. The first error term, handled in (2.28), puts constraints on an auxiliary sequence $\{c_n\}$ while the second error term, handled in (2.29), forces a constraint on ϕ depending upon the $\{c_n\}$. The two constraints are then solved in (3.20) under (1.5). It would be interesting to know lesser conditions on ϕ under which one has convergence in probability but not necessarily almost sure convergence.

1.3. **Article Outline.** Section 2 discusses fundamental results of Fleming-Viot processes. Section 3 focuses on the long-time limit of α-stable Fleming-Viot processes. In Section 4, long-time asymptotic of the occupation time process as well as the related inhabitation time process of an α-stable Fleming-Viot process is investigated.

2. Fleming-Viot processes

We use $\nu(f)$ and $\langle f,\nu\rangle$ to denote $\int f\mathrm{d}\nu$ for a measure ν and integrable function f. Let $(E,\mathcal{E}(E))$ be a Polish space with its Borel σ-algebra $\mathcal{E}(E)$ and $((\xi_t)_{t\geq 0},(P_x)_{x\in E})$ be an E-valued Borel strong Markov process with sample paths in $D(E)$. Hereafter, $D(E)$ is the space of cadlag paths from $\mathbb{R}_+ := [0,\infty)$ to E equipped with the Skorohod J_1 topology. Define the semigroup on $b\mathcal{E}(E)$ (the space of real-valued bounded measurable functions on E) by

$$T_tf(x) = P_xf(\xi_t)\,.$$

and assume that T_t maps $C_b(E)$ (the space of real continuous bounded functions on E) to itself. The right-continuity of ξ implies bp-$\lim_{t\to 0}T_tf = f$ for every $f\in C_b(E)$. We also assume ξ is conservative, i.e. $T_t\mathbf{1}=\mathbf{1}$. Define

$$Af = \text{bp-}\lim_{t\to 0}\frac{T_tf-f}{t}$$

when the limit exists. The domain $\mathcal{D}(A)$ of A contains all functions in $b\mathcal{E}(E)$ such that the above limit exists, including the constant function $\mathbf{1}$ for which $A\mathbf{1}=0$. $(A,\mathcal{D}(A))$ is the so-called weak generator of ξ. It is known ([Per02, Corollary II.2.3]) that $\mathcal{D}(A)$ is bp-dense in $b\mathcal{E}(E)$. We adopt the following standard notation:

- $M_F(E)$, $M_1(E)$ denote the spaces of finite, respectively probability measures.
- $(\Omega_F,\mathcal{F})$, $(\Omega,\mathcal{G})$ are the sample spaces of (compact-open) continuous mappings $(C([0,\infty),M_F(E))$ respectively $C([0,\infty),M_1(E))$ with their respective Borel σ-fields.
- $W_t(\omega)=\omega_t$, $X_t=\omega_t$ denote the coordinate mappings on Ω_F and Ω respectively.
- $\mathcal{F}^0_t=\sigma(W_s : s\leq t)$, $\mathcal{F}_t=\mathcal{F}^0_{t+}$; $\mathcal{G}^0_t=\sigma(X_s : s\leq t)$, $\mathcal{G}_t=\mathcal{G}^0_{t+}$.

2.1. **Martingale problems.** Let $E=\mathbb{R}^d$. For each $\beta\geq 0$, $\eta>0$ and $m\in M_F(\mathbb{R}^d)$, there is a unique probability $\mathbb{P}_m$ on $(\Omega_F,\mathcal{G})$ such that for all $f\in\mathcal{D}(A)$

$$M^W_t(f) = W_t(f) - m(f) - \int_0^t W_s(Af+\beta f)\mathrm{d}s \tag{2.1}$$

is a continuous $(\mathcal{F}_t)$-martingale starting at 0 with quadratic variation

$$\langle M^W(f)\rangle_t = \eta\int_0^t W_s(f^2)\mathrm{d}s\,. \tag{2.2}$$

$\mathbb{P}_m$ is the law of the critical or supercritical A-Dawson-Watanabe process with drift β and branching variance η.

Remark 2.1. There is substantial theory on the existence, uniqueness, path properties and high density limits for Dawson-Watanabe superprocesses under conditions far more general than required here. However, the martingale problem and the connection to finite populations motivate the study of long-time behaviour of our model. Hence, we will expand upon Example 10.1.2.2 in [Daw93] and remind the reader of some basic points in the case $E=\mathbb{R}^d$ while neglecting details similar to those handled in [Daw93] and [EK86]. It follows from the proofs of Theorems 9.4.2 and 9.4.3 of [EK86] that the local martingale problem:

$$\begin{cases} \exp(-W_t(f)) + \int_0^t \exp(-W_s(f))\, W_s\left(Af + \beta f - \frac{\eta}{2} f^2\right)\, \mathrm{d}s \\ \text{is a local martingale for all non-negative } f \in \mathcal{D}(A)\,, \end{cases} \tag{2.3}$$

is well posed. (Technically, one could first follow [EK86, Theorems 9.4.2 and 9.4.3] to get well-posedness of the martingale problem. Then, for a potential local martingale solution W, we realize that $M_t^\lambda \doteq \exp(-\lambda W_t(1)) + \int_0^t \exp(-\lambda W_s(1)) \left(\lambda\beta - \frac{\lambda^2\eta}{2}\right) W_s(1) ds$ is actually a martingale and use the argument on p. 403 of [EK86] to show $\mathbb{P}_m[W_t(1)] \le m(1)\exp(\beta t)$. From there, it follows that any solution to the local martingale problem is a solution to the martingale problem.) As part of justifying the use of these proofs, we note that (2.3) is the high density limit of finite branching population models. For example, using the notation of [EK86] and letting $c \in \left(0, \frac{1}{\eta}\right)$, we find that the population starting with n individuals, undergoing independent A-motions with location-independent lifetime rates $\alpha_n = \frac{n}{c}$ and having offspring probability generating function $\varphi_n(z) = c\left(\frac{\eta}{2} - \frac{\beta}{n}\right) + \left(\frac{c\beta}{n} + 1 - c\eta\right) z + \frac{c\eta}{2} z^2$ is a well defined model for large enough n and these populations converge (pathwise) to the solution of (2.3) as $n \to \infty$. Next, it follows from Corollary 2.3.3 of [EK86] that the local martingale problem in (2.3) is equivalent to the local martingale problem:

$$\begin{cases} \exp\left(-W_t(f) + \int_0^t W_s\left(Af + \beta f - \frac{\eta}{2} f^2\right)\, \mathrm{d}s\right) \\ \text{is a local martingale for all non-negative } f \in \mathcal{D}(A)\,. \end{cases} \tag{2.4}$$

However, to go further, we must ensure that W is continuous. This continuity is shown by following Theorem 4.7.2 of [Daw93] for the case $\beta = 0$ and the case $\beta \neq 0$ is converted to the case $\beta = 0$ by Dawson's Girsanov theorem (Theorem 7.2.2 and Lemma 10.1.2.1 of [Daw93]) with $r(\mu, y) = \beta$ and $Q(\mu; \mathrm{d}x, \mathrm{d}y) = \delta_x(\mathrm{d}y)\mu(\mathrm{d}x)$. (This theorem is stated in terms of a larger domain but we already have uniqueness for the smaller domain in (2.3).) Now, by this continuity, the local martingale problem (2.4) is equivalent to the local martingale problem (2.1-2.2) by e.g. Theorem 6.2 [CW90]. Finally, we show that each $M_t^W(f)$ in (2.1) is a martingale for any continuous local martingale problem solution W by showing $\mathbb{P}_m[W_t(1)] \le m(1)\exp(\beta t)$ through stopping and Gronwall's inequality.

The process $\{W_t(1)\}_{t \ge 0}$ describes the evolution of total mass with life time

$$t_W = \inf\{t > 0 : W_t(1) = 0\}\,.$$

Even in the supercritical regime ($\beta > 0$), t_W is finite with positive probability. Using the martingale structure of W, we can describe the evolution of the normalized process $\overline{W} = \{\frac{W_t}{W_t(1)},\ 0 \le t < t_W\}$ as in the following result.

Lemma 2.2. *Assume that $m \neq 0$. Let $\overline{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(W_s(1) : s \ge 0)$ and $\mu = m/m(1)$. For every $f \in \mathcal{D}(A)$*

$$M_t^{\overline{W}}(f) = \overline{W}_t(f) - \mu(f) - \int_0^t \mathbf{1}(s < t_W)\overline{W}_s(Af)\mathrm{d}s\,, \quad t \ge 0$$

is a continuous $(\overline{\mathcal{F}}_t)-$*martingale starting at 0 such that*

$$\langle M_t^{\overline{W}}(f)\rangle_t = \eta\int_0^t \mathbf{1}(s<t_W)(\overline{W}_s(f^2)-\overline{W}_s(f)^2)\frac{\mathrm{d}s}{W_s(1)} \quad \mathbb{P}_m\text{−a.s.}$$

Proof. The case when $\beta=0$ is proved in Perkins [Per92] using Itô formula. If $\beta>0$, the proof follows analogously, we omit the details. □

Let C_+ be the space of continuous functions $\phi:[0,\infty)\to[0,\infty)$ such that $\phi(t)>0$ if $t\in[0,t_\phi)$ and $\phi(t)=0$ if $t\geq t_\phi$ for some $t_\phi\in(0,\infty]$. Let $Q_{m(1)}$ be the law of $W_\cdot(1)$, i.e.

$$\mathbb{P}_m(W_\cdot(1)\in B)=Q_{m(1)}(B)\,.$$

Theorem 2.3 (Perkins [Per92])**.** *For every* $\phi\in C_+$ *and* $\mu\in M_1(E)$*, there is a unique probability* $\mathbb{P}_\mu^\phi$ *on* $(\Omega,\mathcal{G})$ *such that under* $\mathbb{P}_\mu^\phi$*, for all* $f\in\mathcal{D}(A)$*,*

$$M_t^X(f)=X_t(f)-\mu(f)-\int_0^t X_s(Af)\mathrm{d}s\,,\quad t<t_\phi \tag{2.5}$$

is a continuous $(\mathcal{G}_t)$*-martingale starting at* 0 *and such that*

$$\langle M^X(f)\rangle_t=\eta\int_0^t (X_s(f^2)-X_s(f)^2)\phi(s)^{-1}\mathrm{d}s \quad \forall t<t_\phi \tag{2.6}$$

and $X_t=X_{t_\phi}$ *for all* $t\geq t_\phi$*.*

Remark 2.4. We will use this theorem in Polish spaces $E=\mathbb{R}^d$ and $E=\mathbb{E}$, defined just above (1.17). It is obtained under the assumption that E is locally compact in Theorem 2 (a) of [Per92]. The proof in [Per92] uses detailed arguments, state augmentation and worthy martingale measure representation to change the speed of the sampling martingale relative to the particles motions. This martingale time change argument is then used to infer the existence and uniqueness of $\mathbb{P}_\mu^\phi$ from that of the law of the classical Fleming-Viot process, i.e. $\mathbb{P}_\mu^{\mathbf{1}}$, which was only known on locally compact spaces. This is the only place in [Per92] where locally compactness was used. The existence and uniqueness of Fleming-Viot processes on Polish spaces have been since obtained by Donnelly and Kurtz in [DK96, DK99] based upon earlier ideas of Dawson and Hochberg [DH82]. Therefore, Perkins' argument carries through in the setting of Polish spaces.

The connection between Dawson-Watanabe processes and Fleming-Viot processes with time-varying sampling rates ϕ is as follows.

Theorem 2.5 ([Per92, Theorem 3])**.** *For every* $m\in M_F(E)\setminus\{0\}$*, set* $\mu=m/m(1)$*. For* $Q_{m(1)}$*−a.a.* ϕ*, we have*

$$\mathbb{P}_m\left(\frac{W}{W_\cdot(1)}\in A\middle|W_\cdot(1)=\phi\right)=\mathbb{P}_\mu^\phi(A)\quad\forall A\in\mathcal{G}\,.$$

[Per92, Theorem 3] is in the setting of locally compact E, which is fine for our purposes as we only use this theorem in the case of $E=\mathbb{R}^d$ to motivate our work.

Corollaries 4 and 5 of [Per92] further establish that for every $\phi\in C_+$, $\mathbb{P}_\mu^\phi$ is indeed the regular conditional law $\mathbb{P}_m(\frac{W}{W_\cdot(1)}\in\cdot|W(1)=\phi)$. Without loss of generality, we assume $\eta=1$ hereafter.

2.2. **Long term asymptotics.** Let $E=\mathbb{R}^d$, $\mu\in M_1(E)$ and $\phi\in C_+$ with $t_\phi=\infty$. Let $\mathbb{P}^\phi_\mu$ be the probability law introduced in Theorem 2.3. Recall $\{T_t\}_{t\ge0}$ is the semigroup generated by A. In practice, the semigroup T_t usually satisfies some asymptotical property. One possibility is the following: for each $t>0$, there exist a deterministic positive scaling $c(t)$ and an operator L_t such that

$$\lim_{t\to\infty} c(t)\|T_tf-L_tf\|_{L^\infty(E)}=0\,. \tag{2.7}$$

(2.7) becomes trivial if we choose $L_tf=T_tf$. However, we can choose a different L_tf to our advantage. When T_t is the symmetric stable semigroup considered in Section 3, L_t can be chosen as the projection onto a finite dimensional vector space, whose basis are the partial derivatives of the kernel density $p_t(x)$ (see (3.4) and (3.9) to follow).

In the current section, we present a general procedure to study long term asymptotic for $X_t(f)$ given a test function $f\in b\mathcal{E}(E)$. The method consists of two steps. One first shows that $X_t(f)$ and $X_{\rho(t)}(T_{t-\rho(t)}f)$ have the same asymptotic as $t\to\infty$. Hereafter, $\rho:\mathbb{R}_+\to\mathbb{R}_+$ is an increasing sub-linear function, that is ρ satisfies

$$\lim_{t\to\infty}\frac{\rho(t)}{t}=0\,. \tag{2.8}$$

This step requires a certain integrability condition of the function $1/\phi$ over $\mathbb{R}_+$ (see Proposition 2.7 below). Next, by (2.7), one can further deduce the asymptotic of $X_{\rho(t)}(T_{t-\rho(t)}f)$ from that of $X_{\rho(t)}(L_{t-\rho(t)}f)$. In this second step, having chosen L_t in our favor, we find the asymptotic of $X_{\rho(t)}(L_{t-\rho(t)}f)$ directly by other tools. In Section 3, we explain how the procedure can be applied to study super stable processes and their occupation times.

In the context of Dawson-Watanabe processes with supercritical branching mechanisms, this method goes back to [AH76] and has been extended to treat superprocesses with more general Markovian motions (see for instance [LRS13, CS07]). Until recently, it seemed that Asmussen and Hering's method required a certain spectral gap assumption on the semigroup T_t. However, in [KLS18], the same procedure is applied for supercritical branching Gaussian processes. The treatment presented here contains some simplifications and improvements.

Let us now develop a stochastic integration framework which is an essential tool in our approach. Letting $M^X_t(U)=M^X_t(1_U)$, we note that for every $U,V\in\mathcal{E}(E)$,

$$\langle M^X(U),M^X(V)\rangle_t\le\int_0^t X_s(\mathbf{1}_U\mathbf{1}_V)\frac{\mathrm{d}s}{\phi(s)}\,.$$

In particular, $(M^X_t)_{t\ge0}$ is a worthy martingale measure (see [Wal86, Chapter 2]). For every adapted process $\{g(r,z)=g_r(z):r\ge0,z\in E\}$ satisfying

$$\mathbb{P}^\phi_\mu\int_0^\infty X_r(g_r^2)\frac{\mathrm{d}r}{\phi(r)}<\infty\,,$$

one can construct the stochastic integral $\int_0^\infty\int_E g(r,z)\mathrm{d}M^X(r,z)$ such that

$$\mathbb{P}^\phi_\mu\left(\int_0^\infty\int_E g(r,z)\mathrm{d}M^X(r,z)\right)^2=\mathbb{P}^\phi_\mu\int_0^\infty(X_r(g_r^2)-X_r(g_r)^2)\frac{\mathrm{d}r}{\phi(r)}\,. \tag{2.9}$$

We refer to [Wal86, Chapter 2] for a detailed construction.

This worthy martingale measure representation allows us to extend the martingale problem (2.5,2.6) by an integration by parts argument. In particular, for continuously differentiable f_t in t that satisfies $f_t \in \mathcal{D}(A)$ for all t and $\mathbb{P}^{\phi}_{\mu}\int_0^\infty X_r(f_r^2)\frac{\mathrm{d}r}{\phi(r)} < \infty$, we have that

$$\int_0^t \int_E f_r(z)\mathrm{d}M^X(r,z) = X_t(f_t) - \mu(f_0) - \int_0^t X_r(Af_r)\mathrm{d}r - \int_0^t X_r(\partial_r f_r)\mathrm{d}r \tag{2.10}$$

is a continuous $(\mathcal{G}_t)$-martingale starting at 0 and such that

$$\langle \int_0^\cdot \int_E f_r(z)\mathrm{d}M^X(r,z)\rangle_t = \eta \int_0^t (X_r(f_r^2) - X_r(f_r)^2)\phi(r)^{-1}\mathrm{d}r\,. \tag{2.11}$$

The particular choice $f_s = \int_0^{t-s} T_r f \mathrm{d}r$ for t fixed and $f \in b\mathcal{E}(E)$ gives

$$\int_s^t X_r(f)\mathrm{d}r = X_s\left(\int_0^{t-s} T_r f\mathrm{d}r\right) + \int_s^t \int_E \int_0^{t-r} T_{\bar{r}} f(z)\mathrm{d}\bar{r}\mathrm{d}M^X(r,z)\,. \tag{2.12}$$

Moreover, it follows from (2.5) (and fact $t_\phi = \infty$) that for every $f \in b\mathcal{E}(E)$,

$$X_t(f) = \mu(T_t f) + \int_0^t \int_E T_{t-r} f(z)\mathrm{d}M^X(r,z)\,, \tag{2.13}$$

which is called Green function representation in [Per02, pg. 167]. The representation (2.13) and (2.9) play a central role in our approach. A direct consequence of (2.13) is the following identity

$$X_t(f) - X_s(T_{t-s}f) = \int_s^t \int_E T_{t-r} f(z)\mathrm{d}M^X(r,z)\,, \tag{2.14}$$

which holds for every $0 \le s \le t$ and $f \in b\mathcal{E}(E)$. Another consequence of (2.13) is

$$\mathbb{P}^{\phi}_{\mu} X_t(f) = \mu(T_t f)\,. \tag{2.15}$$

Lemma 2.6. *For every $f \in b\mathcal{E}(E)$ and $t \ge s \ge 0$, we have*

$$\mathbb{P}^{\phi}_{\mu}\left[(X_t(f) - X_s(T_{t-s}f))^2\right] \le \|T_t(f^2)\|_\infty \int_s^t \frac{\mathrm{d}r}{\phi(r)}\,, \tag{2.16}$$

and

$$\mathbb{P}^{\phi}_{\mu}\left[\left(\int_s^t X_r(f)\mathrm{d}r - X_s\left(\int_0^{t-s} T_r f\mathrm{d}r\right)\right)^2\right] \le \int_s^t \left\|\int_0^r T_{\bar{r}} f\mathrm{d}\bar{r}\right\|_\infty^2 \frac{\mathrm{d}r}{\phi(r)}\,. \tag{2.17}$$

Proof. From (2.14), (2.9) and (2.15)

$$\begin{aligned}\mathbb{P}^{\phi}_{\mu}\left[(X_t(f) - X_s(T_{t-s}f))^2\right] &\le \mathbb{P}^{\phi}_{\mu}\int_s^t X_r((T_{t-r}f)^2)\frac{\mathrm{d}r}{\phi(r)} \\ &\le \int_s^t \langle T_r(T_{t-r}f)^2, \mu\rangle \frac{\mathrm{d}r}{\phi(r)}\,.\end{aligned}$$

By Jensen inequality,

$$T_r(T_{t-r}f)^2 \le T_r T_{t-r}(f^2) = T_t(f^2)\,.$$

Hence, $\langle T_r(T_{t-r}f)^2, \mu\rangle \le \mu(T_t(f^2)) \le \|T_t(f^2)\|_\infty$. The estimate (2.16) follows. Showing (2.17) is similar so we omit the detail. □

Convergence along lattice times. Suppose that f is a function in $b\mathcal{E}(E)$. Let $\{t_n\}_{n\geq 1}$ be an increasing sequence diverging to infinity such that

$$\sum_n c(t_n)\|T_{t_n} f^2\|_\infty \int_{\rho(t_n)}^{t_n} \frac{\mathrm{d}s}{\phi(s)} < \infty \tag{2.18}$$

and

$$\lim_{n\to\infty} \frac{c(t_n)}{c(t_n-\rho(t_n))} = 1\,. \tag{2.19}$$

Proposition 2.7. *Assuming* (2.7), (2.8), (2.18) *and* (2.19)*, the following limit holds*

$$\lim_{n\to\infty} c(t_n)|X_{t_n}(f) - X_{\rho(t_n)}(L_{t_n-\rho(t_n)}f)| = 0 \quad \mathbb{P}^\phi_\mu\text{−a.s.} \tag{2.20}$$

Proof. From Lemma 2.6,

$$\sum_n \mathbb{P}^\phi_\mu c(t_n)^2 \left[|X_{t_n}(f) - X_{\rho(t_n)}(T_{t_n-\rho(t_n)}f)|^2\right] \leq \sum_n c(t_n)\|T_{t_n} f^2_{t_n}\|_\infty \int_{\rho(t_n)}^{t_n} \frac{\mathrm{d}s}{\phi(s)} < \infty$$

by condition (2.18). An application of Borel-Cantelli lemma yields

$$\lim_{n\to\infty} \left|c(t_n)X_{t_n}(f) - c(t_n)X_{\rho(t_n)}(T_{t_n-\rho(t_n)}f)\right| = 0 \quad \mathbb{P}^\phi_\mu\text{−a.s.}$$

Moreover, noting that $X_s(1) = 1$ for every $s > 0$, we have

$$\begin{aligned} &c(t_n)|X_{\rho(t_n)}(T_{t_n-\rho(t_n)}f) - X_{\rho(t_n)}(L_{t_n-\rho(t_n)}f)| \\ &\leq c(t_n)X_{\rho(t_n)}(|T_{t_n-\rho(t_n)}f - L_{t_n-\rho(t_n)}f|) \\ &\leq c(t_n)\|T_{t_n-\rho(t_n)}f - L_{t_n-\rho(t_n)}f\|_\infty\,, \end{aligned}$$

which converges $\mathbb{P}^\phi_\mu$-a.s. to 0 by (2.7), (2.8) and (2.19). The identity (2.20) follows. □

From lattice time to continuous time. If the cost of replacing $c(t_n)$ by $c(t)$ for any $t \in [t_n, t_{n+1}]$ is negligible as $n \to \infty$, then previous result can be transfered to continuous time limit. There are several ways to obtain this. One possibility is the following result while Section 4 provides another way. Hereafter, c_n denotes $\sup_{t\in[t_n,t_{n+1}]} c(t)$.

Proposition 2.8. *In addition to the hypothesis in Proposition 2.7, we assume that*

$$\lim_{n\to\infty} c_n \sup_{t\in[t_n,t_{n+1}]} \|T_{t_{n+1}-t}f - f\|_\infty = 0\,, \tag{2.21}$$

$$\lim_{n\to\infty} \frac{c_n}{c(t_n)} = 1\,, \tag{2.22}$$

and

$$\sum_n c_n\|T_{t_{n+1}}(f^2)\|_\infty \int_{t_n}^{t_{n+1}} \frac{\mathrm{d}s}{\phi(s)} < \infty\,. \tag{2.23}$$

Then

$$\lim_{n\to\infty} \sup_{t\in[t_n,t_{n+1})} c(t)|X_t(f) - X_{\rho(t_n)}(L_{t_n-\rho(t_n)}f)| = 0 \quad \mathbb{P}^\phi_\mu\text{−a.s.} \tag{2.24}$$

Proof. We adopt an argument from [LRS13], which utilizes the properties of the semigroup T_t and the martingale M_t^X at the same time. For every $t \in [t_n, t_{n+1})$ we have

$$|X_t(f) - X_t(T_{t_{n+1}-t}f)| \le X_t(|f - T_{t_{n+1}-t}f|) \le \sup_{t\in[t_n,t_{n+1}]} \|T_{t_{n+1}-t}f - f\|_\infty .$$

It follows from (2.21) that

$$\lim_n \sup_{t\in[t_n,t_{n+1})} c(t)|X_t(f) - X_t(T_{t_{n+1}-t}f)| = 0 . \tag{2.25}$$

Hence, to show (2.24), it suffices to prove

$$\lim_{n\to\infty} \sup_{t\in[t_n,t_{n+1})} c(t)|X_t(T_{t_{n+1}-t}f) - X_{\rho(t_n)}(L_{t_n-\rho(t_n)}f)| = 0 \quad \mathbb{P}_\mu^\phi\text{−a.s.} \tag{2.26}$$

From (2.14) we have

$$\begin{aligned} X_t(T_{t_{n+1}-t}f) &= X_{t_n}(T_{t-t_n}T_{t_{n+1}-t}f) + \int_{t_n}^t \int_E T_{t-s}T_{t_{n+1}-t}f(x)\mathrm{d}M^X(s,x) \\ &= X_{t_n}(T_{t_{n+1}-t_n}f) + \int_{t_n}^t \int_E T_{t_{n+1}-s}f(x)\mathrm{d}M^X(s,x) . \end{aligned} \tag{2.27}$$

Similar to (2.25), we have

$$\lim_n \sup_{t\in[t_n,t_{n+1})} c(t)|X_{t_n}(T_{t_{n+1}-t_n}f) - X_{t_n}(f)| = 0.$$

Together with Proposition 2.7 and (2.22), this yields

$$\lim_n \sup_{t\in[t_n,t_{n+1})} c(t)|X_{t_n}(T_{t_{n+1}-t_n}f) - X_{\rho(t_n)}(L_{t_n-\rho(t_n)}f)| = 0 \quad \mathbb{P}_\mu^\phi\text{−a.s.} \tag{2.28}$$

Hence, (2.26) follows from (2.27) and (2.28) if we can show that

$$\lim_n c_n \sup_{t\in[t_n,t_{n+1}]} \left| \int_{t_n}^t \int_E T_{t_{n+1}-s}f(x)\mathrm{d}M^X(s,x) \right| = 0 \quad \mathbb{P}_\mu^\phi\text{−a.s.} \tag{2.29}$$

Fixing $\varepsilon > 0$ and applying the martingale maximal inequality as well as Lemma 2.6 and (2.14), we have

$$\begin{aligned} &\mathbb{P}_\mu^\phi\left(c_n \sup_{t\in[t_n,t_{n+1}]} \left| \int_{t_n}^t \int_E T_{t_{n+1}-s}f(x)\mathrm{d}M^X(s,x) \right| > \varepsilon \right) \\ &\le \varepsilon^{-2} c_n^2 \mathbb{P}_\mu^\phi \left| \int_{t_n}^{t_{n+1}} \int_E T_{t_{n+1}-s}f(x)\mathrm{d}M^X(s,x) \right|^2 \\ &\le \varepsilon^{-2} c_n^2 \|T_{t_{n+1}}f^2\|_\infty \int_{t_n}^{t_{n+1}} \frac{\mathrm{d}s}{\phi(s)} . \end{aligned}$$

Using (2.18), we see that

$$\sum_n \mathbb{P}_\mu^\phi\left(c_n \sup_{t\in[t_n,t_{n+1}]} \left| \int_{t_n}^t \int_E T_{t_{n+1}-s}f(x)\mathrm{d}M^X(s,x) \right| > \varepsilon \right) < \infty .$$

Applying Borel-Cantelli lemma, we find that (2.29) follows and the proof is complete. □

Remark 2.9. In view of Proposition 2.7, to study the long-time asymptotic of $X_t(f)$ for a test function f, we first study the long-time asymptotic of $T_t f$ and identify $c(t)$ and L_t in (2.7). Then, we establish the long-time limit for $X_{\rho(t)}(L_{t-\rho(t)}f)$ for a suitable sublinear function ρ. This procedure will be applied throughout Sections 3 and 4.

2.3. **Finite particle motivation.** The inhabitation time Z_t discussed in the introduction counts the time spent (in sets) by all ancestors of all particles living at time t. It counts common ancestors multiple times. It does not count time for particles with no living descendants. As such it requires genealogical information that is not readily available from the Flemming-Viot process X itself. We need to construct the historical process $\mathbb{X}$ associated with X.

To motivate historical processes and the difference between occupation and inhabitation times, we consider a finite particle approximation. Suppose that $X_t^N = \frac{1}{N}\sum_{\alpha\sim t}\delta_{\xi_t^\alpha}$ is a (Moran particle system empirical measure) pre-high-density limit of X. $\{\xi^\alpha\}_{\alpha\in M}$ are particles that undergo independent A-motions/mutations and are resampled at (time-inhomogeneous) rate proportional to $N(N-1)$. At a resampling time one random particle is selected to move to another random particle's location. This moved particle disowns her ancestors and adopts those of the particle to which it jumped. (This common convention is consistent with Fleming-Viot superprocesses providing distributional information about Dawson-Watanabe superprocess populations. Sampling is simultaneous deaths and generation of offspring from some of the dying particles.) Here, the set of multi-indices α keep track of all particles, whether they are living at t or not, and $\alpha\sim t$ means particle α is alive at time t. Naturally, there are N particles alive at any time so X_t^N is a probability measure but the actual particles that are alive is dependent upon which particles are sampled prior to t and multi-indices α are used to keep track of ancestors. For example, particle $(1,2,3)$ would be the parent ancestor of $(1,2,3,1)$ and $(1,2,3,2)$ for random outcomes where they all exist. Now, let $\xi_{[0,t]}^\alpha$ denote the ancestral path of particle α as a $D(\mathbb{R}^d)$-path held constant after t so $\xi_{[0,t]}^\alpha(u) = \xi_t^\alpha$ for $u\geq t$. Then, our times of interest are:

Occupation: $Y_t^N(1_{\mathcal{O}}) = \dfrac{1}{N}\displaystyle\int_0^t \sum_{\alpha\sim s} 1_{\mathcal{O}}(\xi_s^\alpha)\,\mathrm{d}s$ so $Y_t^N(f) = \int_0^t X_s^N(f)\mathrm{d}s$.

Inhabitation: $Z_t^N(1_{\mathcal{O}}) = \dfrac{1}{N}\displaystyle\int_0^t \sum_{\alpha\sim t} 1_{\mathcal{O}}(\xi_s^\alpha)\,\mathrm{d}s$ so $Z_t^N(f) = \displaystyle\sum_{\alpha\sim t}\int_0^t f(\xi_{[0,t]}^\alpha(s))\,\mathrm{d}s.$

for $\mathcal{O}\subset\mathbb{R}^d$ and $f\in B(\mathbb{R}^d)$. Theorem 1.4 in the introduction states that these two times (after high density limits) only differ by a martingale defined in terms of this function ℓ_f i.e. that the multiple counting of common ancestors is similar to the counting of time spent by dead lineages. Whereas $Y_t^N(f)$ was immediately expressed in terms of the empirical process X^N, one can only easily express the inhabitation time in terms of the

Historical Process: $\mathbb{X}_t^N = \dfrac{1}{N}\displaystyle\sum_{\alpha\sim t}\delta_{(t,\xi_{[0,t]}^\alpha)}$ in $\mathcal{P}(\mathbb{E})$ supported on $\mathbb{E}^t$.

In particular, $Z_t^N(f) = \mathbb{X}_t^N(\ell_f)$, where $\ell_f(t,y^t) = \int_0^t f(y_s^t)\mathrm{d}s$. (Here, $\mathbb{E}$ and $\mathbb{E}^t$ are defined around (1.17) and since $\mathbb{X}_t^N$ is supported on $\mathbb{E}^t$ we also have $Z_t^N(f) = \mathbb{X}_t^N(\ell_f^t)$, where $\ell_f^t(r,y^r) = \int_0^{t\wedge r} f(y_s^r)\mathrm{d}s$.) To relate occupation and inhabitation times, we express Y_t^N in terms of the historical process as well. For $f\in B(\mathbb{R}^d)$, we let $j^*f(r,y^r) \doteq f(y_r^r)\in B(\mathbb{E})$

and find $X_t^N(f) = \mathbb{X}_t^N(j^* f)$ so $Y_t^N(f) = \int_0^t X_s^N(f)\mathrm{d}s = \int_0^t \mathbb{X}_s^N(j^* f)\mathrm{d}s$. Notice, t is included with $\xi^\alpha_{[0,t]}$ in the definition of the historical process. This is to allow time-inhomogeneous generator and to make support properties obvious as will be seen below. The developments of this motivating subsection survive the process of taking high density limits while martingale problem formulation actually gets easier. We will use the historical martingale problem below to relate the occupation and inhabitation times now that we have expressed them both in terms of the historical process. The first step is to define the historical process when there are infinitely many particles.

2.4. **Fleming-Viot Historical processes.** Historical superprocesses were first introduced by Dawson and Perkins [DP91]. To make our presentation manifest, we assume that (ξ_t, P_x) is an $\mathbb{R}^d$-valued Borel strong Markov process with sample path in $D := D(\mathbb{R}^d)$, the Skorohod space defined at the beginning of Section 2. The weak generator of ξ is still denoted by $(A, \mathcal{D}(A))$, $\mu \in M_1(E)$ and $\phi \in C_+$ with $t_\phi = \infty$.

For each $(r, y) \in \mathbb{E}$, we consider the process $(\Xi_t)_{t\geq 0}$ in $\mathbb{E}$ defined by

$$\Xi_t = (r + t, (y \ltimes_r \xi)^{r+t}),$$

where for every $w, w' \in D(\mathbb{R}^d)$, $w \ltimes_r w'$ is the concatenation path

$$w \ltimes_r w'(s) = \begin{cases} w(s) & \text{for } s \in [0, r) \\ w(r) + w'(s-r) & \text{for } s \in [r, \infty). \end{cases}$$

The law of Ξ is denoted by $P_{r,y}$, namely

$$P_{r,y}(\mathcal{O}) = P_0(\Xi \in \mathcal{O}) \quad \forall \mathcal{O} \in \mathcal{E}(D(\mathbb{E})).$$

$((\Xi_t)_{t\geq 0}, P_{r,y})$ is called the historical process, associated to ξ, with initial position $\Xi_0 = (r, y)$. $((\Xi_t)_{t\geq 0}, (P_{r,y})_{(r,y)\in\mathbb{E}})$ is a time-homogeneous Borel strong Markov process in $\mathbb{E}$ with semigroup

$$\begin{aligned} \mathbb{T}_t &: C_b(\mathbb{E}) \to C_b(\mathbb{E}) \\ \mathbb{T}_t f(r, y) &= P_{r,y} f(\Xi_t). \end{aligned} \tag{2.30}$$

(See [Per02, Proposition II.2.5] for a more general result.) It is more convenient to express $\mathbb{T}$ directly through ξ by

$$\mathbb{T}_t f(r, y) = P_{y_r} f(r + t, (y \ltimes_r \xi)^{r+t}). \tag{2.31}$$

We denote by $\mathbb{A}$ the (weak) generator of $\mathbb{T}$. A function $f \in b\mathcal{E}(\mathbb{E})$ belongs to the domain of $\mathbb{A}$, $\mathcal{D}(\mathbb{A})$, iff the limit

$$\text{bp-}\lim_{h\downarrow 0} \frac{1}{h}(\mathbb{T}_h f(r, y) - f(r, y))$$

exists. In such case, we denote the limit as $\mathbb{A}f(r, y)$.

Let $\tau \geq 0$ and χ be a measure in $M_1(D)$ such that $\chi(\{y \in D(\mathbb{R}^d) : y^\tau = y\}) = 1$. Then, $\delta_\tau \times \chi$ is a probability measure on $\mathbb{E}$. By Theorem 2.3 there is a unique solution $(\mathbb{X}, \mathbb{P}^\phi_{\tau,\chi}(\equiv \mathbb{P}^\phi_{\delta_\tau\times\chi}))$ on $(\Omega, \mathcal{G})$ (with $E = \mathbb{E}$) to the $\mathbb{A}$-martingale problem, meaning

$$\mathbb{M}_t(f) = \mathbb{X}_t(f) - \delta_\tau \times \chi(f) - \int_0^t \mathbb{X}_s(\mathbb{A}f)\mathrm{d}s \tag{2.32}$$

is a continuous $(\mathcal{G}_t)$-martingale starting at 0 such that

$$\langle \mathbb{M}(f)\rangle_t = \int_0^t (\mathbb{X}_s(f^2) - \mathbb{X}_s(f)^2)\frac{\mathrm{d}s}{\phi(s)} \tag{2.33}$$

for all $f \in \mathcal{D}(\mathbb{A})$. The process $(\mathbb{X}_t, \mathbb{P}^{\phi}_{\tau,\chi})$ is called the (time-homogeneous) historical Fleming-Viot process. The relations (2.13) and (2.14) in the current context become respectively

$$\mathbb{X}_t(f) = \delta_\tau \times \chi(\mathbb{T}_t f) + \int_0^t \int_{\mathbb{E}} \mathbb{T}_{t-s} f(r,y) \mathrm{d}\mathbb{M}(s,(r,y)) \,, \tag{2.34}$$

$$\mathbb{X}_u(f) - \mathbb{X}_t(\mathbb{T}_{u-t} f) = \int_t^u \int_{\mathbb{E}} \mathbb{T}_{u-s} f(r,y) \mathrm{d}\mathbb{M}(s,(r,y)) \,, \tag{2.35}$$

which hold for every $0 \le t \le u$ and $f \in b\mathcal{E}(\mathbb{E})$. In particular,

$$\mathbb{P}^{\phi}_{\tau,\chi} \mathbb{X}_t(f) = \delta_\tau \times \chi(\mathbb{T}_t f) \quad \forall t \ge 0, \forall f \in b\mathcal{E}(\mathbb{E}) \,. \tag{2.36}$$

It is possible to recover the Fleming-Viot process X from $\mathbb{X}$. We just define the projection

$$\begin{aligned} \jmath &: \mathbb{E} \to \mathbb{R}^d \\ \jmath(r,y) &= y_r \,, \end{aligned}$$

and put $X_t = \mathbb{X}_t \circ \jmath^{-1}$, $M^X_t = \mathbb{M}_t \circ \jmath^{-1}$, respectively the pushforward measures of $\mathbb{X}_t, \mathbb{M}_t$ via $\jmath$. Each function f in $b\mathcal{E}(\mathbb{R}^d)$ induces the function $\jmath^* f$ in $C_b(\mathbb{E})$ by $\jmath^* f(r,y) = f(y_r)$. In addition, for each $f \in b\mathcal{E}(\mathbb{R}^d)$ we have

$$X_t(f) = \mathbb{X}_t(\jmath^* f) \quad \text{and} \quad M^X_t(f) = \mathbb{M}_t(\jmath^* f) \quad \forall t \ge \tau \,. \tag{2.37}$$

If f belongs to the domain of A, then $\jmath^* f$ belongs to the domain of $\mathbb{A}$ and $\mathbb{A}\jmath^* f = Af$. It follows from (2.32) and (2.33) that (X, M^X) is a Fleming-Viot process with law $\mathbb{P}^{\phi}_{\mu}$, where $\mu = (\delta_\tau \times \chi) \circ \jmath^{-1}$.

We give a brief investigation on the support of $\mathbb{X}_t$. Let $\Pi : \mathbb{E} \to D(\mathbb{R}^d)$ be the projection $\Pi(r,y) = y$ and define an $M_1(D)$-valued process $(H_t, t \ge \tau)$ by

$$H_{\tau+t} = \mathbb{X}_t \circ \Pi^{-1} \quad \forall t \ge 0 \,.$$

Define $\mathbb{D}^t = \Pi \mathbb{E}^t = \{y \in D : y^t = y\}$ for each $t \ge 0$ and note $\mathbb{E} = \cup_{t \ge 0} \mathbb{E}^t$. The following result is an analog of [Per02, Lemma II.8.1].

Proposition 2.10. *$\mathbb{X}_t = \delta_{\tau+t} \times H_{\tau+t}$ and $\operatorname{supp} H_{\tau+t} \subset \mathbb{D}^{\tau+t}$ for all $t \ge 0$ $\mathbb{P}^{\phi}_{\tau,\chi}$-a.s.*

Proof. We define

$$\Lambda(t) = \{(r,y) \in \mathbb{E} : r \ne \tau + t\} \,.$$

Then, by (2.34) and (2.31),

$$\begin{aligned} \mathbb{P}^{\phi}_{\tau,\chi} \mathbb{X}_t(\mathbf{1}_{\Lambda(t)}) &= \int_D \mathbb{T}_t \mathbf{1}_{\Lambda(t)}(\tau, y) \mathrm{d}\chi(y) \\ &= \int_D E_0 \mathbf{1}_{\Lambda(t)}(\tau + t, (y \ltimes_\tau \xi)^{\tau+t}) \mathrm{d}\chi(y) = 0 \,. \end{aligned}$$

This shows $\mathbb{X}_t = \delta_{\tau+t} \times H_{\tau+t}$ $\mathbb{P}^{\phi}_{\tau,\chi}$-a.s. for each $t \ge 0$ and hence for all $t \ge 0$ by the right-continuity of both sides. The later assertion in the proposition statement follows from the former. Indeed, for every $\mathcal{O} \in \mathcal{E}(D)$,

$$H_{\tau+t}(\mathcal{O}) = \mathbb{X}_t(\Pi^{-1}\mathcal{O}) = \delta_{\tau+t} \times H_{\tau+t}(\{(r,y) \in \mathbb{E} : y^r \in \mathcal{O}\}) = H_{\tau+t}(\{y \in \mathcal{O} : y^{\tau+t} = y\}) \,,$$

which implies $\operatorname{supp} H_{\tau+t} \subset \mathbb{D}^{\tau+t}$. ∎

Remark 2.11. The process $(H_t)_{t\geq\tau}$ is time inhomogeneous and is called historical superprocess in literature ([DP91, Dyn91]). In the current article, we use its time-homogeneous counter part $(\mathbb{X}_t)_{t\geq 0}$. It is evident from the previous result that under $\mathbb{P}^\phi_{\tau,\chi}$, $\operatorname{supp}\mathbb{X}_t\subset\mathbb{E}^{\tau+t}$. Consequently, for every bounded measurable function f on $\mathbb{E}^{\tau+t}$

$$\mathbb{X}_t(f)=\int_{\mathbb{E}}f(r,y)\mathrm{d}\mathbb{X}_t(r,y)=\int_{\mathbb{E}}f(r,y)\mathbf{1}_{(r=\tau+t)}\mathrm{d}\mathbb{X}_t(r,y) \tag{2.38}$$

and

$$|\mathbb{P}^\phi_{\tau,\chi}\mathbb{X}_t(f)|\leq\|f\|_{L^\infty(\mathbb{E}^{\tau+t})}\,. \tag{2.39}$$

In addition, it is seen from (2.33) that $\operatorname{supp}\mathbb{M}_t\subset\operatorname{supp}\mathbb{X}_t\subset\mathbb{E}^{\tau+t}$.

Our interest is the superprocess $(X_t)_{t\geq0}$ starting from a specified initial measure $X_0=\mu$. Hence, it is natural to simply take $\tau=0$ for the historical process $(\mathbb{X}_t)_{t\geq0}$. In such case, the measure χ can also be constructed (uniquely) from μ by

$$\chi(\mathcal{O})=\mu^*(\mathcal{O})=\mu(\{y(0):y\in\mathcal{O}\})\quad\forall\mathcal{O}\in\mathcal{E}(\mathbb{E})\,.$$

2.5. **Occupation times and inhabitation times.** The *occupation time* process $(Y_t)_{t\geq0}$ associated with $(X_t)_{t\geq0}$ is the measure-valued process defined by

$$Y_t(\mathcal{O})=\int_0^t X_s(\mathcal{O})\mathrm{d}s\quad\forall\mathcal{O}\in\mathcal{E}(\mathbb{R}^d)\,. \tag{2.40}$$

In the context of critical Dawson-Watanabe processes, the occupation time process was introduced and studied by [Isc86] by means of Laplace functionals. Our other time of interest *inhabitation time* is defined through the historical process and the counting function ℓ_f. It is natural to ask whether ℓ_f, defined in (1.12), is measurable when restricted to $\mathbb{E}$.

Proposition 2.12. *For every $f\in b\mathcal{E}(\mathbb{R}^d)$, $\ell_f:(\mathbb{E},\mathcal{E}(\mathbb{E}))\to(\mathbb{R},\mathcal{E}(\mathbb{R}))$ is measurable.*

Proof. First, suppose f is continuous. Then, it follows by Ethier and Kurtz [EK86, Problems 3.11.13 and 3.11.26] that $D(\mathbb{E})\ni y\to\int_0^\cdot f(y_s)\mathrm{d}s\in D(\mathbb{R})$ is continuous and so $(r,y)\to\int_0^r f(y_s)\mathrm{d}s$ is also continuous. Now, let $\mathcal{O}$ be a closed set in $\mathbb{R}^d$. Then, there exist continuous f^n such that $f^n\to\mathbf{1}_{\mathcal{O}}$ pointwise by Billingsley [Bil68, Theorem 1.2] so for every $(r,y)\in\mathbb{E}$,

$$\lim_{n\to\infty}\ell_{f_n}(r,y)=\int_0^r\lim_{n\to\infty}f_n(y_s)\mathrm{d}s=\int_0^r\mathbf{1}_{\mathcal{O}}(y_s)\mathrm{d}s=\ell_{\mathbf{1}_{\mathcal{O}}}(r,y)$$

by dominated convergence and $\ell_{\mathbf{1}_{\mathcal{O}}}$ is measurable. Finally, the family $\mathcal{H}=\{f\in b\mathcal{E}(\mathbb{R}^d):\ell_f\text{ is measurable}\}$ contains $\mathbf{1}_{\mathcal{O}}\in\mathcal{H}$ for every closed set $\mathcal{O}\subset\mathbb{R}^d$ and is closed under additions, scalar multiplications and pointwise limits. Hence, $\mathcal{H}=b\mathcal{E}(\mathbb{R}^d)$ by the monotone class theorem. □

Let $(\mathbb{X}_t)_{t\geq0}$ be the historical Fleming-Viot process constructed in subsection 2.4. The inhabitation time process $(Z_t)_{t\geq0}$ associated with X is the measure-valued process defined by

$$Z_t(\mathcal{O})=\mathbb{X}_t(\ell_{\mathbf{1}_{\mathcal{O}}})\quad\forall t\geq0,\mathcal{O}\in\mathcal{E}(\mathbb{R}^d)\,.$$

$\mathbb{X}_t(\ell_f)$ makes sense at least for non-negative f since ℓ_f is measurable. As mentioned in the introduction $\mathbb{X}_t(\ell_f)$ satisfies martingale problem (1.13,1.14) once we know that each $\ell^t_f\in\mathcal{D}(\mathbb{A})$.

Lemma 2.13. *Let f be a function in $b\mathcal{E}(\mathbb{R}^d)$. Then, for every $t,h>0$ and every (r,y) in $\mathbb{E}$,*

$$\mathbb{T}_h\ell^t_f(r,y)=\ell^t_f(r,y)+\mathbf{1}_{(r<t)}\int_0^{(r+h)\wedge t-r}T_sf(y_r)\mathrm{d}s\,. \tag{2.41}$$

In addition, ℓ^t_f belongs to the domain of $\mathbb{A}$ and

$$\mathbb{A}\ell^t_f(r,y)=f(y_r)\mathbf{1}_{(r<t)}\quad \textit{and}\quad \mathbb{A}\ell_f(r,y)=bp\text{-}\lim_{t\to\infty}\mathbb{A}\ell^t_f(r,y)=f(y_r). \tag{2.42}$$

Proof. We observe that for every path $\omega\in D(\mathbb{R}^d)$

$$\begin{aligned}\ell^t_f(r+h,\omega^{r+h})&=\int_0^{r\wedge t}f(\omega_s)\mathrm{d}s+\mathbf{1}_{(r<t)}\int_r^{(r+h)\wedge t}f(\omega_s)\mathrm{d}s\\&=\ell^t_f(r,\omega^r)+\mathbf{1}_{(r<t)}\int_0^{(r+h)\wedge t-r}f(\omega_{r+s})\mathrm{d}s\,.\end{aligned}$$

This implies that

$$\begin{aligned}\mathbb{T}_h\ell^t_f(r,y)&=P_{r,y}\ell^t_f(r+h,(y\ltimes_r\xi)^{r+h})\\&=\ell^t_f(r,y)+\mathbf{1}_{(r<t)}P_{y_r}\int_0^{(r+h)\wedge t-r}f(\xi_s)\mathrm{d}s\,,\end{aligned}$$

which yields (2.41). Equation (2.42) is obtained by differentiating (2.41) at $h=0$ and then letting $t\to\infty$. ☐

We observe that $Z_0\equiv0$. In comparison with the occupation time process Y defined in (2.40), it is easy to derive from (2.36) that for every $f\in b\mathcal{E}(E)$ and $t\ge0$, $Y_t(f)$ and $Z_t(f)$ have the same mean, that is

$$\mathbb{P}^\phi_\mu Y_t(f)=\mathbb{P}^\phi_\mu Z_t(f)=\mu\left(\int_0^tT_sf\mathrm{d}s\right)\,.$$

In fact, a deeper relation between Z and Y holds.

Proposition 2.14. *For every $f\in b\mathcal{E}(\mathbb{R}^d)$ and $t\ge0$*

$$Z_t(f)=\mathbb{M}_t(\ell_f)+Y_t(f)\,, \tag{2.43}$$

where the process $(\mathbb{M}_t(\ell_f))_{t\ge0}$ is the continuous $(\mathcal{G}_t)$-martingale defined in (1.18), with quadratic variation

$$\langle\mathbb{M}(\ell_f)\rangle_t=\int_0^t\left(\mathbb{X}_s((\ell_f)^2)-\mathbb{X}_s(\ell_f)^2\right)\frac{\mathrm{d}s}{\phi(s)}\quad\forall t\ge0\,. \tag{2.44}$$

Proof. Fix $u>0$ and recall the bounded variant ℓ^u_f of ℓ_f is in $\mathcal{D}(\mathbb{A})$ so $\mathbb{X}_t(\ell_f)=\mathbb{X}_t(\ell^u_f)$, $\mathbb{X}_t(\mathbb{A}\ell_f)=\mathbb{X}_t(\mathbb{A}\ell^u_f)$ and $\langle\mathbb{M}(\ell_f)\rangle_t=\langle\mathbb{M}(\ell^u_f)\rangle_t$ for $t\le u$ by (1.16) and (1.19). In addition, by definition $Z_t(f)=\mathbb{X}_t(\ell_f)$, hence we derive from (1.18) that

$$\mathbb{M}_t(\ell_f)=Z_t(f)-\mathbb{X}_0(\ell_f)-\int_0^t\mathbb{X}_s(\mathbb{A}\ell_f)\mathrm{d}s\,.$$

We know $\mathbb{X}_0(\ell_f)=0$ and have from (2.42), (2.37) that

$$\mathbb{X}_s(\mathbb{A}\ell_f)=\int_Ef(y_r)\mathrm{d}\mathbb{X}_s(r,y)=\mathbb{X}_s(\jmath^*f)=X_s(f)\,,$$

for every s. This yields (2.43) and $(\mathbb{M}_t(\ell_f))_{t\geq 0}$ is a $(\mathcal{G}_t)$-martingale with the required quadratic variation by (2.33) by the arbitrariness of u. □

In relation (2.43), if the long term asymptotics of any two among the three quantities are known, then, this implies the long term asymptotic of the other term. Since $\mathbb{M}_t(\ell_f)$ is a martingale, its analysis is subjected to martingale limit theorems. Depending on the situation at hand, the asymptotic of one of Y_t and Z_t is easier than the other. This is the case for α-stable Fleming-Viot process considered in Section 4 below.

3. Stable Fleming-Viot processes

Hereafter, we consider the specific case when $A = -(-\Delta)^{\frac{\alpha}{2}}$ on $\mathbb{R}^d$ for some $\alpha \in (0, 2]$. The historical α-stable generator is still denoted by $\mathbb{A}$. The motion of each particle has the law of the α stable process in $\mathbb{R}^d$. The associated superprocess $(X_t)_{t\geq 0}$ constructed in Theorem 2.3 is called α-stable Fleming-Viot process. The associated historical superprocess $(\mathbb{X}_t)_{t\geq 0}$ with law $\mathbb{P}^{\phi}_{0,\mu^*}$ constructed in Subsection 2.4 is called historical α-stable Fleming-Viot process. The relation (2.37) describes the connection between X and $\mathbb{X}$. In this section, we develop several intermediate results following the guideline described in Remark 2.9. These considerations eventually lead to the proofs of Theorems 1.1 and 1.2 stated in the Introduction.

3.1. The stable semigroup.

Let T_t be the semigroup corresponding to a symmetric α-stable process. In particular, for each test function f,

$$T_t f(x) = \int_{\mathbb{R}^d} p_t(x-y) f(y) \mathrm{d}y \,, \tag{3.1}$$

where

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\theta} e^{-t|\theta|^\alpha} \mathrm{d}\theta \,. \tag{3.2}$$

Let $\hat{f}$ be the Fourier transform of f, $\hat{f}(\theta) = \int_{\mathbb{R}^d} e^{-i\theta\cdot x} f(x)\mathrm{d}x$. Using Fourier transform, $T_t f$ takes an alternative form

$$T_t f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\theta - t|\theta|^\alpha} \hat{f}(\theta) \mathrm{d}\theta \,. \tag{3.3}$$

We have seen in Subsection 2.2 that the long term asymptotic of $X_t(f)$ depends upon that of $T_t f$. It is therefore natural to study $T_t f$ as $t \to \infty$ for a given test function f. If $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ is a multi-index, we define $\partial^k f = \partial_1^{k_1} \partial_2^{k_1} \cdots \partial_d^{k_d} f$.

Proposition 3.1 (Semigroup expansion)**.** *Let f be a bounded measurable function on $\mathbb{R}^d$ and N be a non-negative integer such that* (1.8) *holds. Then, we have*

$$\lim_{t\to\infty} t^{\frac{N+d}{\alpha}} \sup_{x\in\mathbb{R}^d} \left| T_t f(x) - \sum_{k\in\mathbb{N}^d : |k|\leq N} \frac{(-1)^{|k|}}{k!} \int_{\mathbb{R}^d} f(y) y^k \mathrm{d}y \partial^k p_t(x) \right| = 0 \,. \tag{3.4}$$

Proof. We begin with a rescaled version of (3.3)

$$t^{d/\alpha} T_t f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it^{-1/\alpha} x\cdot\theta - |\theta|^\alpha} \hat{f}(t^{-1/\alpha}\theta) \mathrm{d}\theta \,. \tag{3.5}$$

The condition (1.8) ensures that the derivative $\partial^k\hat f$ exists and is continuous and bounded for every multi-index k such that $|k|\le N$. Hence, we have the following Taylor's expansion for $\hat f(u)$ around $u=0$,

$$\hat f(u)=\sum_{|k|\le N}\frac{\partial^k\hat f(0)}{k!}u^k+R_N(u)\,. \tag{3.6}$$

The remainder term satisfies

$$\lim_{u\to0}|u|^{-N}|R_N(u)|=0\quad\text{and}\quad\sup_{u\in\mathbb{R}^d\setminus\{0\}}\frac{|R_N(u)|}{|u|^N}=O(1)\,. \tag{3.7}$$

The second estimate in (3.7) comes from the first estimate, (3.6) and the fact that $\hat f$ is bounded. Hence, we can rewrite the right-hand side of (3.5) as follows:

$$\sum_{|k|\le N}\frac{\partial^k\hat f(0)}{k!}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{it^{-1/\alpha}x\cdot\theta-|\theta|^\alpha}(t^{-1/\alpha}\theta)^k\mathrm{d}\theta+\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{it^{-1/\alpha}x\cdot\theta-|\theta|^\alpha}R_N(t^{-1/\alpha}\theta)\mathrm{d}\theta\,.$$

Taking into account the facts that

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{it^{-1/\alpha}x\cdot\theta-|\theta|^\alpha}(t^{-1/\alpha}\theta)^k\mathrm{d}\theta=i^{-|k|}t^{d/\alpha}\partial^kp_t(x)$$

and

$$\partial^k\hat f(0)=(-i)^{|k|}\int_{\mathbb{R}^d}f(y)y^k\mathrm{d}y\,, \tag{3.8}$$

we obtain

$$\begin{aligned}t^{d/\alpha}T_tf(x)&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{it^{-1/\alpha}x\cdot\theta-|\theta|^\alpha}\hat f(t^{-1/\alpha}\theta)\mathrm{d}\theta\\&=t^{d/\alpha}\sum_{|k|\le N}\frac{(-1)^{|k|}}{k!}\int_{\mathbb{R}^d}f(y)y^k\mathrm{d}y\ \partial^kp_t(x)+\tilde R_N(x)\,,\end{aligned}$$

where

$$\tilde R_N(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{it^{-1/\alpha}x\cdot\theta-|\theta|^\alpha}R_N(t^{-1/\alpha}\theta)\mathrm{d}\theta\,.$$

Hence, it remains to show $\lim_{t\to\infty}t^{\frac{N}{\alpha}}\|\tilde R_N\|_\infty=0$. In fact, we have

$$t^{\frac{N}{\alpha}}\sup_{x\in\mathbb{R}^d}|\tilde R_N(x)|\lesssim\int_{\mathbb{R}^d}e^{-|\theta|^\alpha}t^{\frac{N}{\alpha}}|R_N(t^{-\frac{1}{\alpha}}\theta)|\mathrm{d}\theta\,,$$

which converges to 0 as $t\to\infty$ by dominated convergence theorem and (3.7). (Here and below, we use $\lesssim$ in the standard way: $A\lesssim B$ means there exists a constant $C>0$ such that $A\le CB$.) □

As an immediate consequence, the stable semigroup T_t satisfies (2.7) with $c(t)=t^{(N+d)/\alpha}$ and

$$L_tf=\sum_{|k|\le N}\frac{(-1)^{|k|}}{k!}\int_{\mathbb{R}^d}f(y)y^k\mathrm{d}y\partial^kp_t\,. \tag{3.9}$$

In view of Proposition 2.8 and (3.9), the long term asymptotic of $X_t(f)$ is reduced to the long term asymptotic along a sequence of

$$X_{\rho(t)}(\partial^k p_{t-\rho(t)})\,,\quad k\in\mathbb{N}^d\,,|k|\le N\,,$$

which we will describe in Subsection 3.2.

3.2. **Limit theorems for super stable processes.** For each $\theta\in\mathbb{R}^d$, we denote $\mathrm{e}_\theta(x)=e^{i\theta\cdot x}$, $\cos_\theta(x)=\cos(\theta\cdot x)$ and $\sin_\theta(x)=\sin(\theta\cdot x)$ and recall the definition of $\vartheta^k_{d,\alpha}$ in (1.6).

Proposition 3.2. *Let ρ be a sublinear function such that $\lim_{t\to\infty}\frac{\rho(t)}{t^{1-\varepsilon_0}}=0$ for some $\varepsilon_0>0$. Suppose that ϕ satisfies*

$$\int_0^\infty\frac{\mathrm{d}s}{\phi(s)}<\infty \tag{3.10}$$

and μ satisfies (1.4). With $\mathbb{P}^\phi_\mu$-probability one, we have for every $k\in\mathbb{N}^d$ that

$$\lim_{t\to\infty}t^{\frac{d+|k|}{\alpha}}X_{\rho(t)}(\partial^k p_{t-\rho(t)})=\begin{cases}0 & \text{if } |k| \text{ is odd}\\ (-1)^{\frac{|k|}{2}}\vartheta^k_{d,\alpha} & \text{if } |k| \text{ is even}.\end{cases} \tag{3.11}$$

Proof. We note that for every function $f\in L^1(\mathbb{R}^d)$, by Fubini's theorem,

$$X_t(f)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}X_t(\mathrm{e}_\theta)\hat f(\theta)\mathrm{d}\theta\,. \tag{3.12}$$

Hence,

$$X_{\rho(t)}(\partial^k p_{t-\rho(t)})=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-(t-\rho(t))|\theta|^\alpha}X_{\rho(t)}(\mathrm{e}_\theta)(i\theta)^k\mathrm{d}\theta\,.$$

In addition, from (2.5), we obtain

$$X_{\rho(t)}(\mathrm{e}_\theta)=\mu(\mathrm{e}_\theta)-|\theta|^\alpha\int_0^{\rho(t)}X_s(\mathrm{e}_\theta)\mathrm{d}s+M^X_{\rho(t)}(\mathrm{e}_\theta)\,. \tag{3.13}$$

It follows that

$$X_{\rho(t)}(\partial^k p_{t-\rho(t)})=I_1+I_2+I_3\,,$$

where

$$\begin{aligned}I_1&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-(t-\rho(t))|\theta|^\alpha}\mu(\mathrm{e}_\theta)(i\theta)^k\mathrm{d}\theta\,,\\ I_2&=-\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-(t-\rho(t))|\theta|^\alpha}\int_0^{\rho(t)}X_s(\mathrm{e}_\theta)\mathrm{d}s|\theta|^\alpha(i\theta)^k\mathrm{d}\theta\,,\\ I_3&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-(t-\rho(t))|\theta|^\alpha}M^X_{\rho(t)}(\mathrm{e}_\theta)(i\theta)^k\mathrm{d}\theta\,.\end{aligned}$$

We will show that

$$\lim_{t\to\infty}t^{\frac{d+|k|}{\alpha}}I_1=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-|\theta|^\alpha}(i\theta)^k\mathrm{d}\theta\quad\text{a.s.}\;, \tag{3.14}$$

$$\lim_{t\to\infty}t^{\frac{d+|k|}{\alpha}}I_2=0\text{ a.s.}\qquad\text{and}\qquad\lim_{t\to\infty}t^{\frac{d+|k|}{\alpha}}I_3=0\quad\text{a.s.} \tag{3.15}$$

By a change of variable, we see that

$$I_1 = t^{-\frac{d+|k|}{\alpha}} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-(1-\frac{\rho(t)}{t})|\theta|^\alpha} \mu(\mathrm{e}_{t^{-1/\alpha}\theta})(i\theta)^k \mathrm{d}\theta \,.$$

This, together with dominated convergence theorem yields (3.14). For I_2, we observe that

$$\begin{aligned}|I_2| &\lesssim \rho(t) \int_{\mathbb{R}^d} e^{-(t-\rho(t))|\theta|^\alpha} |\theta|^{|k|+\alpha} \mathrm{d}\theta \\ &\lesssim \frac{\rho(t)}{t} t^{-\frac{d+|k|}{\alpha}} \int_{\mathbb{R}^d} e^{-(1-\frac{\rho(t)}{t})|\theta|^\alpha} |\theta|^{|k|+\alpha} \mathrm{d}\theta \,,\end{aligned}$$

which due to sublinearity of ρ immediately implies the first assertion in (3.15). For I_3, putting $a_n = e^n$ and utilizing the Borel-Cantelli lemma, we merely need to show

$$\sum_{n\ge1} \mathbb{P}^\phi_\mu \left(\sup_{a_n\le t\le a_{n+1}} t^{\frac{d+|k|}{\alpha}} |I_3| \right)^2 < \infty \,. \tag{3.16}$$

Set $\rho_n = \rho(a_n)$ and note by a change of variables that

$$\int_{\mathbb{R}^d} e^{-(a_n-\rho_{n+1})|\theta|^\alpha} |\theta|^{|k|} \mathrm{d}\theta \lesssim a_n^{-\frac{d+|k|}{\alpha}} \,.$$

By Jensen's inequality, we have

$$\mathbb{P}^\phi_\mu \left(\sup_{a_n\le t\le a_{n+1}} t^{\frac{d+|k|}{\alpha}} |I_3| \right)^2 \lesssim \left(\frac{a_{n+1}^2}{a_n} \right)^{\frac{d+|k|}{\alpha}} \int_{\mathbb{R}^d} e^{-(a_n-\rho_{n+1})|\theta|^\alpha} \mathbb{P}^\phi_\mu \sup_{a_n\le t\le a_{n+1}} |M^X_{\rho(t)}(\mathrm{e}_\theta)|^2 |\theta|^{|k|} \mathrm{d}\theta. \tag{3.17}$$

For each $\theta \in \mathbb{R}^d$, $(M_t(\mathrm{e}_\theta))_{t\ge0}$ is a complex valued martingale with quadratic variations satisfying

$$\begin{aligned}\langle \mathrm{Re}\, M(\mathrm{e}_\theta)\rangle_t &= \int_0^t \left[X_s\left(\cos^2_\theta\right) - X^2_s\left(\cos_\theta\right) \right] \frac{\mathrm{d}s}{\phi(s)} \le \int_0^t X_s((1-\cos_\theta)^2) \frac{\mathrm{d}s}{\phi(s)} \,, \\ \langle \mathrm{Im}\, M(\mathrm{e}_\theta)\rangle_t &= \int_0^t \left[X_s\left(\sin^2_\theta\right) - X^2_s\left(\sin_\theta\right) \right] \frac{\mathrm{d}s}{\phi(s)} \le \int_0^t X_s(\sin^2_\theta) \frac{\mathrm{d}s}{\phi(s)} \,.\end{aligned}$$

Hence, using the elementary identity $1-\cos_\theta = 2\sin^2_{\theta/2}$, we obtain

$$\begin{aligned}\mathbb{P}^\phi_\mu |M^X_t(\mathrm{e}_\theta)|^2 &\lesssim \int_0^t \mathbb{P}^\phi_\mu X_s(\sin^4_{\theta/2} + \sin^2_\theta) \frac{\mathrm{d}s}{\phi(s)} \\ &\lesssim \int_0^t \langle T_s(\sin^4_{\theta/2} + \sin^2_\theta), \mu\rangle \frac{\mathrm{d}s}{\phi(s)} \,.\end{aligned}$$

Note that for every $x \in \mathbb{R}^d$

$$\begin{aligned}2T_s \sin^2_\theta(x) &= 1 - \cos_{2\theta}(x) e^{-s|2\theta|^\alpha} = (1-\cos_{2\theta}(x)) e^{-s|2\theta|^\alpha} + 1 - e^{-s|2\theta|^\alpha} \\ &\lesssim (1\wedge|\theta||x|)^2 + s|\theta|^\alpha \,.\end{aligned}$$

Similarly, $4T_s \sin^4_{\frac{\theta}{2}}(x) = T_s(1-2\cos_\theta + \cos^2_\theta) \le 2T_s(1-\cos_\theta) \lesssim (1\wedge|\theta||x|)^2 + s|\theta|^\alpha$. Using (3.10) and (1.4), it follows that

$$\mathbb{P}^\phi_\mu |M^X_t(\mathrm{e}_\theta)|^2 \lesssim |\theta|^{2\wedge a} + t|\theta|^\alpha \,. \tag{3.18}$$

By martingale maximal inequality

$$\mathbb{P}^\phi_\mu \sup_{a_n\le t\le a_{n+1}} |M^X_{\rho(t)}(\mathrm{e}_\theta)|^2 \lesssim \mathbb{P}^\phi_\mu |M^X_{\rho_{n+1}}(\mathrm{e}_\theta)|^2 \lesssim |\theta|^{2\wedge a} + \rho_{n+1}|\theta|^\alpha \,. \tag{3.19}$$

Applying the above estimate in (3.17) and a change of variables yields

$$\begin{aligned}\mathbb{P}^\phi_\mu \left(\sup_{a_n\le t\le a_{n+1}} t^{\frac{d+|k|}{\alpha}}|I_3|\right)^2 &\lesssim \left(\frac{a_{n+1}^2}{a_n}\right)^{\frac{d+|k|}{\alpha}} \int_{\mathbb{R}^d} e^{-(a_n-\rho_{n+1})|\theta|^\alpha}(|\theta|^{2\wedge a}+\rho_{n+1}|\theta|^\alpha)|\theta|^{|k|}\mathrm{d}\theta \\ &\lesssim e^{\frac{d+|k|}{\alpha}} \int_{\mathbb{R}^d} e^{-(e^{-1}-\frac{\rho_{n+1}}{a_{n+1}})|\theta|^\alpha}(a_{n+1}^{-(2\wedge a)/\alpha}|\theta|^{2\wedge a}+\frac{\rho_{n+1}}{a_{n+1}}|\theta|^\alpha)|\theta|^{|k|}\mathrm{d}\theta \,.\end{aligned}$$

Observing that $\frac{\rho_n}{a_n} \lesssim a_n^{-\varepsilon_0}$ and $\sum_n a_n^{-\varepsilon} < \infty$ for any $\varepsilon > 0$, the above estimate implies (3.16).

Finally, combining (3.14) and (3.15) yields

$$\lim_{t\to\infty} t^{\frac{d+|k|}{\alpha}} X_{\rho(t)}(\partial^k p_{t-\rho(t)}) = \frac{i^{|k|}}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|\theta|^\alpha}\theta^k \mathrm{d}\theta \,.$$

The equality (3.11) follows from here, after observing that $X_{\rho(t)}(\partial^k p_{t-\rho(t)})$ is a real number. □

Proof of Theorem 1.2. We are going to verify the hypotheses in Proposition 2.8. As we have seen previously, the identity (3.4) verifies condition (2.7) with $c(t) = t^{(N+d)/\alpha}$ and L_t defined in (3.9). We choose $\rho(t) = t^\kappa$ and $t_n = n^\delta$ with $\kappa, \delta \in (0,1)$ such that

$$\frac{N+d}{\alpha} + 1 + \varepsilon_0 > \frac{1}{\delta} > \frac{N+d}{\alpha} + 1 \quad \text{and} \quad \left(\frac{2N+d}{\alpha} + 1 + \varepsilon_0\right)\kappa > \frac{N}{\alpha} + \frac{1}{\delta}\,. \tag{3.20}$$

It is easy to verify conditions (2.8), (2.19) and (2.22). To check the condition (2.18), we note that $\|T_t f^2\|_\infty \lesssim t^{-d/\alpha}\|f\|^2_{L^2}$. So we need to verify that

$$\sum_{n=1}^\infty n^{\delta\frac{N}{\alpha}} \int_{n^{\kappa\delta}}^{n^\delta} \frac{\mathrm{d}s}{\phi(s)} < \infty \,.$$

By Tonelli's theorem, the left-hand side above is at most a constant multiple of

$$\int_1^\infty s^{\frac{1}{\kappa}\frac{N}{\alpha}+\frac{1}{\kappa\delta}} \frac{\mathrm{d}s}{\phi(s)} \,.$$

The ranges of κ, δ chosen in (3.20) ensures that $\frac{2N+d}{\alpha} + 1 + \varepsilon_0 > \frac{1}{\kappa}\frac{N}{\alpha} + \frac{1}{\kappa\delta}$. Hence, the above integral is finite due to (1.7) and we have verified condition (2.18). The condition (2.23) is verified analogously. Finally, we verify (2.21). The assumption (1.9) ensures that $f \in \mathcal{D}(A)$ and

$$|Af(x)| = \frac{1}{(2\pi)^d}\left|\int_{\mathbb{R}^d} \hat{f}(\xi)e^{ix\cdot\xi}|\xi|^\alpha d\xi\right| \le \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\hat{f}(\xi)||\xi|^\alpha d\xi \,.$$

It follows that

$$c_n \sup_{t\in[t_n,t_{n+1}]} \|T_{t_{n+1}-t}f - f\|_\infty \lesssim c_n(t_{n+1}-t_n) \lesssim n^{\delta\frac{N+d}{\alpha}+\delta-1}$$

and, hence, (2.21) is satisfied because of our assumption on the range of δ in (3.20). Therefore, applying Proposition 2.8, we find that (2.24) is valid with $c(t) = t^{\frac{N+d}{\alpha}}$ and L_t defined by (3.9).

In particular, we have

$$\lim_n \sup_{t\in[t_n,t_{n+1}]} t^{\frac{N+d}{\alpha}} \left| X_t(f) - \sum_{|k|\le N} \frac{(-1)^{|k|}}{k!} \int_{\mathbb{R}^d} f(y)y^k \mathrm{d}y X_{\rho(t_n)}(\partial^k p_{t_n-\rho(t_n)}) \right| = 0\,.$$

The long-time limit of $X_{\rho(t_n)}(\partial^k p_{t_n-\rho(t_n)})$ is given by Proposition 3.2. This implies (1.10). □

Proof of Theorem 1.1. The class of functions $C^2_c(\mathbb{R}^d)$ strongly separates points in the sense of Ethier and Kurtz [EK86]. From [BK10, Lemma 2], there exists a countable subset $\mathcal{M}$ of $C^2_c(\mathbb{R}^d)$ which strongly separates points and is closed under multiplication. Set $\widetilde{\mathcal{M}} = \{e^{-\varepsilon|\cdot|^2}f : f \in \mathcal{M}, \varepsilon > 0\}$. By Theorem 1.2, we see that with $\mathbb{P}^\phi_\mu$-probability one, (1.10) with $N = 0$ holds for every $f \in \widetilde{\mathcal{M}}$. An application of [KR14, Lemma 7] implies that with $\mathbb{P}^\phi_\mu$-probability one, (1.10) with $N = 0$ holds for every continuous functions g such that $e^{\varepsilon|\cdot|^2}g$ is bounded for some $\varepsilon > 0$. This yields almost-sure shallow convergence of $t^{\frac{d}{\alpha}}X_t$ to $\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|\theta|^\alpha}\mathrm{d}\theta\, \lambda_d$ as $t \to \infty$. □

4. Occupation times of stable Fleming-Viot processes

Let $(X_t)_{t\ge0}$ be the (α, ϕ) Fleming-Viot superprocess and $(\mathbb{X}_t)_{t\ge0}$ be the corresponding (α, ϕ) Fleming-Viot historical process with martingale measure $\mathbb{M}$. Then, we established the occupation time process Y and the inhabitation time process Z for X are connected through $Z_t(f) - Y_t(f) = \mathbb{M}(\ell_f)$, where ℓ_f is defined in (1.12). (See Theorem 1.4**a** and Proposition 2.14.) Using this $Z - Y$ relation and the method described in Subsection 2.2, we are able to obtain long term asymptotics of both time processes. As we saw earlier at the beginning of Section 3, the (α, ϕ) Fleming-Viot superprocess can be recovered from the (α, ϕ) Fleming-Viot historical process so we need only consider one probability measure, $\mathbb{P}_{0,\mu^*}$, which we relabel $\mathbb{P}_\mu$ to ease notation. Recall that $\mathcal{N}_d$ is defined in (1.20) and μ is a probability measure on $\mathbb{R}^d$. The following result, whose proof is presented in Subsection 4.2, is the key step in showing Theorem 1.5.

Proposition 4.1. *Assume that ϕ satisfies (1.22). Let f be a function in $b\mathcal{E}(\mathbb{R}^d)$ such that $\mathcal{N}_d(f) < \infty$. Then, the following assertions hold $\mathbb{P}^\phi_\mu$-a.s.*

(i) (Low and critical dimensions, $d \le \alpha$)

$$\lim_{t\to\infty} \frac{Y_t(f)}{\gamma_d(t)} = \lim_{t\to\infty} \frac{Z_t(f)}{\gamma_d(t)} = \varkappa_d(\alpha) \int_{\mathbb{R}^d} f(x)\mathrm{d}x\,. \tag{4.1}$$

(ii) (High dimension, $d > \alpha$) The limits $Y_\infty(f) := \lim_{t\to\infty} Y_t(f)$, $Z_\infty(f) := \lim_{t\to\infty} Z_t(f)$ and $\mathbb{M}_\infty(\ell_f) := \lim_{t\to\infty} \mathbb{M}_t(\ell_f)$ exist and are finite random variables. In addition, we have the following relation

$$Z_\infty(f) = \mathbb{M}_\infty(\ell_f) + Y_\infty(f)\,.$$

Remark 4.2. The condition $\mathcal{N}_d(f) < \infty$ ensures that $\int_0^t T_s f(x)\mathrm{d}s$ is finite for every $t > 0$ and $x \in \mathbb{R}^d$. This can be seen from the following identity, which is a consequence of (3.3),

$$\int_0^t T_s f(x)\mathrm{d}s = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\theta\cdot x}\frac{1-e^{-t|\theta|^\alpha}}{|\theta|^\alpha}\hat{f}(\theta)\mathrm{d}\theta\,. \tag{4.2}$$

Indeed, when $d < \alpha$, $\frac{1-e^{-t|\theta|^\alpha}}{|\theta|^\alpha}$ is integrable over $\mathbb{R}^d$, then the right-hand side above is bounded above by a multiple constant of $\|f\|_{L^1(\mathbb{R}^d)}$. When $d \geq \alpha$, $\frac{1-e^{-t|\theta|^\alpha}}{|\theta|^\alpha}$ is not integrable as $|\theta| \to \infty$. However, the right-hand side of (4.2) is finite if $\int_{\mathbb{R}^d} |\hat{f}(\theta)||\theta|^{-\alpha}\mathrm{d}\theta$ is finite. The finiteness of $\int_{\mathbb{R}^d} |\hat{f}(\theta)||\theta|^{-\alpha}\mathrm{d}\theta$ is also necessary to control $\int_0^1 T_s f(x)\mathrm{d}s$ when $d = \alpha$.

The following lemma will be useful later.

Lemma 4.3. *Let f be a function in $b\mathcal{E}(\mathbb{R}^d)$ with $\mathcal{N}_d(f) < \infty$.*

(i) If $d \leq \alpha$, then for every $x \in \mathbb{R}^d$,

$$\lim_{t\to\infty} \frac{1}{\gamma_d(t)} \int_0^t T_s f(x)\mathrm{d}s = \varkappa_d(\alpha)\lambda_d(f)\,, \tag{4.3}$$

where we recall that $\varkappa_d$ is defined in (1.21) and λ_d is the Lebesgue measure on $\mathbb{R}^d$.

(ii) If $d > \alpha$, then

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^d} \left| \int_0^t T_s f(x)\mathrm{d}s - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\theta} \hat{f}(\theta)|\theta|^{-\alpha}\mathrm{d}\theta \right| = 0\,. \tag{4.4}$$

Proof. Consider first the case $d < \alpha$. From (4.2), we have

$$\int_0^t T_s f(x)\mathrm{d}s = t^{1-\frac{d}{\alpha}} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it^{-1/\alpha}\theta\cdot x} \frac{1-e^{-|\theta|^\alpha}}{|\theta|^\alpha} \hat{f}(t^{-\frac{1}{\alpha}}\theta)\mathrm{d}\theta\,. \tag{4.5}$$

Using the facts that $\int_{\mathbb{R}^d} \frac{1-e^{-|\theta|^\alpha}}{|\theta|^\alpha}\mathrm{d}\theta$ is integrable and $\lim_{t\to\infty} \hat{f}(t^{-1/\alpha}\theta) = \hat{f}(0) = \lambda_d(f)$, we can derive (4.3) from the dominated convergence theorem.

The case $d = \alpha$ is a bit more subtle. From (3.3), we have

$$\int_1^t T_s f(x)\mathrm{d}s = \frac{1}{(2\pi)^d} \int_1^t \int_{\mathbb{R}^d} e^{ix\cdot\theta - s|\theta|^d} \hat{f}(\theta)\mathrm{d}\theta\mathrm{d}s = \frac{1}{(2\pi)^d} \int_1^t \int_{\mathbb{R}^d} e^{is^{-1/d}x\cdot\theta - |\theta|^d} \hat{f}(s^{-\frac{1}{d}}\theta)\mathrm{d}\theta\frac{\mathrm{d}s}{s}\,,$$

which implies

$$\left\| \int_1^u T_s f\mathrm{d}s \right\|_\infty \lesssim \ln(u)|\hat{f}(0)| \quad \forall u \geq 1\,. \tag{4.6}$$

Now, let ε be a positive number and choose $K > 0$ such that

$$\int_{|\theta|>K} e^{-|\theta|^d}\mathrm{d}\theta \leq \varepsilon$$

and then $u > 1$ such that

$$\sup_{s\geq u} \sup_{|\theta|\leq K} |e^{is^{-1/d}x\cdot\theta} \hat{f}(s^{-\frac{1}{d}}\theta) - \hat{f}(0)| \leq \varepsilon\,.$$

Such a choice is always possible because of the continuity of $\hat{f}$ at 0. It follows that

$$\begin{aligned}
&\left\| \int_u^t T_s f(x)\mathrm{d}s - \hat{f}(0) \int_u^t \int_{\mathbb{R}^d} e^{-|\theta|^d}\mathrm{d}\theta\frac{\mathrm{d}s}{s} \right\|_\infty \\
&\leq \frac{1}{(2\pi)^d} \left(\int_u^t \int_{|\theta|\leq K} + \int_u^t \int_{|\theta|>K} \right) e^{-|\theta|^d} |e^{is^{-1/d}x\cdot\theta} \hat{f}(s^{-\frac{1}{d}}\theta) - \hat{f}(0)|\mathrm{d}\theta\frac{\mathrm{d}s}{s} \\
&\lesssim \varepsilon \int_{\mathbb{R}^d} e^{-|\theta|^d}\mathrm{d}\theta \ln\left(\frac{t}{u}\right) + \varepsilon|\hat{f}(0)| \ln\left(\frac{t}{u}\right).
\end{aligned}$$

Combining with (4.6), one has that

$$\limsup_{t\to\infty}\frac{1}{\ln t}\left\|\int_1^t T_sf(x)\mathrm{d}s-\hat f(0)\int_1^t\int_{\mathbb{R}^d}e^{-|\theta|^d}\mathrm{d}\theta\frac{\mathrm{d}s}{s}\right\|_\infty\lesssim\varepsilon\,.$$

Sending $\varepsilon\to0$, we obtain

$$\lim_{t\to\infty}\frac{1}{\ln t}\int_1^t T_sf(x)\mathrm{d}s=\varkappa_d(\alpha)\lambda_d(f)\,.$$

Finally, since $|\int_0^1 T_sf(x)\mathrm{d}s|\lesssim\int_{\mathbb{R}^d}|\hat f(\theta)||\theta|^{-\alpha}\mathrm{d}\theta$, which is finite, the above implies (4.3).

In case $d>\alpha$, from (4.2), we have

$$\int_0^t T_sf(x)\mathrm{d}s-\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{ix\cdot\theta}\hat f(\theta)|\theta|^{-\alpha}\mathrm{d}\theta=\frac{-1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{ix\cdot\theta-t|\theta|^\alpha}\hat f(\theta)|\theta|^{-\alpha}\mathrm{d}\theta\,.$$

Hence,

$$\sup_{x\in\mathbb{R}^d}\left|\int_0^t T_sf(x)\mathrm{d}s-\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{ix\cdot\theta}\hat f(\theta)|\theta|^{-\alpha}\mathrm{d}\theta\right|\le\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-t|\theta|^\alpha}|\hat f(\theta)||\theta|^{-\alpha}\mathrm{d}\theta\,,$$

which together with the dominated convergence theorem implies (4.4). □

From now on, we assume that f is a bounded measurable function on $\mathbb{R}^d$ such that $\mathcal{N}_d(f)$ is finite. From the proof of Lemma 4.3, it follows that in every dimension,

$$\left\|\int_0^t T_sf\mathrm{d}s\right\|_\infty\lesssim\mathcal{N}_d(f)(\gamma_d(t)\vee1)\quad\forall t\ge0\,.\tag{4.7}$$

By the homogeneous Markov property of ξ, we also have

$$\sup_{x\in\mathbb{R}^d}\left|P_x\left(\int_0^t f(\xi_u)\mathrm{d}u\right)^2\right|=2\sup_{x\in\mathbb{R}^d}\left|\int_0^t\int_0^{t-u}T_u[fT_sf](x)\mathrm{d}s\mathrm{d}u\right|\lesssim\mathcal{N}_d^2(f)(\gamma_d(t)\vee1)^2\tag{4.8}$$

for every $t\ge0$.

4.1. **Martingale corrector.** We investigate the long time limit of the martingale difference $\mathbb{M}_t(\ell_f)$. For each $q>1$ and $n\in\mathbb{N}_0$, define

$$t_n=t_n(q)=\begin{cases}q^{\frac{\alpha}{\alpha-d}n} & \text{if } d<\alpha\\ e^{q^n} & \text{if } d=\alpha\end{cases}\quad\text{so that}\quad\gamma_d(t_n)=q^n\,.\tag{4.9}$$

Proposition 4.4. *Let f be a bounded measurable function on $\mathbb{R}^d$ such that $\mathcal{N}_d(f)<\infty$. Then, (i) $\mathbb{M}_t(\ell_f)$ converges $\mathbb{P}_\mu^\phi$-a.s. and in $L^2(\Omega)$ as $t\to\infty$ if $\int_0^\infty\frac{\gamma_d^2(s)}{\phi(s)}\mathrm{d}s<\infty$.*
(ii) $\lim_{t\to\infty}\frac{\mathbb{M}_t(\ell_f)}{\gamma_d(t)}=0$ $\mathbb{P}_\mu^\phi$-a.s. if (1.22) holds.

Proof. (i) By martingale convergence theorem, it suffices to show

$$\sup_{t\ge0}\mathbb{P}_\mu^\phi[\mathbb{M}_t(\ell_f^t)^2]<\infty\,.\tag{4.10}$$

Indeed, from (2.44) and (2.36) we have that

$$\mathbb{P}_\mu^\phi[\mathbb{M}_t(\ell_f^t)^2]\le\mathbb{P}_\mu^\phi\int_0^t\mathbb{X}_s((\ell_f^s)^2)\frac{ds}{\phi(s)}=\int_0^t\langle\mathbb{T}_s((\ell_f^s)^2),\delta_0\times m\rangle\frac{ds}{\phi(s)}\,.$$

We observe that for every path $\omega \in D(\mathbb{R}^d)$

$$\ell^s_f(r+s,\omega^{r+s}) = \ell^s_f(r,\omega^r) + \mathbf{1}_{(r<s)} \int_0^{s-r} f(\omega_{r+u})du\,.$$

Thus,

$$(\ell^s_f(r+s,\omega^{r+s}))^2 \le 2(\ell^s_f(r,\omega^r))^2 + \mathbf{1}_{(r<s)} 2 \left(\int_0^{s-r} f(\omega_{r+u})du\right)^2 .$$

Together with (4.8), this implies that

$$\begin{aligned}\mathbb{T}_s(\ell^s_f)^2(r,y) &= P_{y_r}\left[(\ell^s_f(r+s,(y\ltimes_r \xi)^{r+s}))^2\right]\\ &\le 2(\ell^s_f(r,y))^2 + \mathbf{1}_{(r<s)} 2P_{y_r}\left(\int_0^{s-r} f(\xi_u)du\right)^2\\ &\lesssim r^2\|f\|^2_\infty + \mathbf{1}_{(r<s)}\mathcal{N}^2_d(f)(\gamma_d(s-r)\vee 1)^2\,. \end{aligned} \tag{4.11}$$

Therefore, we have

$$\int_0^t \langle \mathbb{T}_s(\ell^s_f)^2, \delta_0\times m\rangle \frac{ds}{\phi(s)} \lesssim \mathcal{N}^2_d(f)\int_0^t (\gamma_d(s)\vee 1)^2 \frac{ds}{\phi(s)}\,,$$

which is uniformly bounded in t by our assumptions on f and ϕ. The estimate (4.10) and the convergence of $\mathbb{M}_t(\ell^t_f)$ follow.

(ii) Let $\{t_n\}$ be the sequence defined in (4.9). It suffices to show that

$$\sum_n \frac{1}{\gamma^2_d(t_n)} \mathbb{P}^\phi_\mu\left[\left(\sup_{t\in[t_{n-1},t_n]} \mathbb{M}_t(\ell^t_f)\right)^2\right] < \infty\,.$$

By martingale maximal inequality and the computations in the previous case, we see that

$$\mathbb{P}^\phi_\mu\left[\left(\sup_{t\in[t_{n-1},t_n]} \mathbb{M}_t(\ell^t_f)\right)^2\right] \lesssim \mathbb{P}^\phi_\mu\left[\left(\mathbb{M}_{t_n}(\ell^{t_n}_f)\right)^2\right] \lesssim \int_0^{t_n} (\gamma_d(s)\vee 1)^2 \frac{\mathrm{d}s}{\phi(s)}\,.$$

It remains to show that

$$\sum_n \frac{1}{\gamma^2_d(t_n)} \int_0^{t_n} (\gamma_d(s)\vee 1)^2 \frac{\mathrm{d}s}{\phi(s)} < \infty\,. \tag{4.12}$$

Since $\gamma_d(t_n) = q^n$, $\sum_n q^{-2n} < \infty$ and $\int_0^1 (\gamma_d(s)\vee 1)^2 \frac{\mathrm{d}s}{\phi(s)} < \infty$, we can replace 0 in the lower limit of each integral above by 1. Consider the case $d<\alpha$. Interchanging the order of summation and integration, we see that

$$\sum_n \frac{1}{\gamma^2_d(t_n)} \int_1^{t_n} (\gamma_d(s)\vee 1)^2 \frac{\mathrm{d}s}{\phi(s)} \lesssim \int_1^\infty \sum_{n:\ q^n > s^{1-\frac{d}{\alpha}}} \frac{1}{q^{2n}} (\gamma_d(s)\vee 1)^2 \frac{\mathrm{d}s}{\phi(s)} \lesssim \int_1^\infty \frac{\mathrm{d}s}{\phi(s)}\,.$$

In the second estimate above, we use $\sum_{n:\ q^n>s^{1-\frac{d}{\alpha}}} \frac{1}{q^{2n}} \lesssim \frac{1}{\gamma^2(s)}$. It is straightforward to verify that in the case $d=\alpha$, we have the same estimate. That is

$$\sum_n \frac{1}{\gamma^2_d(t_n)} \int_1^{t_n} (\gamma_d(s)\vee 1)^2 \frac{\mathrm{d}s}{\phi(s)} \lesssim \int_1^\infty \frac{\mathrm{d}s}{\phi(s)}\,.$$

The integral on the right-hand side above is finite by our assumption. Hence, (4.12) follows and so does the result. ☐

4.2. **Limit theorems for occupation times.** We present the proofs of Proposition 4.1 and Theorem 1.5.

Proof of Proposition 4.1(ii). Without loss of generality, we assume that f is non-negative. The process $Y_t(f)$ is nonnegative and increasing. Hence, the limit $\lim_{t\to\infty} Y_t(f)$ exists. In addition, using Tonelli's theorem, (2.15) and (4.4), we have

$$\lim_{t\to\infty}\mathbb{P}^\phi_\mu Y_t(f)=\lim_{t\to\infty}\int_0^t\mu(T_sf)\mathrm{d}s=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mu(\mathrm{e}_\theta)\hat f(\theta)|\theta|^{-\alpha}\mathrm{d}\theta\,.$$

Hence, by Fatou's lemma and the fact that $\mathcal{N}_d(f)<\infty$,

$$\mathbb{P}^\phi_\mu\lim_{t\to\infty}Y_t(f)\le\lim_{t\to\infty}\mathbb{P}^\phi_\mu Y_t(f)<\infty\,.$$

It follows that $\lim_{t\to\infty} Y_t(f)$ is a finite random variable. From Proposition 4.4, the limit $\lim_{t\to\infty}\mathbb{M}_t(\ell^t_f)$ exists and is a finite random variable. Together with the relation (2.43), these observations imply Proposition 4.1(ii). ☐

Proof of Proposition 4.1(i). Without loss of generality, we can assume $f\ge 0$. Let q be at least 1 and $\{t_n\}=\{t_n(q)\}$ be the sequence defined in (4.9).
Step 1. Reduce to subsequence convergence: Suppose that

$$\lim_n\frac{Y_{t_n(q)}(f)}{\gamma_d(t_n(q))}=\varkappa_d(\alpha)\int_{\mathbb{R}^d}f(x)\mathrm{d}x\quad\text{a.s.}\tag{4.13}$$

for all $q>1$. For every $t\in[t_n,t_{n+1})$, by monotonicity of $Y_t(f)$, we see that

$$\frac1q\lim_n\frac{Y_{t_n}(f)}{\gamma_d(t_n)}\le\liminf_t\frac{Y_t(f)}{\gamma_d(t)}\le\limsup_t\frac{Y_t(f)}{\gamma_d(t)}\le q\lim_n\frac{Y_{t_{n+1}}(f)}{\gamma_d(t_{n+1})}\,.$$

By sending $q\downarrow 1$, one has $\lim_{t\to\infty}\frac{Y_t(f)}{\gamma_d(t)}=\varkappa_d(\alpha)\int_{\mathbb{R}^d}f(x)\mathrm{d}x$. Now, Proposition 4.4 (ii) implies (4.1).
Step 2. Reduce to $\mu\left(\int_0^{t_n}T_sf\,\mathrm{d}s\right)$: From Lemma 2.6 and (4.7), we have

$$\mathbb{P}^\phi_\mu\left|\int_0^{t_n}X_s(f)\mathrm{d}s-\mu\left(\int_0^{t_n}T_sf\mathrm{d}s\right)\right|^2\lesssim\mathcal{N}_d^2(f)\int_0^{t_n}\gamma_d^2(s)\frac{\mathrm{d}s}{\phi(s)}\,.$$

It follows that

$$\sum_{n=1}^\infty\frac{1}{\gamma_d^2(t_n)}\mathbb{P}^\phi_\mu\left|\int_0^{t_n}X_s(f)\mathrm{d}s-\mu(\int_0^{t_n}T_sf\mathrm{d}s)\right|^2\lesssim\sum_{n=1}^\infty\frac{1}{q^{2n}}\int_0^{t_n}\gamma_d^2(s)\frac{\mathrm{d}s}{\phi(s)}\,.$$

The series on the right-hand side above appeared earlier in (4.12). The same reasoning as in the proof of Proposition 4.4 shows that the above series is finite under condition (1.22). Hence, the Borel-Cantelli lemma implies

$$\lim_n\frac{1}{\gamma_d(t_n)}\left|\int_0^{t_n}X_s(f)\mathrm{d}s-\mu\left(\int_0^{t_n}T_sf\mathrm{d}s\right)\right|=0\,.$$

Step 3. From Lemma 4.3, (4.6) and dominated convergence theorem, we deduce that

$$\lim_n \frac{1}{\gamma_d(t_n)}\mu\left(\int_0^{t_n} T_s f \mathrm{d}s\right) = \varkappa_d(\alpha)\lambda_d(f)\,.$$

Combining previous steps, one has the result. □

Proof of Theorem 1.5. We note that each function in $C_c^2(\mathbb{R}^d)$ satisfies the hypotheses of Proposition 4.1. Therefore, by an analogous argument as in the proof of Theorem 1.1 on page 24, we can easily deduce Theorem 1.5 from Proposition 4.1. We omit the details. □

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