A Strong Law of Large Numbers for Super-stable Processes

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Abstract

Let ℓ be Lebesgue measure and $X = (X_t, t \geq 0; P_\mu)$ be a supercritical, super-stable process corresponding to the operator $-(-\Delta)^{\alpha/2} u + \beta u - \eta u^2$ on $\mathbb{R}^d$ with constants $\beta, \eta > 0$ and $\alpha \in (0, 2]$. Put $\hat{W}_t(\theta) = e^{(|\theta|^\alpha - \beta)t} X_t(e^{-i\theta\cdot})$, which for each *small* θ is an a.s. convergent complex-valued martingale with limit $\hat{W}(\theta)$ say. We establish for any starting finite measure μ satisfying $\int_{\mathbb{R}^d} |x|\mu(\mathrm{d}x) < \infty$ that $\frac{t^{d/\alpha} X_t}{e^{\beta t}} \to c_\alpha \hat{W}(0)\,\ell$ P_μ-a.s. in a topology, termed the shallow topology, strictly stronger than the vague topology yet weaker than the weak topology, where $c_\alpha > 0$ is a known constant. This result can be thought of as an extension to a class of superprocesses of Watanabe's strong law of large numbers for branching Markov processes.

Key words: Super-stable process, Super-Brownian motion, Strong law of large numbers, Fourier Transform, Vague convergence, Probability measures

2010 MR Subject Classification: 60J68, 60F15, 60G57

1 Introduction

We use $M_F(\mathbb{R}^d)$ to denote the set of finite measures on $\mathbb{R}^d$. We use $\mu(f)$ to denote $\int f d\mu$ for a measure μ and integrable function f. It is clear that $\mu(D) = \mu(I_D)$, where I_D is the indicator function of D. Let $C_c(\mathbb{R}^d)$ denote the set of continuous functions on $\mathbb{R}^d$ with compact support.

In 1967, Watanabe [28] first discussed the strong law of large numbers for branching Brownian motion. Let $(X_t, t \geq 0; P_x)$ be a branching Brownian motion on $\mathbb{R}^d$ ($d \geq 1$) starting from a single point $x \in \mathbb{R}^d$ and corresponding to the operator

$$\frac{1}{2}\triangle u + a(F(u) - u),$$

where a is a positive constant and $F(s) := \sum_{n=0}^{\infty} p_n s^n$, $s \geq 0$, is the generating function of the offspring distribution $\{p_n, n \geq 0\}$. By explicitly using the Gaussian density, Watanabe [28] proved in the supercritical case, i.e. $\beta := a(F'(1) - 1) > 0$, that under the condition $\sum_{n=0}^{\infty} n^2 p_n < \infty$, it follows that

$$\frac{X_t}{e^{\beta t} t^{-d/2}} \to (2\pi)^{-d/2} \ell \cdot W, \qquad P_x - \text{a.s.} \tag{1}$$

as $t \to \infty$ in the sense of vague convergence, where ℓ is the Lebesgue measure on $\mathbb{R}^d$ and W is the limit of the martingale $W_t := e^{-\beta t} X_t(1)$. Later, based on the ideas in [28], Biggins [2] proved a strong law of large numbers for discrete-time branching random walk.

Suppose $(X_t, t \geq 0; P_\mu)$ is a super-Brownian motion on $\mathbb{R}^d$, $d \geq 1$, corresponding to operator $\frac{1}{2}\triangle u + \beta u - \eta u^2$, where $\beta > 0$ and $\eta > 0$ are positive constants, and starting from $\mu \in M_F(\mathbb{R}^d)$. Then, it seems that Englander [11] was the first to discuss the law of large numbers for the

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supercritical super-Brownian motion $(X_t, t \geq 0; P_\mu)$. It was proved in [11] that for any $f \in C_c(\mathbb{R}^d)$,

$$\frac{X_t(f)}{e^{\beta t} t^{-d/2}} \to (2\pi)^{-d/2} \ell(f) \cdot W, \qquad \text{in } P_\mu\text{-probability}, \tag{2}$$

where W is the limit of the martingale $W_t := e^{-\beta t} X_t(1)$. More recently, Wang [27] improved the convergence in (2) from "in probability" to "P_μ-a.s." in the special case that $\mu = \delta_x$, $x \in \mathbb{R}^d$ by combining the Fourier analysis used [28] and the uniform convergence method for martingales used in [2]. Wang's proof depends on the specific density of Brownian motion and the compact support property of super-Brownian motion starting from a compactly supported measure. For more path properties of super-Brownian motion, see Dawson, Iscoe and Perkins [8], Dawson and Perkins [10], and Perkins [24], [25]. But, α-stable processes ($\alpha \in (0,2)$) do not have specific density expressions. More critically, for any $t > 0$, the support of X_t, the super-stable process with index $\alpha \in (0,2)$, is the whole space $\mathbb{R}^d$ even when the starting measure μ has compact support (see Dawson and Perkins [10] or Perkins [25]). Therefore, the methods in Wang [27] do not transfer over to general $\mu \in M_F(\mathbb{R}^d)$ nor to super-stable process with index $\alpha \in (0,2)$.

Note that both for branching Brownian motion and super-Brownian motion, the mean of X_t is described by the linear operator $\frac{1}{2}\triangle + \beta$ on $\mathbb{R}^d$. The denominator $e^{\beta t} t^{-d/2}$ in (1) and (2) is exactly the growth rate of $e^{\beta t} S_t^{\frac{1}{2}\Delta}$, the semigroup corresponding to $\frac{1}{2}\triangle + \beta$ on $\mathbb{R}^d$, as $t \to \infty$. In our more general α-stable case, corresponding to the operator $-(-\triangle)^{\frac{\alpha}{2}} + \beta$, it will again turn out that the correct scaling, $e^{\beta t} t^{-d/\alpha}$, is dictated by the growth rate of $e^{\beta t} S_t^{\triangle^\alpha}$, the semigroup corresponding to $-(-\triangle)^{\frac{\alpha}{2}} + \beta$.

If $\frac{1}{2}\triangle$ is replaced by a diffusion operator L with spatially dependent coefficients or more general operator and β is spatially dependent, the strong (or weak) law of large numbers for branching diffusion (or more general branching Hunt processes) and superdiffusion have been investigated recently by many papers. See [1] and [6] for branching diffusion, [12] for branching Hunt processes, and [5] [11] [14] and [15] and [23] (with general branching mechanism) for superdiffusions. In all of these papers, the mean of the process grows pure exponentially as $e^{\lambda_c t}$ with some positive constant λ_c, usually called the (generalized) principal eigenvalue. The techniques used in these papers can not be applied to handle the case when the mean of the process grows in the non-exponential manner $f(t)e^{\lambda_c t}$, where, for example, $f(t) = t^{-d/\alpha}$ as above.

In this paper, we will prove the strong law of large numbers for super-stable processes with index $\alpha \in (0, 2]$ corresponding to the operator

$$-(-\Delta)^{\alpha/2} u + \beta u - \eta u^2,$$

where β and η are positive constants. In the special case $\alpha = 2$, our results extend the main result Theorem 3.2 in [27]. In particular, we extend the starting measure δ_x, $x \in \mathbb{R}^d$, in [27] to any finite μ on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} |x| \mathrm{d}\mu < \infty$, and the test function $f \in C_c(\mathbb{R}^d)$ in [27] to more general ones (see Theorem 4 below), and moreover, we improve Wang's result from one specific f to shallow convergence (see Theorem 8 below), which implies vague convergence. Our proof depends mainly on Fourier analysis and stochastic calculations, advancing the methods introduced in [3] in the discussion of Hölder continuity for general measure-valued Markov processes including superprocesses. We incorporate the core ideas of Watanabe [28] and one could consider our main contribution as showing that these original ideas carry over to superdiffusions. Still, it should be mentioned that our developments are simpler and more extendable than those in [28], [2] and [27]. Indeed, based upon the fundamental role of the Fourier transform in pde and our initial investigation we believe that our methods can be extended to more general operators and branching mechanisms.

The spine method recently developed for measure-valued Markov processes is a powerful probabilistic tool in studying properties of the processes, see [11], [12], [13] [17] and [22] (to list a few but not all). Englander, Harris and Kyprianou [12] used the martingale change of measure and spine decomposition to prove the SLLN for branching diffusions. Their proof depends on how the support of branching diffusion expands (see condition (iii) on page 282 of [12]). But as mentioned above, the support of a super-stable process with index $\alpha \in (0,2)$ expands to the whole space $\mathbb{R}^d$ immediately, so we can not expect to extend the method in [12] to superprocesses with general underlying processes, like α-stable process. The purpose of this paper is to generalize Watanabe's results in [27] from discrete particle systems to superprocesses using techniques from Fourier transform theory and stochastic calculations. We emphasize that we consider all $\alpha \in (0,2]$ and do not assume our starting measure has compact support. Our only assumption on μ is that $\int_{\mathbb{R}^d} |x|\mu(\mathrm{d}x) < \infty$.

2 Notation and Model

Recall that we use $\mu(f)$ to denote $\int f \mathrm{d}\mu$ for a measure μ and integrable function f. For simplicity, we let $\mu_r = \int |x|^r \mu(\mathrm{d}r)$ and $\cos_\theta$ denote the function $x \to \cos(\theta \cdot x)$ below. We also use the following extended Vinogradov symbol (also used in [19]): Suppose $a(n,m), b(n,m)$ are expressions depending upon two sets of variables n, m. Then,

$$a(n,m) \overset{n}{\ll} b(n,m) \text{ means } \exists\ c_m > 0 \text{ such that } a(n,m) \leq c_m b(n,m) \quad \forall\ n, m.$$

For clarity, c_m depends only on m. We will use this extended Vinogradov symbol with various parameters below not just n, m. For example,

$$a(\lambda,\theta) \overset{\lambda,\theta}{\ll} b(\lambda,\theta) \text{ means } \exists\ c > 0 \text{ such that } a(\lambda,\theta) \leq c b(\lambda,\theta) \quad \forall\ r, \theta,$$

where c is a constant which does not depend on λ and θ.

Throughout this paper, we assume $\mu \in M_F\left(\mathbb{R}^d\right)$ such that $\mu_0, \mu_1 < \infty$. We consider the measure-valued Markov process $X = (X_t, t \geq 0; P_\mu)$ on $\mathbb{R}^d$ such that

$$X_t(f) = \mu(f) + \int_0^t X_s\left(\left(-(-\Delta)^{\alpha/2} + \beta\right) f\right) \mathrm{d}s + M_t(f) \tag{3}$$

for all f bounded and continuous functions with bounded and continuous partial derivatives of order $k \leq 2$, where $M_t(f)$ is a martingale with quadratic variation

$$[M(f)](t) = \int_0^t X_s\left(\eta f^2\right) \mathrm{d}s,$$

and $\eta > 0$ and $\beta > 0$ are positive constants. Note that X starts from μ, the particles move independently according to a symmetric α-stable process on $\mathbb{R}^d$ with generator $-(-\Delta)^{\alpha/2}$ with $\alpha \in (0, 2]$, and the branching mechanism is given by $\eta z^2 - \beta z$. Since $\beta > 0$, X is supercritical.

Substituting $f(x) = e^{-i\theta x}$ in (3) and letting $\hat{X}(t,\theta) = X_t(\cos_\theta) - iX_t(\sin_\theta)$, we get

$$\hat{X}(t,\theta) = \hat{X}(0,\theta) + \int_0^t (-|\theta|^\alpha + \beta)\hat{X}(s,\theta)\,\mathrm{d}s + \hat{M}(t,\theta) \tag{4}$$

for all $\theta \in \mathbb{R}^d$, where $\hat{M}(t,\theta)$ is a complex martingale with quadratic variations and covariations:

$$\left[\mathrm{Re}\ \hat{M}(\cdot,\theta)\right](t) = \int_0^t X_s\left(\eta \cos^2_\theta\right) \mathrm{d}s;$$

$$\left[\mathrm{Im}\ \hat{M}(\cdot,\theta)\right](t) = \int_0^t X_s\left(\eta \sin^2_\theta\right) \mathrm{d}s;$$

$$\left[\hat{M}(\cdot,0), \mathrm{Re}\ \hat{M}(\cdot,\theta)\right](t) = \int_0^t X_s(\eta \cos_\theta)\,\mathrm{d}s;$$

$$\left[\hat{M}(\cdot,0), \mathrm{Im}\ \hat{M}(\cdot,\theta)\right](t) = \int_0^t X_s(\eta \sin_\theta)\,\mathrm{d}s.$$

Using variations of constants, we get

$$\hat{X}(t,\theta) = e^{(\beta-|\theta|^\alpha)t}\hat{X}(0,\theta) + \int_0^t e^{(\beta-|\theta|^\alpha)(t-s)}\hat{M}(\mathrm{d}s,\theta). \tag{5}$$

Define

$$\hat{W}_t(\theta) = \hat{W}(t,\theta) = e^{(|\theta|^\alpha - \beta)t}\hat{X}(t,\theta) = e^{(|\theta|^\alpha-\beta)t}X_t(e^{-i\theta\cdot}). \tag{6}$$

Then, $\hat{W}(t,\theta)$ is a complex martingale for any $\theta \in \mathbb{R}^d$.

3 Results

Our first result describes the limiting object of our scaled super-stable process in frequency domain. It will be used in the subsequent results herein.

Theorem 1 *Suppose $\alpha \in (0,2]$ and $\kappa \in \left(0, \frac{\beta}{2}\right)$. Then, $\hat{W}_t(\theta)$ converges almost surely and in the mean-square sense to limit $\hat{W}(\theta)$ for each $\theta \in \mathbb{R}^d$ satisfying $|\theta|^\alpha < \frac{\beta}{2}$. Moreover, $\hat{W}(\theta)$ is jointly-measurable in ω and θ, and satisfies*

$$P_\mu\left[\left|\hat{W}(\lambda) - \hat{W}(\theta)\right|^2\right] \overset{\lambda,\theta}{\ll} |\theta-\lambda|^{1\wedge\alpha}. \tag{7}$$

for all $|\lambda|^\alpha, |\theta|^\alpha \le \kappa$.

Remark 1 *Previous authors (e.g. Biggins [2] and Wang [27]) developed clever methods to show uniform convergence (over the equivalent of $|\theta|^\alpha \le \kappa$ for some $\kappa < \frac{\beta}{2}$) of $\hat{W}_u(\theta)$ to $\hat{W}(\theta)$ in order to obtain a single null set such that $\hat{W}_u(\theta) \to \hat{W}(\theta)$ for all such θ on this null set. However, these methods do not apply to our super-stable process setting due to the lack of the compact support property and the authors are not even sure that this uniform convergence holds. Fortunately, there is no need for this single null set. Indeed, our proofs below show that it is enough to have a jointly measurable $(\omega,\theta) \to \hat{W}(\theta)$ such that $\hat{W}_u(\theta) \to \hat{W}(\theta)$ in the mean-square sense.*

Proof. Let $\epsilon = \beta - 2\kappa$. Note that $\hat{W}_t(\theta)$ and $\hat{W}_t(\theta) - \hat{W}_t(\lambda)$ are complex martingales with quadratic variations satisfying

$$\left[\mathrm{Re}\ \hat{W}(\theta)\right](t) = \int_0^t e^{2(|\theta|^\alpha - \beta)s} X_s \left(\eta \cos^2_\theta\right) \mathrm{d}s;$$

$$\left[\mathrm{Im}\ \hat{W}(\theta)\right](t) = \int_0^t e^{2(|\theta|^\alpha - \beta)s} X_s \left(\eta \sin^2_\theta\right) \mathrm{d}s;$$

$$\left[\mathrm{Re}\ (\hat{W}(\theta) - \hat{W}(\lambda))\right](t) = \int_0^t e^{-2\beta s} X_s \left(\eta (e^{|\theta|^\alpha s} \cos_\theta - e^{|\lambda|^\alpha s} \cos_\lambda)^2\right) \mathrm{d}s;$$

$$\left[\mathrm{Im}\ (\hat{W}(\theta) - \hat{W}(\lambda))\right](t) = \int_0^t e^{-2\beta s} X_s \left(\eta (e^{|\theta|^\alpha s} \sin_\theta - e^{|\lambda|^\alpha s} \sin_\lambda)^2\right) \mathrm{d}s.$$

By the martingale property of $\hat{W}_t(0) = e^{-\beta t}\hat{X}(t,0) = e^{-\beta t} X_t(1)$, we have for $0 \le u < t$ that

$$\begin{aligned} P_\mu\left[\left|\hat{W}_t(\theta) - \hat{W}_u(\theta)\right|^2\right] &= \int_u^t e^{2(|\theta|^\alpha - \beta)s} \eta P_\mu[X_s(1)] \mathrm{d}s \\ &= \eta\mu(1) \int_u^t e^{(2|\theta|^\alpha - \beta)s} ds \\ &= \begin{cases} \frac{\eta\mu(1)}{2|\theta|^\alpha - \beta}\left(e^{(2|\theta|^\alpha - \beta)t} - e^{(2|\theta|^\alpha - \beta)u}\right), & \text{if } 2|\theta|^\alpha \neq \beta, \\ \eta\mu(1)(t-u), & \text{if } 2|\theta|^\alpha = \beta. \end{cases} \end{aligned} \tag{8}$$

Therefore, letting $u = 0$, we find $0 < \sup_{t \ge 0} P_\mu\left[\left|\hat{W}_t(\theta)\right|^2\right] < \infty$ if $2|\theta|^\alpha < \beta$ (since $\mu_0 \doteq \mu(1)$ and $P_\mu\left[\left|\hat{X}(0,\theta)\right|^2\right] \le |\mu(\sin(\theta))|^2 + |\mu(\cos(\theta))|^2 \le \mu_0^2 < \infty$). An application of the martingale convergence theorem yields $\lim_{t\to\infty} \hat{W}_t(\theta)$ exists almost surely and in mean-square sense for each $\theta \in \mathbb{R}^d$ such that $|\theta|^\alpha < \frac{\beta}{2}$. We define a jointly measurable in ω, θ version of this limit via

$$\hat{W}(\theta) \doteq \begin{cases} \lim_{t\to\infty} \hat{W}_t(\theta) & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

Next, we show the Hölder continuity in mean property for $\hat{W}$. $\hat{W}_t(0)$ is a non-negative martingale starting at $\hat{X}(0,0) = \mu_0$ and satisfying

$$\left[\hat{W}(0)\right]_t = \int_0^t \eta e^{-\beta s} \hat{W}_s(0) \,\mathrm{d}s.$$

Hence, we have by the Burkholder-Davis-Gundy inequality that

$$
\begin{aligned}
& P_\mu\left[\sup_{u\geq 0}\left|\hat{W}_u(\lambda)-\hat{W}_u(\theta)-\hat{X}(0,\lambda)+\hat{X}(0,\theta)\right|^2\right] \\
\overset{\lambda,\theta}{\ll}\ & P_\mu\left[\left|\int_0^\infty e^{-2\beta s}X_s\left(e^{2|\lambda|^\alpha s}+e^{2|\theta|^\alpha s}-2e^{|\lambda|^\alpha s+|\theta|^\alpha s}\cos_{\theta-\lambda}\right)\mathrm{d}s\right|\right] \\
\overset{\lambda,\theta}{\ll}\ & \int_0^\infty\Big[e^{-\beta s}(e^{2|\lambda|^\alpha s}+e^{2|\theta|^\alpha s}-2e^{|\lambda|^\alpha s+|\theta|^\alpha s})P_\mu(\hat{W}_s(0)) \\
+\ & P_\mu\left|e^{(|\lambda|^\alpha+|\theta|^\alpha-2\beta)s}X_s\left(1-\cos_{\theta-\lambda}\right)\right|\Big]\mathrm{d}s \\
\overset{\lambda,\theta}{\ll}\ & \int_0^\infty e^{-\beta s}(e^{|\lambda|^\alpha s}-e^{|\theta|^\alpha s})^2\mathrm{d}s \\
+\ & \int_0^\infty e^{-\epsilon s}\left(\mu(1)-e^{-|\theta-\lambda|^\alpha s}P_\mu\left[\mathrm{Re}\hat{W}_s(\theta-\lambda)\right]\right)\mathrm{d}s,
\end{aligned} \tag{10}
$$

where in the last inequality we used the facts that $\epsilon=\beta-2\kappa$ and $|\lambda|^\alpha,|\theta|^\alpha\leq\kappa$. However,

$$
\begin{aligned}
\mu(1)-e^{-|\theta-\lambda|^\alpha s}P_\mu\left[\mathrm{Re}\hat{W}_s(\theta-\lambda)\right] &= e^{-|\theta-\lambda|^\alpha s}\mu\left(1-\cos_{\theta-\lambda}\right)+\mu(1-e^{-|\theta-\lambda|^\alpha s}) \\
&\leq \mu\left(1-\cos_{\theta-\lambda}\right)+|\theta-\lambda|^\alpha s\mu\left(1\right),
\end{aligned} \tag{11}
$$

$$
P_\mu\left[\left|\hat{X}(0,\lambda)-\hat{X}(0,\theta)\right|^2\right]\overset{\lambda,\theta}{\ll}\mu\left(1-\cos_{\theta-\lambda}\right), \tag{12}
$$

and it follows by Taylor's theorem that

$$
\left|1-\cos\left(\left(\theta-\lambda\right)x\right)\right|\leq|\theta-\lambda|\,|x|\,, \tag{13}
$$

and

$$
\left|e^{|\lambda|^\alpha s}-e^{|\theta|^\alpha s}\right|^2\overset{s,\lambda,\theta}{\ll}s^2e^{2\kappa s}\left(|\lambda|^\alpha-|\theta|^\alpha\right)^2\overset{s,\lambda,\theta}{\ll}s^2e^{2\kappa s}\left|\lambda-\theta\right|^\alpha \tag{14}
$$

since if $|\theta|>|\lambda|$, then $|\theta|^\alpha-|\lambda|^\alpha\leq\left(|\theta|^2-|\lambda|^2\right)^{\frac{\alpha}{2}}\leq 2\kappa^{\frac{\alpha}{2}}\left|\theta-\lambda\right|^{\frac{\alpha}{2}}$.

Substituting bounds (11)-(14) above into (10), we find by the Burkholder-Davis-Gundy inequality that

$$
P_\mu\left[\sup_{u\geq 0}\left|\hat{W}(u,\lambda)-\hat{W}(u,\theta)\right|^2\right]\overset{\lambda,\theta}{\ll}|\theta-\lambda|^\alpha+|\theta-\lambda|\,. \tag{15}
$$

and, letting $u \to \infty$, we get (7). ■

Next, we convert our "frequency domain" result to a SLLN for super-stable processes. Since both the limit and prelimit are measures, we introduce test functions f.

Theorem 2 *Suppose $\kappa \in \left(0, \frac{\beta}{2}\right)$ and f is such that its Fourier transform $\hat{f}$ exists and*

$$\hat{c} \doteq \int_{\mathbb{R}^d} e^{\epsilon|\theta|^\alpha} \left|\hat{f}(\theta)\right| \frac{\mathrm{d}\theta}{(2\pi)^d} < \infty \tag{16}$$

for $\epsilon = \beta - 2\kappa$. Then, for any $\delta \in (0, \beta\kappa - 2\kappa^2)$ there is a constant $c > 0$ and a random variable $C_\delta > 0$ such that

$$P_\mu\left[\max_{n\epsilon \le t \le (n+1)\epsilon} \left|\frac{X_t(f)}{e^{\beta t} t^{-\frac{d}{\alpha}}} - \int_{|\theta|^\alpha \le \kappa} e^{-t|\theta|^\alpha} \hat{W}(\theta) \hat{f}(\theta) \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right|\right] \le c\sqrt{n} e^{-\left(\beta\kappa - 2\kappa^2\right)n} \tag{17}$$

and

$$\left|\frac{X_t(f)}{e^{\beta t} t^{-\frac{d}{\alpha}}} - \int_{|\theta|^\alpha \le \kappa} e^{-t|\theta|^\alpha} \hat{W}(\theta) \hat{f}(\theta) \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right| \le C_\delta e^{-\delta t} \quad P_\mu\text{-a.s.}, \tag{18}$$

where $\hat{W}$ is defined in the previous theorem.

Remark 2 *This result directly generalizes Wang [27, Theorem 3.1]. The condition (16) ensures that $\hat{f} \in L_1$. Hence, f can be taken to be a continuous function that vanishes at ∞ by the Riemann-Lebesgue lemma so $X_t(f)$ is well defined and finite. Moreover, this condition ensures that $X_t(\int_{\mathbb{R}^d} |\hat{f}(\theta) e^{-i\theta \cdot x}| d\theta) < \infty$ so (19) below easily holds by Fubini's theorem. Condition (16) is satisfied by any rapidly decreasing function as well as many stable densities. For example, (16) follows if f is the product of d identical scalar symmetric stable densities with stability parameter greater than α or equal to α if the scale parameter is large enough.*

Proof. We first note that under condition (16)

$$\frac{X_t(f)}{e^{\beta t}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\theta|^\alpha t} \hat{W}_t(\theta) \hat{f}(\theta) \,\mathrm{d}\theta. \tag{19}$$

By Doob's L_2-inequality and (8)

$$P_\mu\left[\sup_{t>u}\left|\hat{W}_t(\theta)-\hat{W}_u(\theta)\right|^2\right]\leq 4\frac{\eta\mu(1)}{\beta-2|\theta|^\alpha}e^{(2|\theta|^\alpha-\beta)u},$$

provided $2|\theta|^\alpha<\beta$. Letting $t\to\infty$ above, we get

$$P_\mu\left[\left|\hat{W}(\theta)-\hat{W}_u(\theta)\right|^2\right]\leq 4\frac{\eta\mu(1)}{\beta-2|\theta|^\alpha}e^{(2|\theta|^\alpha-\beta)u}$$

if $2|\theta|^\alpha<\beta$, and combining the last two equations, we get

$$P_\mu\left[\sup_{t\geq u}\left|\hat{W}_t(\theta)-\hat{W}(\theta)\right|^2\right]\leq 16\frac{\eta\mu(1)}{\beta-2|\theta|^\alpha}e^{(2|\theta|^\alpha-\beta)u}$$

provided $2|\theta|^\alpha<\beta$. Letting $u=n\epsilon$, we get

$$\begin{aligned}
&\int_{|\theta|^\alpha\leq\kappa}\sqrt{P_\mu\left(\sup_{t\geq n\epsilon}|\hat{W}_t(\theta)\hat{f}(\theta)-\hat{W}(\theta)\hat{f}(\theta)|^2e^{-2t|\theta|^\alpha}\right)}\mathrm{d}\theta\\
\leq&\int_{|\theta|^\alpha\leq\kappa}\sqrt{P_\mu\left(\sup_{t\geq n\epsilon}|\hat{W}_t(\theta)-\hat{W}(\theta)|^2\right)}|\hat{f}(\theta)|e^{-n\epsilon|\theta|^\alpha}\mathrm{d}\theta\\
\leq&\int_{|\theta|^\alpha\leq\kappa}4\sqrt{\frac{\eta\mu(1)}{\beta-2|\theta|^\alpha}}e^{\left(|\theta|^\alpha-\frac{\beta}{2}\right)n\epsilon}|\hat{f}(\theta)|e^{-n\epsilon|\theta|^\alpha}\mathrm{d}\theta\\
\leq&4\sqrt{\frac{\eta\mu(1)}{\beta-2\kappa}}e^{-\frac{\beta}{2}n\epsilon}\int_{|\theta|^\alpha\leq\frac{\beta}{2}}|\hat{f}(\theta)|\mathrm{d}\theta\\
\overset{n}{\ll}&e^{\left(-\frac{\beta^2}{2}+\beta\kappa\right)n}
\end{aligned}$$

since $\epsilon=\beta-2\kappa$. Moreover, by (8) and Doob's L_2-inequality

$$\begin{aligned}
&\int_{|\theta|^\alpha>\kappa}\sqrt{P_\mu\left(\sup_{n\epsilon\leq t\leq(n+1)\epsilon}|\hat{W}_t(\theta)\hat{f}(\theta)|^2e^{-2t|\theta|^\alpha}\right)}\mathrm{d}\theta\\
\leq&\int_{|\theta|^\alpha>\kappa}\sqrt{P_\mu\left(\sup_{n\epsilon\leq t\leq(n+1)\epsilon}|\hat{W}_t(\theta)-\hat{W}_0(\theta)|^2\right)}|\hat{f}(\theta)|e^{-n\epsilon|\theta|^\alpha}\mathrm{d}\theta\\
&+\int_{|\theta|^\alpha>\kappa}\sqrt{P_\mu(|\hat{W}_0(\theta)|^2)}|\hat{f}(\theta)|e^{-n\epsilon|\theta|^\alpha}\mathrm{d}\theta\\
\leq&\int_{|\theta|^\alpha>\kappa}\sqrt{\frac{\eta\mu(1)}{2|\theta|^\alpha-\beta}\left(e^{(2|\theta|^\alpha-\beta)(n+1)\epsilon}-1\right)}|\hat{f}(\theta)|e^{-n\epsilon|\theta|^\alpha}\mathrm{d}\theta\\
&+\mu(1)\int_{|\theta|^\alpha>\kappa}|\hat{f}(\theta)|e^{-n\epsilon|\theta|^\alpha}\mathrm{d}\theta
\end{aligned}$$

Using Taylor's theorem, we continue the above estimate to get

$$\int_{|\theta|^\alpha>\kappa}\sqrt{P_\mu\left(\sup_{n\epsilon\le t\le(n+1)\epsilon}|\hat{W}_t(\theta)\hat{f}(\theta)|^2e^{-2t|\theta|^\alpha}\right)}\mathrm{d}\theta$$

$$\le\left(\sqrt{\eta(n+1)\mu_0\epsilon}\right)\cdot\left[e^{-\frac{\beta}{2}(n+1)\epsilon}\int_{|\theta|^\alpha\ge\beta}e^{|\theta|^\alpha\epsilon}\left|\hat{f}(\theta)\right|\mathrm{d}\theta+e^{-(n+1)\epsilon\kappa}\int_{\kappa<|\theta|^\alpha<\beta}e^{|\theta|^\alpha\epsilon}\left|\hat{f}(\theta)\right|\mathrm{d}\theta\right]$$

$$+\mu_0e^{-(n+1)\epsilon\kappa}\hat{c}$$

$$\overset{n}{\lll}\left(\sqrt{n}\right)e^{-n\epsilon\kappa}.$$

Hence, by the previous equations and Cauchy-Schwarz' inequality

$$P_\mu\left[\sup_{n\epsilon\le t\le(n+1)\epsilon}\left|\frac{X_t(f)}{e^{\beta t}}-\int_{|\theta|^\alpha\le\kappa}e^{-t|\theta|^\alpha}\hat{W}(\theta)\hat{f}(\theta)\frac{\mathrm{d}\theta}{(2\pi)^d}\right|\right]$$

$$\le\frac{1}{(2\pi)^d}\int_{|\theta|^\alpha\le\kappa}P_\mu\left(\sup_{t\ge n\epsilon}\left|\hat{W}_t(\theta)\hat{f}(\theta)-\hat{W}(\theta)\hat{f}(\theta)\right|e^{-t|\theta|^\alpha}\right)\mathrm{d}\theta$$

$$+\frac{1}{(2\pi)^d}\int_{|\theta|^\alpha>\kappa}P_\mu\left(\sup_{n\epsilon\le t\le(n+1)\epsilon}\left|\hat{W}_t(\theta)\hat{f}(\theta)\right|e^{-t|\theta|^\alpha}\right)\mathrm{d}\theta$$

$$\overset{n}{\lll}\sqrt{n}e^{-(\beta\kappa-2\kappa^2)n}$$

using $\epsilon=\beta-2\kappa$. Then (17) holds. Multiplying both sides by $t^{\frac{d}{\alpha}}$ and fixing $\delta\in(0,\beta\kappa-2\kappa^2)$, we get that

$$\sum_{n=1}^\infty P_\mu\sup_{n\epsilon\le t\le(n+1)\epsilon}\left[\left|\frac{X_t(f)}{e^{\beta t}t^{-\frac{d}{\alpha}}}-\int_{|\theta|^\alpha\le\kappa}e^{-t|\theta|^\alpha}\hat{W}(\theta)\hat{f}(\theta)\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right|e^{\delta t}\right]<\infty.$$

So there is a random $C_\delta>0$ such that

$$\left|\frac{X_t(f)}{e^{\beta t}t^{-\frac{d}{\alpha}}}-\int_{|\theta|^\alpha\le\kappa}e^{-t|\theta|^\alpha}\hat{W}(\theta)\hat{f}(\theta)\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right|\le C_\delta e^{-\delta t}\quad P_\mu\text{-a.s.}$$

■

Finally, we can state our first SLLN (not in frequency domain). The following lemma will be immediately improved by the theorem to follow thereafter. The constant

$$c_\alpha = \int_{\mathbb{R}^d} e^{-|y|^\alpha} \frac{\mathrm{d}y}{(2\pi)^d}.$$

will appear in the following lemma and several results thereafter.

Lemma 3 *Suppose f has Fourier transform $\hat{f}$ that satisfies*

$$\int_{\mathbb{R}^d} e^{\epsilon|\theta|^\alpha} \left|\hat{f}(\theta)\right| \mathrm{d}\theta < \infty \tag{20}$$

for all $\epsilon < \beta$. Then,

(1) existence of a $\kappa_0 < \frac{\beta}{2}$ such that $\sup_{|\theta|^\alpha \le \kappa_0} |\hat{f}(\theta)| < \infty$ implies that

$$\lim_{t\to\infty} \left| \frac{X_t(f)}{e^{\beta t} t^{-\frac{d}{\alpha}}} - \hat{W}(0) \int e^{-t|\theta|^\alpha} \hat{f}(\theta) \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} \right| = 0 \quad P_\mu\text{-a.s.}$$

(2) continuity at 0 of $\hat{f}$ implies that

$$\lim_{t\to\infty} \frac{X_t(f)}{e^{\beta t} t^{-\frac{d}{\alpha}}} = c_\alpha \, \hat{W}(0)\hat{f}(0) \quad P_\mu\text{-a.s.}$$

Remark 3 *1) It is clearly sufficient that*

$$\int_{\mathbb{R}^d} e^{\beta|\theta|^\alpha} \left|\hat{f}(\theta)\right| \frac{\mathrm{d}\theta}{(2\pi)^d} < \infty.$$

2) $c_2 = 2^{-d}\pi^{-\frac{d}{2}}$.

3) The Fourier transform is defined in a different manner for each $L_p(\mathbb{R}^d)$ with $p \in [1,2]$. (Each can be thought of as an extension of the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$, the set of rapidly decreasing functions (see [26] for definition).) If $f \in L_1(\mathbb{R}^d)$, then $\hat{f}$ is continuous and $\hat{f}(0) = \int_{\mathbb{R}^d} f(x)dx$.

Proof. We let $a_j = j^{\frac{3\alpha}{1\wedge\alpha}-1}$ and $s_n = \sum_{j=1}^{n} a_j$. By (7), we have that $P_\mu \left|\hat{W}(\theta) - \hat{W}(0)\right| \overset{\theta}{\ll} |\theta|^{\frac{1\wedge\alpha}{2}}$ for $|\theta|^\alpha \leq \kappa_0$ so

$$P_\mu \left[\max_{s_n \leq t \leq s_{n+1}} \int_{|\theta|^\alpha \leq \kappa} e^{-t|\theta|^\alpha} \left|\hat{W}(\theta)\hat{f}(\theta) - \hat{W}(0)\hat{f}(\theta)\right| \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right] \tag{21}$$
$$\overset{n,\kappa}{\ll} (s_{n+1})^{\frac{d}{\alpha}} \int_{|\theta|^\alpha \leq \kappa} e^{-s_n|\theta|^\alpha} P_\mu \left|\hat{W}(\theta) - \hat{W}(0)\right| \frac{\mathrm{d}\theta}{(2\pi)^d} \sup_{|\theta|^\alpha \leq \kappa} |\hat{f}(\theta)|$$
$$\overset{n,\kappa}{\ll} (s_{n+1})^{\frac{d}{\alpha}} \int_{|\theta|^\alpha \leq \kappa} e^{-s_n|\theta|^\alpha} |\theta|^{\frac{1\wedge\alpha}{2}} \frac{\mathrm{d}\theta}{(2\pi)^d}$$
$$\overset{n,\kappa}{\ll} \left(\frac{s_{n+1}}{s_n}\right)^{d/\alpha} |s_n|^{-\frac{1\wedge\alpha}{2\alpha}}$$

for all $\kappa \leq \kappa_0$ and $n = 1, 2, ...$ Moreover, by (20)

$$P_\mu|\hat{W}(0)| \cdot \max_{s_n \leq t \leq s_{n+1}} \int_{|\theta|^\alpha > \kappa} e^{-t|\theta|^\alpha} |\hat{f}(\theta)| \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} \overset{n,\kappa}{\ll} (s_{n+1})^{\frac{d}{\alpha}} \int_{|\theta|^\alpha > \kappa} e^{-s_n|\theta|^\alpha} |\hat{f}(\theta)| \mathrm{d}\theta \tag{22}$$
$$\overset{n,\kappa}{\ll} (s_{n+1})^{\frac{d}{\alpha}} e^{-(s_n+\epsilon)\kappa},$$

where $\epsilon = \beta - 2\kappa$ as in Theorem 2. From (17) of Theorem 2 one finds that

$$P_\mu \left[\max_{s_n \leq t \leq s_{n+1}} \left|\frac{X_t(f)}{e^{\beta t} t^{-\frac{d}{\alpha}}} - \int_{|\theta|^\alpha \leq \kappa} e^{-t|\theta|^\alpha} \hat{W}(\theta)\hat{f}(\theta) \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right|\right] \tag{23}$$
$$\leq \sum_{j=\lfloor s_n/\epsilon \rfloor}^{\lfloor s_{n+1}/\epsilon \rfloor} P_\mu \left[\max_{j\epsilon \leq t \leq (j+1)\epsilon} \left|\frac{X_t(f)}{e^{\beta t} t^{-\frac{d}{\alpha}}} - \int_{|\theta|^\alpha \leq \kappa} e^{-t|\theta|^\alpha} \hat{W}(\theta)\hat{f}(\theta) \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right|\right]$$
$$\overset{n,\kappa}{\ll} (s_{n+1} - s_n)\sqrt{s_n} e^{-\left(\frac{\beta^2}{2} - \beta\kappa\right)s_n}$$

for $n = 1, 2, ...$ Therefore, we have by the previous three equations that

$$\sum_{n=1}^{\infty} P_\mu \left[\max_{s_n \leq t \leq s_{n+1}} \left|\frac{X_t(f)}{e^{\beta t} t^{-\frac{d}{\alpha}}} - \hat{W}(0) \int e^{-t|\theta|^\alpha} \hat{f}(\theta) \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right|\right] < \infty \tag{24}$$

and so

$$\frac{X_t(f)}{e^{\beta t}t^{-\frac{d}{\alpha}}} - \hat{W}(0)\int_{\mathbb{R}^d} e^{-t|\theta|^\alpha}\hat{f}(\theta)\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} \to 0 \qquad P_\mu\text{-a.s.} \tag{25}$$

Next, given $\gamma > 0$ we have by the continuity of $\hat{f}(\theta)$ at 0 that there is a $\kappa_0 \in (0, \frac{\beta}{2})$ satisfying $\sup_{|\theta|^\alpha \le \kappa_0}\left|\hat{f}(\theta) - \hat{f}(0)\right| < \gamma$, which, together with (20), implies that

$$\begin{aligned} &\int e^{-t|\theta|^\alpha}|\hat{f}(\theta) - \hat{f}(0)|\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} \qquad (26)\\ &= \int_{|\theta|^\alpha \le \kappa_0} e^{-t|\theta|^\alpha}|\hat{f}(\theta) - \hat{f}(0)|\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} + \int_{|\theta|^\alpha > \kappa_0} e^{-t|\theta|^\alpha}|\hat{f}(\theta) - \hat{f}(0)|\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\\ &\overset{\gamma,t}{\ll} \gamma + e^{-(t+\epsilon)\kappa_0}\int_{|\theta|^\alpha > \kappa_0} e^{\epsilon|\theta|^\alpha}|\hat{f}(\theta)|\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} + |\hat{f}(0)|\int_{|\theta|^\alpha > \kappa_0} e^{-t|\theta|^\alpha}\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} \end{aligned}$$

and the result follows from the fact that

$$\int_{|\theta|^\alpha > \kappa_0} e^{-t|\theta|^\alpha}\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} = \int_{|y|^\alpha > t\kappa_0} e^{-|y|^\alpha}\frac{\mathrm{d}y}{(2\pi)^d} \to 0$$

as $t \to \infty$. ■

Starting from Watanabe, everybody considered continuous, compactly supported f. It is interesting to see how far we can relax the assumptions on f.

Theorem 4 *Suppose that f is such that its Fourier transform $\hat{f}$ exists, $\hat{f}$ is continuous at 0 and there is an $\epsilon > 0$ such that*

$$\int_{\mathbb{R}^d} e^{\epsilon|\theta|^\alpha}|\hat{f}(\theta)|\mathrm{d}\theta < \infty.$$

Then,

$$\frac{X_t(f)}{e^{\beta t}t^{-\frac{d}{\alpha}}} \to c_\alpha\hat{W}(0)\hat{f}(0), \quad P_\mu\text{-a.s.}$$

Remark 4 *The Fourier transform is only defined as an element of $L_p\left(\mathbb{R}^d\right)$ for some $p \in [1,2]$ and hence almost everywhere, so continuous at 0 should be interpreted as 'there is* *a version that is continuous at zero'. Compared to the previous theorems, $\epsilon > 0$ can be arbitrarily small.*

Proof. We define $\widehat{\phi^\beta}(\theta) = \begin{cases} 1, & |\theta|^\alpha \le \beta, \\ e^{\beta(\beta - |\theta|^\alpha)}, & |\theta|^\alpha > \beta, \end{cases}$ and $\widehat{f^\beta} = \widehat{f}\widehat{\phi^\beta}$. Then, $\widehat{f^\beta}$ is in $L_1\left(\mathbb{R}^d\right)$, so its inverse Fourier transform f^β exists as a continuous function that vanishes at ∞ and

$$\int_{\mathbb{R}^d} e^{\beta|\theta|^\alpha} |\widehat{f^\beta}(\theta)| \mathrm{d}\theta \le e^{\beta^2} \int_{|\theta|^\alpha \le \beta} |\widehat{f}(\theta)| \mathrm{d}\theta + e^{\beta^2} \int_{|\theta|^\alpha > \beta} |\widehat{f}(\theta)| \mathrm{d}\theta < \infty.$$

Therefore, by Lemma 3, we have that

$$\frac{t^{\frac{d}{\alpha}} X_t(f^\beta)}{e^{\beta t}} \to c_\alpha \hat{W}(0) \hat{f}^\beta(0) = c_\alpha \hat{W}(0) \hat{f}(0), \quad P_\mu\text{-a.s.} \tag{27}$$

Let $a_j = \frac{1}{\sqrt{j}}$ and $s_n = \sum_{j=1}^{n} a_j$ so $s_n \nearrow \infty$. Then, from (19) we also have that

$$\begin{aligned} &(2\pi)^d P_\mu \left[\sup_{s_n \le t \le s_{n+1}} \left| \frac{t^{\frac{d}{\alpha}} X_t(f - f^\beta)}{e^{\beta t}} \right| \right] \\ \le\, & P_\mu \left[\sup_{s_n \le t \le s_{n+1}} \left| t^{\frac{d}{\alpha}} \int_{|\theta|^\alpha > \beta} e^{-|\theta|^\alpha t} (\hat{W}_t - \hat{W}_0)(\theta) \left(\hat{f} - \hat{f}^\beta \right)(\theta) \, \mathrm{d}\theta \right| \right] \\ &+ P_\mu \left[\sup_{s_n \le t \le s_{n+1}} \left| t^{\frac{d}{\alpha}} \int_{|\theta|^\alpha > \beta} e^{-|\theta|^\alpha t} \hat{W}_0(\theta) \left(\hat{f} - \hat{f}^\beta \right)(\theta) \, \mathrm{d}\theta \right| \right]. \end{aligned}$$

For the first term, we find by Doob's L_2-inequality, (8) and Taylor's theorem (in the second last inequality) that

$$
\begin{aligned}
&P_\mu\left[\sup_{s_n\le t\le s_{n+1}}\left|t^{\frac{d}{\alpha}}\int_{|\theta|^\alpha>\beta}e^{-|\theta|^\alpha t}(\hat{W}_t-\hat{W}_0)\left(\theta\right)\left(\hat{f}-\hat{f}^\beta\right)(\theta)\,\mathrm{d}\theta\right|\right]\\
&\le\ (s_{n+1})^{\frac{d}{\alpha}}\int_{|\theta|^\alpha>\beta}e^{-|\theta|^\alpha s_n}P_\mu^{\frac{1}{2}}\left[\sup_{s_n\le t\le s_{n+1}}\left|\hat{W}_t-\hat{W}_0\right|^2(\theta)\right]\left|\hat{f}-\hat{f}^\beta\right|(\theta)\,\mathrm{d}\theta\\
&\le\ 2\sqrt{\eta\mu_0}(s_{n+1})^{\frac{d}{\alpha}}\int_{|\theta|^\alpha>\beta}e^{-|\theta|^\alpha s_n}\sqrt{\frac{e^{(2|\theta|^\alpha-\beta)s_{n+1}}-1}{2|\theta|^\alpha-\beta}}\left|\hat{f}-\hat{f}^\beta\right|(\theta)\,\mathrm{d}\theta\\
&\le\ 2\sqrt{\eta\mu_0}(s_{n+1})^{\frac{d}{\alpha}+\frac{1}{2}}e^{-\frac{\beta}{2}s_{n+1}}\int_{|\theta|^\alpha>\beta}e^{a_{n+1}|\theta|^\alpha}\left|\hat{f}-\hat{f}^\beta\right|(\theta)\,\mathrm{d}\theta\\
&\overset{n}{\ll}\ (s_{n+1})^{\frac{d}{\alpha}+\frac{1}{2}}\sqrt{e^{-\beta s_{n+1}}}.
\end{aligned}
$$

(Here, we used the fact that $a_{n+1}\le\epsilon$ for large n in the last bound.) For the second term, we find

$$
\begin{aligned}
&P_\mu\left[\sup_{t\ge s_n}\left|t^{\frac{d}{\alpha}}\int_{|\theta|^\alpha>\beta}e^{-|\theta|^\alpha t}\hat{W}_0\left(\theta\right)\left(\hat{f}-\hat{f}^\beta\right)(\theta)\,\mathrm{d}\theta\right|\right]\\
&\le\sup_{t\ge s_n}\left|t^{\frac{d}{\alpha}}e^{-\beta t}\int_{|\theta|^\alpha>\beta}\mu\left(e^{-i\theta\cdot(\cdot)}\right)\left(\hat{f}-\hat{f}^\beta\right)(\theta)\,\mathrm{d}\theta\right|\\
&\le s_n^{\frac{d}{\alpha}}e^{-\beta s_n}\mu_0\int\left(1-\hat{\phi}^\beta\right)\left|\hat{f}\left(\theta\right)\right|\mathrm{d}\theta
\end{aligned}
$$

for large enough n. Combining the previous three inequalities, we find

$$
\sum_{n=1}^\infty P_\mu\left[\sup_{s_n\le t\le s_{n+1}}\left|\frac{t^{\frac{d}{\alpha}}X_t(f-f^\beta)}{e^{\beta t}}\right|\right]<\infty,
$$

which implies that $\frac{t^{\frac{d}{\alpha}}X_t(f-f^\beta)}{e^{\beta t}}\to 0$ P_μ-a.s. (since $s_n\to\infty$) and therefore by (27)

$$
\frac{t^{\frac{d}{\alpha}}X_t(f)}{e^{\beta t}}\to c_\alpha\hat{W}\left(0\right)\hat{f}\left(0\right),\quad P_\mu\text{-a.s.}
$$

■

Now that we removed the β-dependence on the decay on $\hat{f}$, we can easily generalize Wang's and Watanabe's works from a single continuous, compactly supported function to vague convergence and beyond. We start by considering the case where $f\in L_1$. (Until now, we only assumed existence of the Fourier transform.)

Let

$$\mathcal{G}^{\alpha}=\left\{g:\ g\in L^{1}(\mathbb{R}^{d})\text{ such that }\int e^{\epsilon|\theta|^{\alpha}}\hat{g}\left(\theta\right)\mathrm{d}\theta<\infty\text{ for some }\epsilon>0\right\}$$

For any $g\in\mathcal{G}^{\alpha}$, it follows that the Fourier transform $\hat{g}$, is continuous by the L_1-property.

Corollary 5 *Suppose ℓ is Lebesgue measure, that $f\in L_1$ and, for each $\gamma>0$, there exists $f_1,f_2\in\mathcal{G}^{\alpha}$ such that $f_1\leq f\leq f_2$ and $\ell\left(f_2-f_1\right)<\gamma$. Then,*

$$\frac{t^{\frac{d}{\alpha}}X_t(f)}{e^{\beta t}}\rightarrow c_{\alpha}\hat{W}\left(0\right)\int_{\mathbb{R}^d}f\left(x\right)\mathrm{d}x\quad P_{\mu}\text{-}a.s.$$

Proof. By Theorem 4, we have that

$$\frac{t^{\frac{d}{\alpha}}X_t(f_i)}{e^{\beta t}}\rightarrow c_{\alpha}\hat{W}\left(0\right)\lambda(f_i)\quad P_{\mu}\text{-a.s.}$$

for $i=1,2$. However, this then implies

$$c_{\alpha}\hat{W}\left(0\right)\ell(f_1)\leq\liminf_{t\to\infty}\frac{t^{\frac{d}{\alpha}}X_t(f)}{e^{\beta t}}\leq\limsup_{t\to\infty}\frac{t^{\frac{d}{\alpha}}X_t(f)}{e^{\beta t}}\leq c_{\alpha}\hat{W}\left(0\right)\ell(f_2)$$

and the Corollary follows. ■

A further useful corollary follows:

Corollary 6 *For any $f\in C_c\left(\mathbb{R}^d\right)$, it follows that*

$$\frac{t^{\frac{d}{\alpha}}X_t(f)}{e^{\beta t}}\rightarrow c_{\alpha}\hat{W}\left(0\right)\int_{\mathbb{R}^d}f\left(x\right)\mathrm{d}x\quad P_{\mu}\text{-}a.s.$$

Proof. Let $M=\sup_x|f\left(x\right)|$, $K=\sup\{|x|:f(x)\neq0\}$ and $\gamma\in(0,1)$. Then, by uniform continuity there is a $\delta>0$ (with $\delta<K$) such that $|f\left(x\right)-f\left(y\right)|<\frac{\gamma}{8}$ for all $|x-y|<\delta$ and an $r\in(0,1)$ such that $2M\int_{B(0,\delta)^c}\phi_{r\delta}\left(y\right)\mathrm{d}y<\frac{\gamma}{8}$, where $\phi_p\left(y\right)=\frac{1}{\left(\sqrt{2\pi}p\right)^d}e^{-\frac{|y|^2}{2p^2}}$. Finally, there is an $n>1$ such that $\int_{B(0,nK)}\phi_{r\delta}\left(x-y\right)\mathrm{d}y>\frac{1}{2}$ for all $|x|\leq K$. Now, we define

$$f_2(x) = \int_{B(0,nK)} \left(\frac{\gamma}{2} + f(y)\right) \phi_{r\delta}(x-y)\,\mathrm{d}y$$

$$f_1(x) = \int_{B(0,nK)} \left(f(y) - \frac{\gamma}{2}\right) \phi_{r\delta}(x-y)\,\mathrm{d}y$$

Then, noting

$$\begin{aligned} f(x) &= \int_{\mathbb{R}^d} f(x)\phi_{r\delta}(x-y)\mathrm{d}y \\ &= \int_{B(0,nK)} f(x)\phi_{r\delta}(x-y)\mathrm{d}y + \int_{B(0,nK)^c} f(x)\phi_{r\delta}(x-y)\mathrm{d}y, \end{aligned}$$

we have

$$\begin{aligned} & f_2(x) - f(x) \\ =& \frac{\gamma}{2}\int_{B(0,nk)} \phi_{r\delta}(x-y)\mathrm{d}y + \int_{B(0,nk)\cap B(x,\delta)} (f(y)-f(x))\phi_{r\delta}(x-y)\mathrm{d}y \\ & + \int_{B(0,nk)\cap B(x,\delta)^c} (f(y)-f(x))\phi_{r\delta}(x-y)\mathrm{d}y - \int_{B(0,nk)^c} f(x)\phi_{r\delta}(x-y)\mathrm{d}y \\ \geq& \frac{\gamma}{4} - \int_{B(x,\delta)} \underbrace{|f(y)-f(x)|}_{<\frac{\gamma}{8}} \phi_{r\delta}(x-y)\,\mathrm{d}y - 2M\int_{B(x,\delta)^c} \phi_{r\delta}(x-y)\,\mathrm{d}y \\ >& 0. \end{aligned}$$

Similarly we have $f_1 \leq f$. By construction, we have that $f_1, f_2 \in \mathcal{G}^\alpha$, $f_1 \leq f \leq f_2$ and $\ell(f_2 - f_1) < \gamma$. Hence, this corollary follows from the previous one. ∎

We will use the following lemma to go from single f convergence to vague convergence and beyond by setting $\mathcal{M}$ to be a countable subset of $C_c\left(\mathbb{R}^d\right)$ that generates the Borel topology on $\mathbb{R}^d$. In what follows, $(E,\mathcal{T})$ will denote a topological space, and $B(E)$ and $\overline{C}(E)$ will denote the bounded Borel measurable and the bounded continuous $\mathbb{R}$-valued functions on E, respectively.

Lemma 7 *Suppose that* $(E,\mathcal{T})$ *is a topological space with a countable base, and* $\{\mu_t\}\cup\{\mu\}$ *are (possibly non-finite) Borel measures;* $f \in B(E)$ *satisfies* $0 < \mu(f) < \infty$*;* $\mathcal{M} \subset B(E)$

strongly separates points, is countable and is closed under multiplication; and

$$\mu_t(gf) \to \mu(gf)$$

for all $g \in \mathcal{M} \cup \{1\}$. *Then,*

$$\mu_t(gf) \to \mu(gf)$$

for all $g \in \overline{C}(E)$.

Proof. We define the probability measures by

$$\nu_t(g) = \tfrac{\mu_t(gf)}{\mu_t(f)} \text{ and } \nu(g) = \tfrac{\mu(gf)}{\mu(f)}$$

for all $g \in B(E)$ and find by hypothesis that $\nu_t(g) \to \nu(g)$ for all $g \in \mathcal{M}$. Now, it follows from Blount and Kouritzin [4, Theorem 6] that

$$\nu_t \to \nu \text{ weakly as } t \to \infty$$

or, equivalently $\mu_t(gf) \to \mu(gf)$ as $t \to \infty$, for all $g \in \overline{C}(E)$. ■

Definition 1 We call $\mathcal{H} = \{h \in C(\mathbb{R}^d) : \exists\epsilon > 0 \text{ so that } \sup_{x\in\mathbb{R}^d} e^{\epsilon|x|^2}|h(x)| < \infty\}$ the *swiftly decreasing* functions on $\mathbb{R}^d$ and say Borel measures $\{\mu_t\}$ *converge shallowly* to Borel measure μ if $\mu_t(h) \to \mu(h)$ as $t \to \infty$ for all $h \in \mathcal{H}$.

Theorem 8 $\dfrac{t^{\frac{d}{\alpha}}X_t}{e^{\beta t}} \to c_\alpha \hat{W}(0)\,\ell$, *$P_\mu$-a.s. in the shallow topology, where ℓ is Lebesgue measure.*

Proof. Let $f_n(x) = \left(\frac{1}{\sqrt{\pi n}}\right)^d e^{-\frac{|x|^2}{n}}$ so $\hat{f}_n(\theta) = e^{-n|\theta|^2}$ and

$$\frac{t^{\frac{d}{\alpha}}X_t(f_n)}{e^{\beta t}} \to c_\alpha \hat{W}(0)\,\ell(f_n) \quad P_\mu\text{-a.s.}$$

by Theorem 4. Moreover, $C_c\left(\mathbb{R}^d\right)$ is an algebra that strongly separates points. Therefore, it follows by Blount and Kouritzin [4, Lemma 2] that there is a countable subcollection $\mathcal{M}$ that strongly separates points and is closed under multiplication. From Corollary 6, we have that

$$\frac{t^{\frac{d}{\alpha}}X_t(gf_n)}{e^{\beta t}} \to c_\alpha \hat{W}(0)\, l(gf_n) \quad P_\mu\text{-a.s.}$$

for all $g \in \mathcal{M}$. Fix an ω such that convergence takes place for all f_n and $g \in \mathcal{M}$. Now, it follows by Lemma 7 that

$$\frac{t^{\frac{d}{\alpha}}X_t(gf_n)}{e^{\beta t}} \to c_\alpha \hat{W}(0)\, l(gf_n) \text{ for all } g \in \overline{C}(E) \text{ and } n = 1, 2, ..., \quad P_\mu\text{-a.s.}$$

The theorem follows. ■

An immediate corollary of this Theorem is the following analog of Watanabe's result:

Corollary 9 $\frac{t^{\frac{d}{\alpha}}X_t}{e^{\beta t}} \to c_\alpha \hat{W}(0)\, \ell$ P_μ*-a.s. in the vague topology, where* ℓ *is Lebesgue measure.*

4 Concluding Remarks

At the request of our referees and at the risk of later being proved wrong, we make the following speculations that may be important to future development.

Remark 5 *Our techniques may even be useful in Watanabe's [28] classical branching Brownian motion setting. In particular, our methods avoid a widely-known concern about Watanabe's work regarding an apparent assumption that an uncountable union of measure zero sets is again measure zero and our methods seamlessly incorporate stable motion.*

Remark 6 *For general Lévy-type branching mechanism, the second moment does not exist, and we need to find method to control some p-th moment with* $p \in (0, 2)$*. Along this direction, the paper of Le Gall and Mytnik [18] could be a helpful reference to handle the p-th moment. Then, our most basic Fourier transform and Doob's* L_p*-inequality techniques may still hold provided* $p > 1$*.*

Remark 7 *One may consider limit theorems under different spatial motion conditions. Two possible approaches occur to us here. In the first method, the Fourier transform could be replaced by the so called generalized eigenfunctions. In earlier works [22] and [23], under an intrinsically ultracontractive condition on the Feynman-Kac semigroup of diffusion, a strong law of large numbers for superdiffusion was obtained using the first eigenfunction (or generalized eigenfunction). It should be noted though that generalized eigenfunctions are often not easy to obtain in higher dimensions. The second potential method appears untried for superdiffusions. It tracks the development of classical solutions to parabolic partial differential equations. In particular, Fourier transform techniques are core to the Parametrix method of constructing fundamental solutions under uniform elliptic assumptions. The idea is to replace a differential (or pseudodifferential) operator with just the highest order portion with a fixed coefficient, use Fourier transform techniques on this simpler equation and treat the remainder of the operator as a type of perturbation that can be handled by an infinite series expansion (see Friedman [16], Kouritzin [20] and Kouritzin [21]). In this manner, it is conceivable that one could first prove a SLLN for a more complicated motion model by first considering a simple motion model as we have done herein and then handle the difference as some type of perturbation. It is not clear exactly how to do this but it would make sense to first look at the one dimensional settings where the superdiffusion can be described by a stochastic partial differential equation so the parametrix construction should essentially be that of a fundamental solution. While certainly not in the current setting, Dawson and Kouritzin [9] have already followed a similar path in establishing their invariance principle. Finally, moving from speculation to fantasy, we mention that these methods hold for higher order parabolic equations so they might have a role if people start considering measure-valued processes with iterated or composite process motions.*

Acknowledgements The research of the second author is supported by NSFC (Grant No. 11271030 and 11128101) and Specialized Research Fund for the Doctoral Program of Higher

Education.

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