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Exact Infinite Dimensional Filters and Explicit Solutions

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This work is dedicated to Donald Dawson not only in recognition of his outstanding mathematical career but also for his generosity with his time and advice.

ABSTRACT. Previously, we defined *infinite dimensional exact filters* as nonlinear filters which can be conveniently reduced without approximation to a single convolution (plus a simple transformation and substitution). We showed that such problems do exist and the observation process can be far more general than those for exact *finite dimensional* filters like the Kalman and Benes filters. Moreover, our infinite dimensional exact filters compare favorably in terms of time efficiency and accuracy to other methods except for the finite dimensional exact filters that have limited utility. Herein, we broaden the realm of applicability for our infinite dimensional exact filters including problems with new nonlinear drifts and nonlinear dispersion coefficients. In particular, we investigate the problem of determining which scalar continuous-discrete filtering problems can be solved with essentially a single convolution with respect to a standard normal distribution. This leads to a particularly simple filtering algorithm because the Fourier transform of the standard normal distribution is known in closed form and very well behaved.

1. Introduction

The classical scalar nonlinear filtering problem is concerned with estimating functions of the current state of a scalar *signal* diffusion process

$$X_t = X_0 + \int_0^t \alpha(X_s)ds + \int_0^t \sigma(X_s)dW_s \tag{1.1}$$

based upon noise-corrupted, distorted observations

$$Y_t = Y_0 + \int_0^t h\left(X_s\right) ds + B_t, \tag{1.2}$$

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where $\{(W_t, B_t),\ t \geq 0\}$ is a standard $\mathbb{R}^2$-valued Brownian motion and X_0 and Y_0 are random variables such that $\{X_0, Y_0, \{(W_t, B_t),\ t \geq 0\}\}$ are mutually independent. More precisely, one wishes to determine the conditional expectation

$$E\left[\varphi(X_t)|\mathcal{B}(\{Y_s,\ 0 \leq s \leq t\})\right] \tag{1.3}$$

for a large class of Borel measurable functions φ or, equivalently, the conditional probability

$$P\left[X_t \in \Gamma|\mathcal{B}(\{Y_s,\ 0 \leq s \leq t\})\right] \tag{1.4}$$

for all Borel sets Γ. This problem is solved mathematically under appropriate regularity conditions by the Kushner-Fujisaki-Kallianpur-Kunita (KFKK), the Duncan-Mortensen-Zakai (DMZ), and the Kallianpur-Striebel (KS) equations. Indeed, many authors have generalized these equations to allow for infinite dimensional signal models in lieu of (1.1) and more general Itô equation models for the observations (1.2). However, regardless of model generality, these (KFKK), (DMZ), and (KS) equations are not readily implementable on computers and further developments have been sought. There are two basic approaches: i) Allow general models within the above Markov framework and introduce approximations to allow computer implementation, or ii) Impose certain restrictions on the model so that the filtering problem degenerates into a computer-convenient algorithm. We are concerned herein with the later approach called *exact filtering.*

Exact filtering refers to the determination of filtering problems that yield a (nearly) readily implementable solution without approximations as well as their implementation schemes. The Kalman filter is the most celebrated exact filter and was thought to be the only exact filter for many years. Here, one considers linear equations of the form

$$X_t = X_0 + \int_0^t A X_s ds + \int_0^s \sigma dW_s \tag{1.5}$$

$$Y_t = Y_0 + \int_0^t C X_s ds + B_t, \tag{1.6}$$

(or a time-inhomogeneous variation) and the idea is to track the conditional mean and error covariance by solving a linear stochastic differential equation and a quadratic ordinary differential equation. However, nowadays, there are other known *finite-dimensional* exact filters where the filtering problem also degenerates to the solution of finite-dimensional stochastic and ordinary differential equations. For examples of such filters the reader is referred to the works of e.g. Benes (1981), Daum (1988) and Leung and Yau (1992).

The finite-dimensionality necessarily imposes stringent conditions on *both* signal and observation processes since there has to be a fixed finite number of sufficient statistics that can be evolved according to a small number of (stochastic and ordinary) differential equations. To avoid some of these restrictions, Kouritzin (1998) introduced the concept of *infinite-dimensional* exact filters, where the filtering problem degenerates into a convolution plus Bayesian update instead of finite dimensional equations. The main benefit of infinite over finite dimensional exact filters initially appeared to be the fact that extremely general observations are allowed at least in the discrete-time observation setting. However, another inherent advantage of this method is it uses highly developed, fast Fourier transform computer algorithms to implement the convolutions. Indeed, in Kouritzin's work all convolution is done with respect to a standard Gaussian distribution so one has control over errors and can save both time and space by using the closed form for the Fourier transform of a standard Gaussian distribution.

Whereas the observations are taken to be a continuous time Itô equation in most mathematical studies, the case of discrete observations may be of more practical importance at least for tracking problems in, for example, the air traffic management and defense industries. Therefore, we return to the general nonlinear diffusion (1.1) but now replace (1.2) with discrete-time measurement Y_j taken at time t_j with $0 < t_1 < t_2 < \ldots$ Then, continuous-discrete non-linear filtering is concerned with obtaining the distribution of X_t conditioned on the observations $\{Y_j;\ t_j \le t\}$ i.e. finding $P(X_t \,|\, \mathcal{B}\{Y_j,\, t_j \le t\})$. We let $P_{Y_j|X_{t_j}}(A,x)$ denote a function on $\mathcal{B}(\mathbb{R}) \times \mathbb{R}$ such that (i) for fixed x, $A \to P_{Y_j|X_{t_j}}(A,x)$ is a probability measure, (ii) for each $A \in \mathcal{B}(\mathbb{R})$, $x \to P_{Y_j|X_{t_j}}(A,x)$ is measurable, and (iii) for each $A \in \mathcal{B}(\mathbb{R})$, $P_{Y_j|X_{t_j}}(A, X_{t_j}(\omega))$ is a P-version of $P\left\{Y_j \in A|\mathcal{B}(X_{t_j})\right\}(\omega)$. (Such a function exists by standard methods see e.g. Theorem 33.3 of Billingsley (1986).) Then, for simplicity, we assume:

(F1): The conditional probability measure $A \to P_{Y_j|X_{t_j}}(A,x)$ has a density $p_{Y_j|X_{t_j}}(\cdot|x)$ with respect to Lebesgue measure for almost all $\left[PX_{t_j}^{-1}\right]$ x such that $x \to p_{Y_j|X_{t_j}}(y|x)$ is continuous and bounded for almost all $y \in \mathbb{R}$.

Then, for any $A \in \mathcal{B}(\mathbb{R})$

$$\int_{\mathbf{A}} p_{Y_j|X_{t_j}}(y|X_{t_j})dy = P\left\{Y_j \in A|\mathcal{B}(X_{t_j})\right\} \text{ a.s.} \tag{1.7}$$

We use the boundedness of $x \to p_{Y_j|X_{t_j}}(y|x)$ to ensure that $p_{X_{t_{j+1}}|\mathcal{Y}_{j+1}}$ is bounded in (1.14) below.

Next, we notice that we are not specifying any specific model for our observations. Still, it would be contrary to the theory and practice of filtering to allow these observations to help predict the future. Therefore, it is reasonable to assume that given $\mathcal{X}_{t_j} \doteq \mathcal{B}\left(X_s, s \le t_j\right)$ conditioning on $\mathcal{Y}_j = \mathcal{B}\left(\{Y_i;\ i \le j\}\right)$ does not provide any extra information in the sense that:

(F2): For all $A \in \mathcal{B}(\mathbb{R})$ and $t \ge t_j$, we have that $E\left[1_{X_t \in A}|\mathcal{X}_{t_j} \vee \mathcal{Y}_j\right] = E\left[1_{X_t \in A}|\mathcal{X}_{t_j}\right]$ almost surely for each $j = 1, 2, \ldots$

Moreover, in order to incorporate new observations in a tractable (Bayesian) manner, we assume that *given the current state* the new observation is independent of the past observations in the sense that:

(F3): For each $j = 0, 1, 2, \ldots$ and $y \in \mathbb{R}$, we have that

$$p_{Y_{j+1}|X_{t_{j+1}}}(y|X_{t_{j+1}}) = p_{Y_{j+1}|(X_{t_{j+1}},\mathcal{Y}_j)}(y|X_{t_{j+1}}) \quad \text{a.s.,} \tag{1.8}$$

where $p_{Y_{j+1}|(X_{t_{j+1}},\mathcal{Y}_j)}(y|X_{t_{j+1}})$ denotes the conditional density of Y_{j+1} given $\mathcal{B}(X_{t_{j+1}}, \mathcal{Y}_j)$

This type of assumption is quite mild and suitable for filtering theory. We still maintain that our assumptions on the observations are quite general. Indeed, we can have observations of the form

$$Y_k = h(X_{t_k}, V_k), \tag{1.9}$$

where h is a nonlinear function and $\{V_k,\ k = 1, 2, 3, ...\}$ is any sequence of independent random variables.

Next, for a matter of convenience, we assume that D is an open interval, e.g. $(-\infty, \infty)$, $(0, \infty)$, or $(0, 1)$, in $\mathbb{R}$ and $P(X_t \in D) = 1$ for all $t \geq 0$.

(F4): The initial law $\mathcal{L}(X_0)$ of the signal has a density p_{X_0} that is bounded, continuous, and zero off of D.

and

(F5): The coefficients $\alpha : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to [0, \infty)$ are continuous and one-time respectively two-times continuously differentiable on D. Moreover, there is a fundamental solution Γ to the Cauchy initial data problem for

$$\begin{aligned} &\partial_t u(t,x) = a(x)\partial_x^2 u(t,x) + b(x)\partial_x u(t,x) + c(x)u(t,x), \\ &u(t)|_{\partial D} = 0,\ u(0,x) = \varphi(x) \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} &a(x) = \frac{\sigma^2(x)}{2},\ b(x) = 2\sigma(x)\sigma'(x) - \alpha(x), \\ &c(x) = (\sigma'(x))^2 + \sigma(x)\sigma''(x) - \alpha'(x), \end{aligned} \tag{1.11}$$

such that $E\left[1_{X_t \in A} | \mathcal{X}_{t_j}\right] = \int_A \Gamma(t - t_j; y, X(t_j))dy$ for all $A \in \mathcal{B}(D)$, $t > t_j$ and $(t, y) \to \int_D \Gamma(t - t_j; y, x)\,\varphi(x)\,dx$ is the unique continuous, bounded solution to (1.10) subject to $u(t_j, x) = \varphi(x)$ for any continuous, bounded $\varphi : D \to \mathbb{R}$.

Then, it follows from the tower property of conditional expectation as well as other standard results on conditional probability (see e.g. Billingsley (1986) Theorem 33.3 and 34.5) that the continuous/discrete filtering problem is solved by using the following three steps:

STEP 1 Set $t_0 = 0$, $\mathcal{Y}_0 = \{\emptyset, \Omega\}$, and

$$p_{X_{t_0}|\mathcal{Y}_0}(x) = p_{X_0}(x), \tag{1.12}$$

the density of initial state X_0;

For all $j = 0, 1, 2, ...$,

STEP 2 Solve for $p_{X_{t_{j+1}}|\mathcal{Y}_j}(x) = q(t_{j+1}, x)$ from the Cauchy problem

$$\begin{aligned} &\partial_t q_j(t,x) = a(x)\partial_x^2 q_j(t,x) + b(x)\partial_x q_j(t,x) + c(x)q_j(t,x), \\ &q_j(t)|_{\partial D} = 0, \quad q_j(t_j, x) = p_{X_{t_j}|\mathcal{Y}_j}(x) \end{aligned} \tag{1.13}$$

STEP 3 Using Bayes' formula with (1.7) at observation time t_{j+1}, we have that

$$p_{X_{t_{j+1}}|\mathcal{Y}_{j+1}}(x) = \frac{p_{Y_{j+1}|X_{t_{j+1}}}(Y_{j+1}|x)p_{X_{t_{j+1}}|\mathcal{Y}_j}(x)}{\int_D p_{Y_{j+1}|X_{t_{j+1}}}(Y_{j+1}|\xi)p_{X_{t_{j+1}}|\mathcal{Y}_j}(\xi)d\xi}, \quad \forall x \in D, \tag{1.14}$$

where $p_{Y_j|X_{t_j}}(\cdot|\cdot)$ and $p_{X_{t_{j+1}}|\mathcal{Y}_j}(\cdot)$ denote the conditional densities of Y_j given X_{t_j} and $X_{t_{j+1}}$ given $\mathcal{Y}_j = \mathcal{B}(\{Y_i;\ i \leq j\})$ respectively.

In particular, (1.14) follows under our conditions from a slight modification of Problem 33.17 in Billingsley (1986) with

$$f(x) = p_{X_{t_{j+1}}|\mathcal{Y}_j}(x), \qquad g_x(y) = p_{Y_{j+1}|X_{t_{j+1}}}(y|x)$$

and

$$p_{Y_{j+1}}(x) = p_{X_{t_{j+1}}|\mathcal{Y}_{j+1}}(x) = p_{X_{t_{j+1}}|\mathcal{B}(\mathcal{Y}_j, Y_{j+1})}(x).$$

The above three-step algorithm is also discussed in Chapter 6 of Jazwinski (1970).

The Kolmogorov equation in Step 2 of the above continuous-discrete filtering algorithm is solved subject to random initial data so it can not be solved off line but rather must be solved in real time (between observations) on a computer. Naturally, this can often be done using time stepping with sophisticated *multi-grid* or *splitting up method* on-line elliptic equation solvers. However, such methods require very complicated computer codes and are usually less efficient than the method presented here, when our method applies. Moreover, many filtering problems result in parabolic equations in Step 2 whose elliptic operators are neither self-adjoint nor diffusive or drift dominant and these features make evaluation through such traditional elliptic equation solvers more difficult. Finally, many signals evolve over infinite domains so (1.13) should be solved over a domain like $\mathbb{R}$, which is impossible for these equation solvers and some artificial boundary conditions must be added.

To motivate our strategy, we let $\Gamma(t-\tau; y, x)$ be a fundamental solution to Kolmogorov's forward equation for $\mathbf{X} = \{X_t;\ 0 \le t < \infty\}$ (i.e. the density for $\mathbf{X}$'s transition probability function) of (1.1). Then,

$$P\left[X_t^x \in dy\right] = \Gamma(t; y, x)\, dy, \tag{1.15}$$

where X^x is defined by

$$X_t^x = x + \int_0^t \alpha\left(X_s^x\right) ds + \int_0^t \sigma\left(X_s^x\right) dW_s, \tag{1.16}$$

and Step 2 above can be rewritten as

$$p_{X_{t_{j+1}}|\mathcal{Y}_j}(\xi) = \int_D \Gamma(t_{j+1} - t_j; \xi, x) p_{X_{t_j}|\mathcal{Y}_j}(x) dx. \tag{1.17}$$

Then, an alternative strategy for solving Step 2 would be to perform the indicated integration in (1.17). However, this would require a separate integration (for each ξ) and storage of Γ for all $x \in D$ (or subdomain where $X_t^{(x)}$ lives) and ξ "close to x" unless convolution can somehow be used.

Herein, we investigate exact infinite dimensional continuous-discrete filters in the single dimensional case further and find explicit representations of (1.16) for which the filtering problem degenerates into a single convolution with respect to a Gaussian kernel. Specifically, we show how to determine scalar (non-linear) Itô equations which have solutions such that (1.17) can be represented by (multiplication, substitution, and) a single convolution with respect to a Gaussian kernel. Our focus is more on exposition than complete generality.

Lockheed Martin is investigating using these methods on their search, surveillance, narcotic smuggling prevention, and military problems. In particular, our convolutional methods are being evaluated for use on long range electro-optical and infrared search and tracking problems by Lockheed Martin Tactical Defense Systems-Eagan. All of these real world problems exhibit discrete-time observations and a continuous-time signal. Also, there are simulations of our infinite-dimensional filters on a problem with a two-dimensional signal in Kouritzin (1998).

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2. Motivation through Gaussian signals

To motivate the method, we take the very simple case where the signal is characterized by affine drift and constant dispersion σ

$$X_t = X_0 + \int_0^t [\alpha_0 + \alpha_1 X_s]\,ds + \sigma W_t \qquad \forall t \geq 0. \tag{2.1}$$

Although this signal model does not preclude solution from a Kalman filter our general observation model does since it allows for non-additive, non-Gaussian noise (see e.g. (1.9)). Hence, we use the above continuous-discrete filtering algorithm with $D = \mathbb{R}$ and find that the Kolmogorov equation in Step 2 becomes

(2.2)

$$\partial_t q_j(t,x) = \frac{\sigma^2}{2}\partial_x^2 q_j(t,x) - [\alpha_0 + \alpha_1 x]\partial_x q_j(t,x) - \alpha_1 q_j(t,x) \quad q_j(t_j,x) = p_{X_{t_j}|\mathcal{Y}_j}(x),$$

and we can apply Feynman-Kac's formula (whether $\alpha_1 \geq 0$ or not) in order to find

$$p_{X_{t_{j+1}}|\mathcal{Y}_j}(x) = q_j(t_{j+1},x) = E\left[p_{X_{t_j}|\mathcal{Y}_j}(Z^x_{\delta t_j})\exp\{-\alpha_1 \times \delta t_j\}\right], \tag{2.3}$$

where $\delta t_j = t_{j+1} - t_j$ and

$$Z_t^x = x - \int_0^t [\alpha_0 + \alpha_1 Z_s^x]\,ds + \sigma W_t. \tag{2.4}$$

However, Z^x has an *explicit solution*

$$\begin{aligned} Z_t^x &= \exp[-\alpha_1 t]\left\{x - \alpha_0\int_0^t \exp\{\alpha_1 s\}\,ds + \sigma\int_0^t \exp\{\alpha_1 s\}\,dW_s\right\} \\ &\doteq \varphi\left(t, x - \int_0^t f_s dW_s\right) \qquad \forall t \geq 0, \end{aligned} \tag{2.5}$$

where

$$\varphi(t,u) \doteq \exp[-\alpha_1 t]\left\{u - \alpha_0\int_0^t \exp\{\alpha_1 s\}\,ds\right\}, \quad f_s = -\sigma\exp\{\alpha_1 s\}. \tag{2.6}$$

This means that we can represent $p_{X_{t_{j+1}}|\mathcal{Y}_j}(x)$ in terms of $\delta t_j \doteq t_{j+1} - t_j$ and the standard normal density $\Phi(y) \doteq \frac{1}{\sqrt{2\pi}}\exp\{-y^2/2\}$ as

$$\begin{aligned} p_{X_{t_{j+1}}|\mathcal{Y}_j}(x) &= \exp\{-\alpha_1\delta t_j\}\,E\left[p_{X_{t_j}|\mathcal{Y}_j}(\varphi(\delta t_j, x - \int_0^{\delta t_j} f_s dW_s))\right] \\ &= \exp\{-\alpha_1\delta t_j\}\int_{\mathbb{R}} \Upsilon_{\delta t_j}(v_x - \xi)\Phi(\xi)d\xi \end{aligned} \tag{2.7}$$

by a change of variables, where

$$v_x = x\Big/\sqrt{\int_0^{\delta t_j} f_s^2 ds} \text{ and } \Upsilon_{\delta t_j}(y) \doteq p_{X_{t_j}|\mathcal{Y}_j}(\varphi(\delta t_j, \sqrt{\int_0^{\delta t_j} f_s^2 ds} \times y)). \tag{2.8}$$

This means that $p_{X_{t_{j+1}}|\mathcal{Y}_j}$ can be evaluated by convolution, substitution, and post multiplication as follows

$$p_{X_{t_{j+1}}|\mathcal{Y}_j}(x) = \exp\{-\alpha_1 \delta t_j\} \left[\left(\Upsilon_{\delta t_j} * \Phi \right)(v_x) \right]. \tag{2.9}$$

(Since $\Phi \in \mathcal{S}(\mathbb{R})$, the space of rapidly decreasing functions, it is enough that $\Upsilon_{\delta t_j}$ is a continuous, bounded function whence a tempered distribution for convolution to be defined.) From a computer solvability point of view this convolution representation is a dramatic improvement over the partial differential equation since now we can replace specialized on-line equations solvers and time stepping with two fast Fourier transforms, i.e., we can evaluate

(2.10)
$$p_{X_{t_{j+1}}|\mathcal{Y}_j}(x) = \exp\{-\alpha_1 \delta t_j\} \mathcal{F}^{-1} \left[\mathcal{F}\Upsilon_{\delta t_j} \cdot \mathcal{F}\Phi \right](v_x), \ \mathcal{F}\Phi(\xi) = \exp\left\{ -\frac{\xi^2}{2} \right\}$$

where $\mathcal{F}$ represents Fourier transform and $\cdot$ denotes pointwise multiplication. The point of this note is that $p_{X_{t_{j+1}}|\mathcal{Y}_j}$ can be calculated in this manner for a class of signal models with *nonlinear* drift and dispersion coefficients. We need only change the functions $\exp\{-\alpha_1 \delta t_j\}$, $\Upsilon_{\delta t_j}$, and v_x.

The case where X has constant dispersion and certain class of non-linear drift has been handled in Kouritzin (1998), even in the vector-valued case. Herein, we consider both nonlinear drift and nonlinear dispersion.

3. Explicit Solutions

In our motivation, we used Feynman-Kacs and then an *explicit* solution for a simple Itô equation with affine drift and constant dispersion coefficients. We now return to more general nonlinear models as in (1.1) and recall the definitions in (1.11). We investigate *explicit* solutions of the form

$$\zeta_t^x = \varphi\left(x, t, \int_0^t f_s dW_s \right) \qquad \forall t \geq 0 \tag{3.1}$$

for some functions $\varphi \in D \times [0, \infty) \times \mathbb{R} \to \overline{\mathbb{R}}$ (the extended real line) and $f \in C(\mathbb{R}; (0, \infty))$, where ζ^x satisfies

$$\zeta_t^x = x + \int_0^t \beta(\zeta_s^x) ds + \int_0^t \sigma(\zeta_s^x) dW_s \qquad \forall t \geq 0. \tag{3.2}$$

Our main contributions of the present section are: (i) to show that any explicit solution to (3.1) with σ satisfying (3.16) below is actually a diffeomorphism $\zeta_t^x = \Lambda^{-1}\left(Z^\kappa(t, \Lambda(x)) \right)$ of a linear stochastic differential equation $dZ_t^\kappa = (-\kappa Z_t^\kappa + x)dt + dW_t$, $z_0^\kappa = \Lambda(x)$ (the reverse implication that $\Lambda^{-1}\left(Z^\kappa(t, \Lambda(x)) \right)$ satisfies (3.1) is obvious), (ii) to characterize the equations (3.2) that yield such explicit solutions, and (iii) state the results that will be required in the next section on infinite dimensional exact filters.

We will not concern ourselves with which explicit solutions can be used immediately within our filtering development or exactly how they would be used in filtering until the next section. Instead, we just fix some open interval D, most typically $(-\infty, \infty)$, $(0, \infty)$, or $(0, 1)$, and construct explicit solutions on D up to $\tau = \inf\left\{ t > 0 : \varphi(x, t, \int_0^t f_s dW_s) \in D^c \right\}$. We assume that $\frac{d}{dt}\varphi(x, t, u)$, $\frac{d}{du}\varphi(x, t, u)$, and $\frac{d^2}{du^2}\varphi(x, t, u)$ exist and are continuous on $D \times (0, \tau) \times \mathbb{R}$. Moreover, we define

$\tau_\infty = \inf\left\{t > 0 : \varphi(x, t, \int_0^t f_s dW_s) = \pm\infty\right\}$ and assume that $\varphi(x, t, \int_0^t f_s dW_s)$ is almost surely continuous on $D \times (0, \tau_\infty)$. Our first result will establish a condition on β and σ which will provide a solution to (3.2) in the form (3.1) for each $x \in D$. Throughout this section we assume that $\beta : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to [0, \infty)$ are continuous functions which are one-time respectively two-times continuously differentiable on D. We *do not* require that β or σ satisfy a linear growth condition.

LEMMA 3.1. *A necessary and sufficient condition for* (3.2) *to have an explicit (strong) solution of the form* (3.1) *up until* τ *with* f *continuous and positive is that*

$$\frac{\sigma'(y)\beta(y)}{\sigma(y)} + \frac{\sigma''(y)\sigma(y)}{2} = \beta'(y) + \kappa \tag{3.3}$$

for all $y \in D$ *such that* $\sigma(y) \neq 0$, *any constant* $\kappa \in \mathbb{R}$ *and*

$$\sigma'(y)\beta(y) = 0 \tag{3.4}$$

for all $y \in D$ *such that* $\sigma(y) = 0$.

PROOF. We note that $t \to \varphi(x, t, \int_0^t f_s dW_s)$ is a continuous stochastic process on $\{t < \tau_\infty\}$, find D^c to be closed, and use the stopping times (with respect to right continuous filtration $\{\mathcal{F}_{t+}^W, t \geq 0\}$) $\tau_N = \inf\{t > 0 : \varphi(x, t, \int_0^t f_s dW_s) \in D^c \bigcup (-\infty, -N] \bigcup [N, \infty)\}$. Then, we find $\{\tau > t\} = \bigcup_{N=1}^\infty \{\tau_N > t\}$ and τ is a stopping time. Next, we use Itô's formula on (3.1) to find that

$$\begin{aligned} d\zeta_t^x &= \left[\varphi_t\left(x, t, \int_0^t f_s dW_s\right) + \frac{1}{2} f^2(t)\varphi_{uu}\left(x, t, \int_0^t f_s dW_s\right)\right] dt \\ &\quad + f(t)\varphi_u\left(x, t, \int_0^t f_s dW_s\right) dW_t, \end{aligned} \tag{3.5}$$

on $\{t < \tau\}$, which matches (3.2) if $\varphi(x, 0, 0) = x$ for all $x \in D$ and

$$\text{(i) } f(t)\frac{\partial \varphi}{\partial u} = \sigma(\varphi) \text{ and (ii) } \frac{\partial \varphi}{\partial t} = \beta(\varphi) - \frac{1}{2} f^2(t) \frac{\partial^2 \varphi}{\partial u^2}. \tag{3.6}$$

Now, suppose $\zeta_{t\wedge\tau}^x = \int_0^{t\wedge\tau} \widehat{\beta}(\zeta_s^x) ds + \int_0^{t\wedge\tau} \widehat{\sigma}(\zeta_s^x) dW_s$ for some continuous $\widehat{\beta}, \widehat{\sigma}$. Then,

$$0 = \int_0^{t\wedge\tau} [\beta(\zeta_s^x) - \widehat{\beta}(\zeta_s^x)] ds + \int_0^{t\wedge\tau} [\sigma(\zeta_s^x) - \widehat{\sigma}(\zeta_s^x)] dW_s \tag{3.7}$$

and the first term is a continuous local martingale with zero quadratic variation. Therefore, both terms are zero for all $t > 0$. This with continuity is enough to conclude $\beta = \widehat{\beta}$ and $\sigma = \widehat{\sigma}$ on $\left\{y \in D : P(\bigcup_{s \in Q, s < \tau^+} \{|\zeta_s^x - y| < \frac{1}{n}\}) > 0 \, \forall n = 1, 2, 3, \ldots\right\}$, where Q^+ denotes the non-negative rationals. Moreover, we can use (3.6) (i) to find that

$$f^2(t)\frac{\partial^2 \varphi}{\partial u^2} = f(t)\frac{\partial \sigma(\varphi)}{\partial \varphi}\frac{\partial \varphi}{\partial u} = \frac{\partial \sigma(\varphi)}{\partial \varphi}\sigma(\varphi), \tag{3.8}$$

which can be used to reduce (3.6) (ii) and we find that (3.2) has an explicit solution of the form (3.1) if and only if $\varphi(x, 0, 0) = x$ and

$$\text{(i) } \frac{\partial \varphi}{\partial u} = \frac{\sigma(\varphi)}{f(t)} \text{ and (ii) } \frac{\partial \varphi}{\partial t} = \beta(\varphi) - \frac{\sigma'(\varphi)\sigma(\varphi)}{2}. \tag{3.9}$$

However, this is equivalent to saying that the 1-form

$$d\varphi \doteq \frac{\sigma(\varphi)}{f(t)}du + \left[\beta(\varphi) - \frac{\sigma'(\varphi)\sigma(\varphi)}{2}\right] dt \tag{3.10}$$

is exact whence closed over $\mathbb{R} \times (0, \infty)$. Therefore, the (3.2) has an explicit solution of the form (3.1) if and only if $\varphi(x, 0, 0) = x$ and

$$\frac{d}{dt}\frac{\sigma(\varphi(x,t,u))}{f(t)} = \frac{d}{du}\left[\beta(\varphi(x,t,u)) - \frac{\sigma'(\varphi(x,t,u))\sigma(\varphi(x,t,u))}{2}\right]. \tag{3.11}$$

Yet, the left hand side of (3.11) is easily calculated (with aid of (3.9)) to be

$$\frac{d}{dt}\frac{\sigma(\varphi)}{f(t)} = \frac{\sigma'(\varphi)}{f(t)}\beta(\varphi) - \frac{(\sigma'(\varphi))^2\sigma(\varphi)}{2f(t)} - \frac{\sigma(\varphi)f'(t)}{f^2(t)} \tag{3.12}$$

and the right hand side of (3.11) to be

$$\frac{\beta'(\varphi)\sigma(\varphi)}{f(t)} - \frac{\sigma''(\varphi)(\sigma(\varphi))^2}{2f(t)} - \frac{(\sigma'(\varphi))^2\sigma(\varphi)}{2f(t)}. \tag{3.13}$$

Comparing these last two equations and making cancellations, we find that

$$\frac{\sigma'(\varphi)}{\sigma(\varphi)}\beta(\varphi) - \frac{f'(t)}{f(t)} = \beta'(\varphi) - \frac{\sigma''(\varphi)\sigma(\varphi)}{2} \tag{3.14}$$

whenever $\sigma(\varphi) \neq 0$. Since this equation must hold for all t and φ with $\sigma(\varphi) \neq 0$ we find that

$$\frac{f'(t)}{f(t)} = \kappa \text{ or } f(t) = f(0)\exp(\kappa t) \tag{3.15}$$

for some constant $\kappa \in \mathbb{R}$. The case where $\sigma(y) = 0$ follows easily by setting the right hand side of (3.12) to be zero and noting that (3.13) is also zero. □

This lemma allows us to check whether a given SDE has an explicit solution of the form given above. However, from the our filtering point of view, we need to actually construct the explicit solutions and determine if convolution can be used. This problem was already partially solved in Kouritzin and Li (1998) and, some interesting examples are given there. Other work on explicit solutions has been done by Doss (1977) and Sussmann (1978).

Now, we will assume that $\sigma(x) > 0$ almost everywhere on D. We know by continuity that our explicit solutions can not explode i.e. hit $\pm\infty$ in finite time unless $\tau_\infty < \infty$. However, our construction does not preclude a possible escape from D when D is a strict subset of $\mathbb{R}$. In this case, our explicit representation may end at the escape time τ.

PROPOSITION 3.2. *Suppose that $\sigma(\cdot) \in C(\mathbb{R}; [0,\infty)) \cap C^1(D, (0,\infty))$ is a given dispersion coefficient and $\lambda \in \overline{D}$ (the closure of D) are such that*

$$\Lambda_\lambda(y) \doteq \int_\lambda^y \frac{dx}{\sigma(x)} < \infty \qquad \forall y \in D. \tag{3.16}$$

($\Lambda_\lambda(y)$ is negative when $y < \lambda$ and monotonically increasing.) Then, (3.2) has an explicit solution of form (3.1) up to $\tau = \inf\left\{t > 0 : \varphi(x, t, \int_0^t f_s dW_s) \in D^c\right\}$ if and only if

$$\beta(x) = \left(\chi - \kappa\Lambda_\lambda(x) + \frac{1}{2}\sigma'(x)\right)\sigma(x) \qquad \forall x \in D, \tag{3.17}$$

for some constants $\chi, \kappa \in \mathbb{R}$. In this case, we let $(a,b) \doteq \Lambda_\lambda(D)$ be the range of Λ_λ for some $-\infty \le a < b \le \infty$, set $f_s = \exp\{\kappa s\}$, and take

$$\varphi(x,t,u) = \begin{cases} \Lambda_\lambda^{-1}\left(\frac{u+\chi\int_0^t f_s ds+\Lambda_\lambda(x)}{f_t}\right) & a < \frac{u+\chi\int_0^t f_s ds+\Lambda_\lambda(x)}{f_t} < b \\ \inf\left\{\Lambda_\lambda^{-1}(y) : a<y<b\right\} & \frac{u+\chi\int_0^t f_s ds+\Lambda_\lambda(x)}{f_t} \le a \\ \sup\left\{\Lambda_\lambda^{-1}(y) : a<y<b\right\} & \frac{u+\chi\int_0^t f_s ds+\Lambda_\lambda(x)}{f_t} \ge b \end{cases}. \tag{3.18}$$

PROOF. Inasmuch as the value of λ does not effect the proof we will consider the case $\lambda = 0$ and let $\Lambda = \Lambda_0$ to ease the notation. Clearly, $\beta(x)\sigma'(x) = 0$ whenever $x \in D$ and $\sigma(x) = 0$. Moreover, when $\sigma(x) \neq 0$ we find that

$$\begin{aligned} \beta'(x) + \kappa &= (\chi - \kappa\Lambda(x) + \frac{1}{2}\sigma'(x))\sigma'(x) + \frac{1}{2}\sigma''(x)\sigma(x) \\ &= \frac{\sigma'(x)}{\sigma(x)}\beta(x) + \frac{\sigma''(x)\sigma(x)}{2}. \end{aligned} \tag{3.19}$$

Therefore, it follows from the previous lemma that there is an explicit solution. Now, suppose we have such a solution. Then, by the previous result again it must satisfy the overall equality in (3.19), which we can simplify in the case $\sigma(x) > 0$ to

$$\frac{d}{dx}\left(\frac{\beta(x)}{\sigma(x)}\right) = \frac{\sigma''(x)}{2} - \frac{\kappa}{\sigma(x)} \tag{3.20}$$

or taking anti-derivatives to

$$\beta(x) = \left[\frac{\sigma'(x)}{2} - \kappa\Lambda(x) + \chi\right]\sigma(x), \tag{3.21}$$

for any constant $\chi \in \mathbb{R}$. Now, since $\int_0^y \frac{dx}{\sigma(x)} < \infty$ for all $y \in D$ the set $\{x : \sigma(x) = 0\}$ must have measure zero and (3.21) holds for all $x \in D$ by continuity. Then, using the fact $\frac{d\Lambda(x)}{dx} = \frac{1}{\sigma(x)}$, we find that $\left.\frac{d\Lambda^{-1}(y)}{dy}\right|_{y=\Lambda(\varphi)} = \sigma(\varphi)$ and this can be used to show that φ, as defined in (3.18), is a solution to

$$d\varphi = \frac{\sigma(\varphi)}{f(t)}du + \left[\beta(\varphi) - \frac{\sigma'(\varphi)\sigma(\varphi)}{2}\right]dt \tag{3.22}$$

and wherefore by the proof of the previous result is a valid function to represent our explicit solution as $\zeta_t^x = \varphi\left(x, t, \int_0^t f_s dW_s\right)$. Since $\varphi(x,0,0) = x$, ζ_t^x starts in D and will leave D if $\tau < \infty$. □

EXAMPLE 3.3. We first take $D = \mathbb{R}$ and consider the case $\sigma(x) = \sigma$ is a constant. Then, we find that (3.2) has an explicit solution of the form (3.1) if and only if

$$\beta(x) = \chi\sigma - \kappa x \qquad \forall x \in \mathbb{R} \tag{3.23}$$

and some constants $\chi, \kappa \in \mathbb{R}$.

Naturally, this just corresponds to the class of Gaussian solution that we discussed in the previous section. However, we can also consider examples with different dispersion coefficients.

EXAMPLE 3.4. We consider the dispersion coefficient of a Feller branching diffusion

$$\sigma(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{3.24}$$

with $D = (0, \infty)$. In this case, $\Lambda_0(x) = 2\sqrt{x}$ for all $x > 0$ and so (3.2) has an explicit solution of the form (3.1) if and only if

$$\beta(x) = \left(\chi\sqrt{x} - 2\kappa x + \frac{1}{4}\right) \qquad \forall x > 0 \tag{3.25}$$

and some constants $\chi, \kappa \in \mathbb{R}$. This positive drift of $\frac{1}{4}$ when $x = 0$ will not allow 0 to be an absorbing state for (3.2). However, the noise is not "turned off" fast enough to prevent our explicit solutions

$$\zeta_t^x = \frac{1}{4}\left(\int_0^t e^{-\kappa(t-s)}[dW_s + \chi ds] + 2\sqrt{x}e^{-\kappa t}\right)^2 \tag{3.26}$$

from hitting zero.

We can also consider the case of the dispersion coefficient corresponding to geometric Brownian motion. In this case, we can not use $\lambda = 0$.

EXAMPLE 3.5. Suppose that $D = (0, \infty)$ and

$$\sigma(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}. \tag{3.27}$$

Then, we find that $\Lambda_1(x) = \ln(x)$ for all $x > 0$. Therefore, (3.2) has an explicit solution of the form (3.1) if and only if

$$\beta(x) = \left(\chi x - \kappa x \ln(x) + \frac{x}{2}\right) \qquad \forall x > 0 \tag{3.28}$$

and some constants $\chi, \kappa \in \mathbb{R}$. Our solution $\zeta_t^x = \exp\left[\int_0^t e^{-\kappa(t-s)}[dW_s + \chi ds] + e^{-\kappa t}\ln(x)\right]$ remains positive and does not escape D.

Finally, we can consider an example with a finite interval.

EXAMPLE 3.6. Suppose $D = (0, 1)$ and σ is the dispersion coefficient corresponding to the Wright-Fisher models in population genetics:

$$\sigma(x) = \begin{cases} \sqrt{x(1-x)} & 0 < x < 1 \\ 0 & x \leq 0, x \geq 1 \end{cases} \tag{3.29}$$

Then,

$$\Lambda_0(x) = \cos^{-1}\left(\frac{1/2 - x}{1/2}\right) \text{ and } \lim_{x \nearrow 1} \Lambda_0(x) = \pi. \tag{3.30}$$

Hence,

$$\beta(x) = \left(\chi - \kappa \cos^{-1}\left(\frac{1/2 - x}{1/2}\right)\right)\sqrt{x(1-x)} + \frac{1}{4}(1 - 2x). \tag{3.31}$$

Since $\beta(0) = \frac{1}{4}$ and $\beta(1) = -\frac{1}{4}$ there are no absorbing states. However, our explicit solutions $\zeta_t^x = [1 - \cos\left(\int_0^t e^{-\kappa(t-s)}[dW_s + \chi ds] + \cos^{-1}\left(\frac{1/2-x}{1/2}\right)\right)]/2$ will hit 0 and 1.

4. Filtering algorithms

We will now tie our explicit solution ideas into our filtering interests through the general framework set forth in the Motivation through Gaussian signals section. We will show that there are at least two forms of potential generality for filtering problems that can be solved by exact infinite-dimensional filters, one through parameters in the Riccati equation solutions below and the other through choice of explicit solution. Alternatively, these generalities can be thought of as ways of choosing and decomposing fundamental solutions to Step 2.

REMARK 4.1. Clearly, the most obvious method of obtaining an interesting class of filtering problems from our explicit solutions is through state space diffeomorphism, i.e. to form signals from $\Lambda^{-1}\big(Z^{\kappa}(t,\Lambda(x))\big)$ for some linear stochastic differential equation Z^{κ} and observations $Y_j = h\big(\Lambda(X_{tj}, V_j\big) = \widetilde{h}(Z_{tj}, V_j)$. However, this method would not include analogs to the important Benes filter nor seem to require much discussion. Therefore, we embark along a somewhat different path more aligned with substitution of variables and decomposition of fundamental solutions. Indeed, it is possible by solution of the Riccati equations below to come up with exact infinite dimensional filters with constant dispersion coefficient and nonlinear drift signals. Example 3.3 shows that these can not be handled by diffeomorphism to linear stochastic differential equation alone.

Following the ideas of Section 2, we would be tempted to apply Feynman-Kac's formula immediately on (1.13) to find that

$$p_{X_{t_{j+1}}|\mathcal{Y}_j}(x) = q_j(t_{j+1},x) = E\left[p_{X_{t_j}|\mathcal{Y}_j}(Z^x_{\delta t_j})\exp\left\{\int_0^{\delta t_j} c(Z^x_s)ds\right\}|\mathcal{Y}_j\right], \tag{4.1}$$

where $\delta t_j = t_{j+1} - t_j$ and Z^x solves

$$Z^x_t = x + \int_0^t b(Z^x_s)ds + \int_0^t \sigma(Z^x_s)dW_s \quad \forall x \in D. \tag{4.2}$$

REMARK 4.2. A sufficient condition to apply Feynman-Kac's formula in the present setting with $\tau_N \doteq \{t \geq 0 : |Z^x_t| \geq N\}$ is that

$$\lim_{M\to\infty} E \sup_{N\geq M}\left[\exp\left\{\int_0^{\tau_N} c(Z^x_s)ds\right\} 1_{\tau_N<\delta t_j}\right] < \infty, \tag{4.3}$$

which is certainly true when $c : \mathbb{R} \to (-\infty, C]$ for any $C > 0$.

However, returning to (4.1), we see that there now is dependence upon the whole process distribution for $\mathbf{Z}^x$ in (4.1) meaning that convolution will not be immediately possible in Step 2 of the Introduction. Hence, our first task will be to remove the dependence on Z^x_s for $s \neq t$. This is most readily done by changing the potential term in Equation (1.13) of Step 2 into a constant term through a substitution of variables and Bernoulli order reduction:

LEMMA 4.3. *Suppose that* $T > 0$, $\eta \in \mathbb{R}$, $m(x) \doteq \frac{b(x)}{2a(x)}$ *and* v *solves the equation*

$$\begin{aligned} \partial_t r(t,x) &= a(x)\partial_x^2 r(t,x) + \beta(x)\partial_x r(t,x) + \eta r(t,x), \\ r(0,x) &= \phi(x) \end{aligned} \tag{4.4}$$

on D or some open interval containing D, where $\beta(x) \doteq 2\mu(x)a(x)$ and μ solves the ordinary differential equation

$$\frac{d\mu(x)}{dx} = \frac{dm(x)}{dx} - \mu^2(x) + m^2(x) + \frac{\eta - c(x)}{a(x)} \tag{4.5}$$

almost everywhere on D. Moreover, assume that $\int_\psi^x |\mu(z) - m(z)|\, dz < \infty$ for some $\psi \in \mathbb{R}$ and all $x \in D$ and

$$\lim_{x \to y} v(t,x) \exp\left\{\int_\psi^x \mu(z) - m(z)\, dz\right\} = 0 \tag{4.6}$$

for each $y \in \partial D$, $t \in [0,T]$. Then, $u(t,x) = v(t,x)\exp\left\{\int_\psi^x \mu(z) - m(z) dz\right\}$ solves the equation

$$\begin{aligned} \partial_t q(t,x) &= a(x)\partial_x^2 q(t,x) + b(x)\partial_x q(t,x) + c(x)q(t,x), \\ q(0,x) &= \phi(x)\exp\left\{\int_\psi^x \mu(z) - m(z) dz\right\}, \ q(t)|_{\partial D} = 0 \end{aligned} \tag{4.7}$$

on $[0,T] \times D$.

Naturally, this result is proved by substitution and cancellation. Preceding formally for the moment, we know from (F5) that Equation (1.13) in Step 2 of the introduction with the added Dirichlet boundary condition has a unique solution, which must be given by:

$$p_{X_{t_{j+1}}|\mathcal{Y}_j}(x) = v_j(t_{j+1}, x)\exp\left\{\int_\psi^x \mu(z) - m(z)dz\right\}, \tag{4.8}$$

where $m(x) \doteq \frac{b(x)}{2a(x)}$, μ is a continuous solution to (4.5), $\beta(x) \doteq 2\mu(x)a(x)$, and

$$\begin{aligned} \partial_t v_j(t,x) &= a(x)\partial_x^2 v_j(t,x) + \beta(x)\partial_x v_j(t,x) + \eta v_j(t,x), \\ v_j(t_j, x) &= p_{X_{t_j}|\mathcal{Y}_j}(x)\exp\left\{\int_\psi^x m(z) - \mu(z)dz\right\}. \end{aligned} \tag{4.9}$$

Now, we can apply Feynman-Kac's formula to v_j to find that

$$p_{X_{t_{j+1}}|\mathcal{Y}_j}(x) = E\left[\theta(\zeta_{\delta t_j}^x)\exp\{\eta \times \delta t_j\}\right]\exp\left\{\int_\psi^x \mu(z) - m(z)dz\right\}, \tag{4.10}$$

for each $x \in D$, where

$$\theta(\xi) = p_{X_{t_j}|\mathcal{Y}_j}(\xi) \times \exp\left\{\int_\psi^\xi m(z) - \mu(z)dz\right\} \quad \forall \xi \in D, \tag{4.11}$$

$$\zeta_t^x = x + \int_0^t \beta(\zeta_s^x)ds + \int_0^t \sigma(\zeta_s^x)dW_s \qquad \forall t \geq 0. \tag{4.12}$$

However, suppose

$$\frac{\sigma'(y)\beta(y)}{\sigma(y)} + \frac{\sigma''(y)\sigma(y)}{2} = \beta'(y) + \kappa \tag{4.13}$$

for all $y \in D$ such that $\sigma(y) \neq 0$ and some constant $\kappa \in \mathbb{R}$,

$$\sigma'(y)\beta(y) = 0 \tag{4.14}$$

for all y such that $\sigma(y) = 0$, and $\int_\lambda^x \frac{dy}{\sigma(y)} < \infty$ for all $x \in D$ and some $\lambda \in \mathbb{R}$. Then, ζ^x has an explicit solution for each $x \in D$ of form

$$\zeta_t^x = \Lambda_\lambda^{-1}\left(\frac{\chi \int_0^t f_s ds + \Lambda_\lambda(x) - \int_0^t (-f_s) dW_s}{f_t}\right) \quad \forall 0 \le t \le \tau, \tag{4.15}$$
$$\tau = \inf\left\{t > 0 : \varphi(x,t,\int_0^t f_s dW_s) \in D^c\right\},$$

where $f_s = \exp\{\kappa s\}$ and $\Lambda_\lambda(y) = \int_\lambda^y \frac{dx}{\sigma(x)}$ for all $y \in D$. Thus, if $P(\tau > \delta t_j) = 1$ we find that $p_{X_{t_{j+1}}|\mathcal{Y}_j}(x)$ can be evaluated by

$$p_{X_{t_{j+1}}|\mathcal{Y}_j}(x) = E\left[\theta(\Lambda_\lambda^{-1}\left(\frac{\chi \int_0^{\delta t_j} f_s ds + \Lambda_\lambda(x) - \int_0^{\delta t_j}(-f_s)dW_s}{f_{\delta t_j}}\right))\right] \times H_\psi(\delta t_j, x), \tag{4.16}$$

where

$$H_\psi(t,x) = \exp\left\{\eta \times t + \int_\psi^x \mu(z) - m(z) dz\right\} \quad \forall t \ge 0, x \in D. \tag{4.17}$$

However, letting

$$\Upsilon_{\delta t_j}(y) = \theta(\Lambda_\lambda^{-1}\left(\frac{\chi \int_0^{\delta t_j} f_s ds + \sqrt{\int_0^{\delta t_j} f_s^2 ds} \times y}{f_{\delta t_j}}\right)), \quad v_x = \frac{\Lambda_\lambda(x)}{\sqrt{\int_0^{\delta t_j} f_s^2 ds}}, \tag{4.18}$$

we find that

$$E\left[\theta(\Lambda_\lambda^{-1}\left(\frac{\chi \int_0^{\delta t_j} f_s ds + \Lambda_\lambda(x) - \int_0^{\delta t_j}(-f_s)dW_s}{f_{\delta t_j}}\right))\right] \tag{4.19}$$
$$= \int_{\mathbb{R}} \Upsilon_{\delta t_j}(v_x - \xi)\Phi(\xi)\, d\xi = (\Upsilon_{\delta t_j} * \Phi)(v_x),$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\}$, is again evaluated via convolution.

Now, we suppose again that our signal is

$$X_t = X_0 + \int_0^t \alpha(X_s) ds + \int_0^t \sigma(X_s) dW_s,\ X_t \subset D \tag{4.20}$$

and determine a class of α corresponding to a particular σ for which filtering through convolution with a standard normal distribution is possible. In particular, we assume that Conditions (F1-F5) are true, so we can use the Three-Step Algorithm with

$$a(x) = \frac{\sigma^2(x)}{2},\ b(x) = 2\sigma(x)\sigma'(x) - \alpha(x),\ c(x) = (\sigma'(x))^2 + \sigma(x)\sigma''(x) - \alpha'(x). \tag{4.21}$$

Then, it follows from (4.21) and $m(x) = b(x)/\sigma^2(x)$ that

$$\frac{dm(x)}{dx} + m^2(x) - \frac{2c(x)}{\sigma^2(x)} = \frac{d}{dx}\left(\frac{\alpha(x)}{\sigma^2(x)}\right) + \frac{\alpha^2(x)}{\sigma^4(x)} \quad \forall x \in D. \tag{4.22}$$

Moreover, we know from our section on explicit solution that

$$\beta(x) = (\chi - \kappa\Lambda_\lambda(x) + \frac{1}{2}\sigma'(x))\sigma(x), \quad \mu(x) = \beta(x)/\sigma^2(x), \tag{4.23}$$

on D so

(4.24)
$$\frac{d\mu(x)}{dx}+\mu^2(x)=\frac{1}{\sigma^2(x)}\left[\frac{\sigma''(x)\sigma(x)}{2}-\kappa-\frac{(\sigma'(x))^2}{4}+(\chi-\kappa\Lambda_\lambda(x))^2\right]\quad\forall x\in D.$$

This can be substituted into (4.5) together with (4.22) and the definitions in (4.21) to find that

(4.25)
$$\frac{d}{dx}\frac{\alpha(x)}{\sigma^2(x)}+\left(\frac{\alpha(x)}{\sigma^2(x)}\right)^2=Q(x),$$
$$Q(x)=\frac{\frac{\sigma''(x)\sigma(x)}{2}-\frac{(\sigma'(x))^2}{4}-2\eta-\kappa+(\kappa\Lambda_\lambda(x)-\chi)^2}{\sigma^2(x)},$$

which is a Riccati equation in $\frac{\alpha(x)}{\sigma^2(x)}$ to be solved on D. Furthermore, (4.23) and $\varphi=p_{X_{t_j}|\mathcal{Y}_j}$ can be substituted into (4.9) to find that

(4.26)
$$\partial_t v_j(t,x)=\frac{\sigma^2(x)}{2}\partial_x^2 v_j(t,x)+(\chi-\kappa\Lambda_\lambda(x)+\frac{\sigma'(x)}{2})\sigma(x)\partial_x v_j(t,x)+\eta v_j(t,x),$$

$$v_j(t_j,x)=\varphi(x)\exp\left\{\int_\psi^x m(z)-\mu(z)dz\right\}\quad\forall x\in D.$$

To implement Step 2 as a convolution, we require a solution for (4.25,4.26) for *some* constants $\kappa,\eta,\chi,\lambda,\psi\in\mathbb{R}$. However, to be a valid signal process we also require that (4.20) have a unique-in-law weak solution. Finally, we also need to be able to apply Feynman-Kac's formula in (4.10), satisfy (4.6) with v_j in place of v and have $P(\tau>\delta t_j)=1$. These criteria motivate the following definition.

Definition 4.4. Suppose D is some open interval, $\sigma\in C(\mathbb{R},[0,\infty))$, and $\sigma|_D\in C^2(D,[0,\infty))$. Then, $\alpha:\mathbb{R}\to\mathbb{R}$ corresponds to an infinite-dimensional exact Gaussian filter if: (i) $\frac{\alpha}{\sigma^2(x)}$ is a $C^1(D,\mathbb{R})$-solution to (4.25) for some collection of constants $\kappa,\eta,\chi,\lambda\in\mathbb{R}$, (ii) $\int_\psi^x\left|\frac{\chi-\kappa\Lambda_\lambda(z)-\frac{3}{2}\sigma'(z)}{\sigma(z)}+\frac{\alpha(z)}{\sigma^2(z)}\right|dz<\infty$ for all $x\in D$, the same κ,χ,λ, and some $\psi\in\mathbb{R}$ (iii) there exists a solution v_j to (4.26) such that

$$\lim_{x\to y}v_j(t,x)\exp\left\{\int_\psi^x\frac{\chi-\kappa\Lambda_\lambda(z)-\frac{3}{3}\sigma'(z)}{\sigma(z)}+\frac{\alpha(z)}{\sigma^2(z)}dz\right\}=0$$
$$\forall y\in\partial D,t\in[t_j,t_{j+1}]$$

for these constants $\kappa,\eta,\chi,\lambda,\psi$ and every continuous, bounded, non-negative φ on D with $\varphi|_{\partial D}=0$, (iv) there is a (unique-in-law) weak solution X to (1.1) such that $X_t\subset D$ and Condition (F5) is true, and (v) our explicit solution from (4.15) satisfies

(4.27)
$$\lim_{M\to\infty}E\left[\sup_{N\geq M}\exp\left[2\int_\psi^{\zeta^x(\tau_N)}m(z)-\mu(z)dz\right]1_{\tau_N<\delta t_j}\right]<\infty$$

for each $x\in D$ with $\tau_N^x\doteq\inf\{s\geq t:|\zeta_s^x|>N\}$ in order that the Feynman-Kac formula applies in (4.10) and $P(\tau>t_j)=1$ with τ as defined in (4.15).

REMARK 4.5. (ii) is precisely the condition in Lemma 4.3. Suppose $\psi \in D$. Then, $\int_\psi^x \left|\frac{\alpha(z)}{\sigma^2(z)}\right| dz < \infty$ for each $x \in D$ by (i). To utilize (v) in Feynman-Kac's formula at (4.10), we use the fact that $v_j(t,x) \times \exp\left[\int_\psi^x \mu(z) - m(z)dz\right]$ is bounded. We are investigating new methods that may combine and reduce conditions (ii), (iv), and (v).

This definition allows us to convert the previous formal development into a theorem and a procedure:

THEOREM 4.6. *Suppose Conditions* (F1-F4) *are true,* $\sigma \in C(\mathbb{R}, [0,\infty))$, $\sigma|_D \in C^2(D, [0,\infty))$, *and* α *corresponds to an infinite-dimensional exact Gaussian filter. Then,* $P_{X_{t_j}|\mathcal{Y}_j}(\cdot)$ *has a density that can be calculated via the following algorithm:*

1. Store the functions

$$F(x) \doteq \exp\left\{\int_\psi^{\Lambda_\lambda^{-1}(x)} m(z) - \mu(z)dz\right\} \quad \forall x \in \mathbb{R} \tag{4.28}$$

$$H_\psi(\delta t_j, x) \doteq \exp\left\{\eta \times \delta t_j + \int_\psi^x \mu(z) - m(z)dz\right\} \quad \forall x \in \mathbb{R}$$

$$v_x \doteq \Lambda_\lambda(x) / \sqrt{\int_0^{\delta t_j} f_s^2 ds} \quad \forall x \in \mathbb{R}$$

2. Store the constants

$$a = \frac{\sqrt{\int_0^{\delta t_j} f_s^2 ds}}{f_{\delta t_j}} \text{ and } b = \frac{\chi \int_0^{\delta t_j} f_s ds}{f_{\delta t_j}} \tag{4.29}$$

3. $p_{X_{t_0}|\mathcal{Y}_0}(x) = p_{X_0}$ (* Step 1 of Introduction *)
4. Do for $j = 0, 1, 2, ...$
   (a) Form $\Xi(x) = p_{X_{t_j}|\mathcal{Y}_j}(\Lambda_\lambda^{-1}(x)) \times F(x)$
   (b) Take a (Fast) Fourier transform $\widehat{\Xi}$ of Ξ
   (c) Pointwise multiply in frequency domain
   $\widehat{\Psi}(\xi) = \left[\frac{1}{a}\exp\left(i\frac{b}{a}\xi\right)\widehat{\Xi}(\xi/a)\right]\exp\left[-\frac{\xi^2}{2}\right]$
   (d) Take inverse (Fast) Fourier transform Ψ of $\widehat{\Psi}$
   (e) Substitute in and post multiply $p_{X_{t_{j+1}}|\mathcal{Y}_j}(x) = \Psi(v_x)H_\psi(\delta t_j, x)$
   (f) Wait for the next observation $Y_{j+1}(\omega)$.
   (g) Evaluate the Bayes' rule update (* Step 3 of Introduction *)

$$p_{X_{t_{j+1}}|\mathcal{Y}_{j+1}}(x) = \frac{p_{Y_{j+1}|X_{t_{j+1}}}(Y_{j+1}|x)p_{X_{t_{j+1}}|\mathcal{Y}_j}(x)}{\int_\mathbb{R} p_{Y_{j+1}|X_{t_{j+1}}}(Y_{j+1}|z)p_{X_{t_{j+1}}|\mathcal{Y}_j}(z)dz}. \tag{4.30}$$

In particular, to save time, one can work with unnormalized conditional densities and avoid the calculation of and division by $\int_\mathbb{R} p_{Y_{j+1}|X_{t_{j+1}}}(Y_{j+1}|z)p_{X_{t_{j+1}}|\mathcal{Y}_j}(z)dz$ until one wishes to integrate with the density $p_{X_{t_{j+1}}|\mathcal{Y}_{j+1}}$ to e.g. form the conditional mean. Indeed, it could be avoided entirely if one only wishes to obtain a maximum a posteriori estimator of $X_{t_{j+1}}$.

Finally, we turn to question of finding $\alpha's$ corresponding to infinite dimensional Gaussian filters. We know that (4.25) is a Riccati equation with a continuous right hand side Q. Such Riccati equations have been heavily studied. Most solutions on

D are not solvable by quadrature. There are some well known methods to employ in trying to establish solutions. First, if the solution to the linear equation

$$\begin{bmatrix} z' \\ w' \end{bmatrix} = \begin{bmatrix} 0 & Q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}, \quad \begin{bmatrix} z(x_0) \\ w(x_0) \end{bmatrix} = \begin{bmatrix} y(x_0) \\ 1 \end{bmatrix} \tag{4.31}$$

satisfies $w(x) > 0$ for all $x \in D$. Then, $y = z/w$ is a solution to (4.25). Once, one solution y_1 (say) has been found others can be found by solving (the linear equation)

$$v' = 2y_1 v + 1 \tag{4.32}$$

and setting $y = y_1 + \frac{1}{v}$ provided $v \neq 0$ on D.

Most often one solves (4.25) or (4.31) on a computer. Naturally, in our application this can be done off-line long before the filtering procedure starts. We give a couple of simple examples of solutions:

EXAMPLE 4.7. Suppose that $D = \mathbb{R}$ and $Q(x) = (c_1)^2$ for all $x \in \mathbb{R}$ and some constant c_1. Then, $y(x) = c_1 \tanh(c_1 x)$ solves (4.25). Comparing $Q(x) = (c_1)^2$ to (4.25), one finds that this is possible when $\sigma(x) = c_2$ (some constant) as in Example 3.3. This present example is related to the work in Kouritzin (1998) and the original filter of Benes (1981).

EXAMPLE 4.8. Suppose that $D = (0, \infty)$, $0 < x_0 < \infty$, and $Q(x)$ is non-negative and continuous on D. Then, there exists a unique continuous solution to

$$\frac{d}{dx}(y(x)) + (y(x))^2 = Q(x), \qquad y(x_0) = 0 \tag{4.33}$$

on D. This solution satisfies $y(x) \geq 0$ for all $x \geq x_0$ and $y(x) \leq 0$ for all $x \leq x_0$. The condition $2\eta + \kappa \leq -\frac{1}{4}$ will ensure that Q remains non-negative for $\sigma(x) = x$ as in Example 3.5. There are better results on Riccati equations in many textbooks. We have just given a simple result whose proof is immediate.

In the above example (as well as on different domains), it is well known that there still can be solutions without the constraint that Q is non-negative provided that it does not become "too negative". Still, even when there is a solution y to (4.25) one must check that $\alpha \doteq y \cdot \sigma^2$ corresponds to an infinite-dimensional exact Gaussian filter. This has been done for simple multidimensional examples in Kouritzin (1998) together with simulations and more examples are being worked on.

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