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Hölder continuity for spatial and path processes via spectral analysis

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Abstract. For $\nu(d\theta)$, a σ-finite Borel measure on R^d, we consider $L^2(\nu(d\theta))$-valued stochastic processes $Y(t)$ with the property that $Y(t) = y(t,\cdot)$ where $y(t,\theta) = \int_0^t e^{-\lambda(\theta)(t-s)} dm(s,\theta)$ and $m(t,\theta)$ is a continuous martingale with quadratic variation $[m](t) = \int_0^t g(s,\theta)\, ds$. We prove timewise Hölder continuity and maximal inequalities for Y and use these results to obtain Hilbert space regularity for a class of superprocesses as well as a class of stochastic evolutions of the form $dX = AXdt + GdW$ with W a cylindrical Brownian motion. Maximal inequalities and Hölder continuity results are also proven for the path process $\mathbf{Y}_t(\tau) \overset{\circ}{=} Y(\tau t \wedge t)$.

0. Introduction

Stochastic evolution equations in infinite dimensions are used to model space-time phenomena in many scientific disciplines. Examples and background material can be found in references such as Da Prato and Zabczyk (1992), Kallianpur and Xiong (1995), and Walsh (1984). An important generic example consists of the formal equation

$$dX(t) = AX(t)dt + dM(t), \tag{0.1}$$

where A is a linear operator. For appropriate test functions φ, (0.1) is shorthand for the rigorous one dimensional equation

$$X(t,\varphi) = X(0,\varphi) + \int_0^t X(s, A^*\varphi)ds + M(t,\varphi), \tag{0.2}$$

where $M(t,\varphi)$ is a continuous martingale, and $X(t, f)$ represents X acting on the function f.

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A natural question is whether X can be interpreted as a stochastic process taking values in a function space such as a Hilbert space. Typically the "nicest" possible spaces are desired, and, in addition, regularity properties such as Hölder continuity of the sample paths are sought.

When φ is an eigenfunction of A^* with eigenvalue $a(\varphi)$, (0.2) becomes

$$X(t,\varphi) = X(0,\varphi) + a(\varphi)\int_0^t X(s,\varphi)ds + M(t,\varphi). \tag{0.3}$$

Applying variation of constants gives

$$X(t,\varphi) = e^{a(\varphi)}X(0,\varphi) + \int_0^t e^{a(\varphi)(t-s)}dM(s,\varphi). \tag{0.4}$$

If the class of eigenfunctions is rich enough, then X can be investigated using the Ornstein-Uhlenbeck like properties of these scalar equations. Although requiring eigenfunctions obviously restricts the class of solutions of (0.1) that can be investigated, we obtain regularity results for two important examples. One case is when $dM(s) = G(s)dW(s)$ with $G(s)$ being a random operator and W a cylindrical Hilbert space valued Brownian motion. Our other example is a class of measure-valued diffusions known as $(\alpha, d, 1)$ superprocesses, for which there are dimension-dependent difficulties in establishing such white noise representations. However, an advantage of our method is that Hilbert space regularity for both these examples follows from the same general result. In addition we obtain maximal inequalities for

$$E\Big[\sup_{s\le u,v\le t} \|Y(u) - Y(v)\|^{2r}\Big]$$

with $\|\cdot\|$ being the Hilbert space norm and $Y(t,\varphi) \overset{\circ}{=} \int_0^t e^{a(\varphi)(t-s)}dM(s,\varphi)$ being the stochastic convolution part of (0.4). We also obtain Hölder continuity results for the path process

$$\mathbf{Y}_t = \begin{cases} Y(\tau t) & 0 \le \tau \le 1 \\ \\ Y(t) & \tau \ge t. \end{cases}$$

For this we rely on a maximal inequality from Kouritzin and Heunis (1994). Another important inequality we use to obtain Hölder continuity results is an extension of Kolmogorov's inequality to Banach space valued random variables found in Revuz and Yor (1994).

In our final remarks we discuss the relationship of our results to those of other authors. Also, see Remark 1.2.

1. Space-time regularity results for a class of stochastic convolutions

During the course of a proof we use the same symbol c for constants, although the exact value of the constant may change. We show the dependence of c on relevant parameters unless suppression causes no confusion. All processes are defined on a probability space, $(\Omega, \mathscr{F}, P)$, but our notation surpresses dependence on $w \in \Omega$.

Lemma 1.1. *Assume $m(t)$ is a continuous martingale with quadratic variation process*

$$[m](t) = \int_0^t g(s)ds,$$

with $\sup_{0\le u\le t} E[g(u)^r] < \infty$ *for some* $r \ge 1$. *Then,*

$$y(t) = \int_0^t e^{-\lambda(t-s)} dm(s)$$

satisfies, for $0 \le s \le t$,

$$E[|y(t) - y(s)|^{2r}] \le C \sup_{0\le u\le t} E[g(u)^r] \left| \frac{1 - e^{-2\lambda|t-s|}}{2\lambda} \right|^r;$$

where $C = c(r)$ *if* $\lambda \ge 0$, $C = c(r)e^{2|\lambda|rt}$ *if* $\lambda < 0$, *and we set* $(1 - e^{-2\lambda|t-s|})/(2\lambda) = |t-s|$ *if* $\lambda = 0$.

Proof. For $0 \le s \le t$ and $\lambda \ne 0$,

$$y(t) - y(s) = (e^{-\lambda(t-s)} - 1)y(s) + \int_s^t e^{-\lambda(t-u)} dm(u).$$

Successively applying Burkholder's and then Jensen's inequality (with probability measure $\frac{e^{2\lambda u}2\lambda}{e^{2\lambda s}-1}\, du$ for the first term), we obtain

$$\begin{aligned} E[|y(t) - y(s)|^{2r}] &\le c(r) \left[\begin{array}{c} |1 - e^{-\lambda(t-s)}|^{2r} E\left[\left(\int_0^s e^{-2\lambda(s-u)} g(u)du\right)^r\right] \\ + E\left[\left(\int_s^t e^{-2\lambda(t-u)} g(u)du\right)^r\right] \end{array} \right] \\ &\le c(r) \left[\begin{array}{c} |1 - e^{-\lambda(t-s)}|^{2r} \left(\frac{1 - e^{-2\lambda s}}{2\lambda}\right)^r \sup_{0\le u\le s} E[g(u)^r] \\ + \left(\frac{1 - e^{-2\lambda(t-s)}}{2\lambda}\right)^r \sup_{s\le u\le t} E[g(u)^r] \end{array} \right]. \end{aligned}$$

Now we apply $a^2 \le (a+b)^2 - b^2$ with $a = 1 - e^{-\lambda(t-s)}$, $b = e^{-\lambda(t-s)}$ if $\lambda > 0$ and $a = e^{-\lambda(t-s)} - 1$, $b = 1$ if $\lambda < 0$ to find

$$|1 - e^{-\lambda(t-s)}|^2 \le |1 - e^{-2\lambda(t-s)}|.$$

If $\lambda = 0$, the proof follows directly from Burkholder's and Jensen's inequality. □

Let ν be a σ-finite Borel measure, and let $\lambda(\theta) \ge \Lambda$ be a Borel measurable function of $\theta \in R^d$. Let $m(t, \theta)$, for fixed $\theta \in R^d$, be a continuous martingale with quadratic variation process

$$[m(\cdot, \theta)](t) = \int_0^t g(s, \theta)ds, \quad \text{where } g(w, s, \theta)$$

is measurable in (w, s, θ). Also, we assume that

$$\sup_{0 \le u \le t} E[g(u, \theta)^r]^{1/r} \le G(r, t, \theta)$$

and G is measurable in θ for each $t \le T$ and $r \ge 1$.

Lemma 1.2. *Let* $r \ge 1$ *and*

$$y(t, \theta) = \int_0^t e^{-\lambda(\theta)(t-s)} dm(s, \theta).$$

Assume for some $a \ge 0$ *and* $\delta \in (0, 1]$ *that*

$$\int_{\lambda(\theta) \le a} G(r, T, \theta)\nu(d\theta) < \infty$$

and

$$\int_{\lambda(\theta) > a} G(r, T, \theta)\lambda^{\delta-1}(\theta)\nu(d\theta) < \infty.$$

Then, for $0 \le s \le t \le T$,

$$E\left[\left(\int_{R^d} |y(t, \theta) - y(s, \theta)|^2 \nu(d\theta)\right)^r\right] \le C(T, \delta, r, a, \Lambda)|t - s|^{r\delta}.$$

Proof. Successively applying Lemma A.1 (to follow) and Lemma 1.1 to $|y(t, \theta, w) - y(s, \theta, w)|^2$ we obtain

$$\begin{aligned}
&E\left[\left(\int_{R^d} |y(t, \theta) - y(s, \theta)|^2 \nu(d\theta)\right)^r\right] \\
&\le \left(\int_{R^d} E[|y(t, \theta) - y(s, \theta)|^{2r}]^{1/r} \nu(d\theta)\right)^r \\
&\le \left(\int_{R^d} C^{\frac{1}{r}}\, G(r, T, \theta)\left(\frac{1 - e^{-2\lambda(\theta)|t-s|}}{2\lambda(\theta)}\right)\nu(d\theta)\right)^r \\
&\le c(r)\left[\begin{array}{l} \left(\int_{\lambda(\theta) \le a} e^{4|\Lambda|T} G(r, T, \theta)|t - s|\nu(d\theta)\right)^r \\ + \left(\int_{\lambda(\theta) > a} G(r, T, \theta)|t - s|^\delta \lambda^{\delta-1}(\theta)\nu(d\theta)\right)^r \end{array}\right];
\end{aligned}$$

where we've used the fact that, for $\beta \ge 0$, $(1 - e^{-\beta|t-s|})/\beta \le$ minimum $(|t - s|, \beta^{\delta-1}|t - s|^\delta)$ for $\delta \in (0, 1]$; and for $\beta < 0$, $(1 - e^{-\beta|t-s|})/\beta \le e^{|\beta||t-s|}|t - s|$. We've also used the value of C given by Lemma 1.1. Noting $|t - s|^r + |t - s|^{r\delta} = |t - s|^{r\delta}(|t - s|^{r(1-\delta)} + 1)$ completes the proof. □

Recalling our previous notation, we let

$$L^2(\nu(d\theta)) = \{f(\theta) : \|f\|^2 = \int_{R^d} |f(\theta)|^2 \nu(d\theta) < \infty\}.$$

Under the assumptions of Lemma 1.2, we can define

$$Y(t) = y(t, \cdot) \quad \text{with} \quad E[\|Y(t)\|^2] < \infty$$

for each t. By applying Lemma 1.2, together with Theorem I.2.1 of Revuz and Yor (1994) which is an extension of Kolmogorov's criterion, we immediately obtain:

Theorem 1.1. *If the assumptions of Lemma 1.2 hold with $r\delta > 1$, then Y has a modification $\tilde{Y}$ satisfying*

$$E[\sup_{0 \le s < t \le T} (\|\tilde{Y}(t) - \tilde{Y}(s)\| / |t - s|^{\beta})^{2r}] < \infty$$

for any $\beta \in [0, (\delta/2) - (1/2r))$; and $\widetilde{Y}$ has Hölder continuous paths of any order $\beta \in (0, (\delta/2) - (1/2r))$.

Remark 1.1. In the sequel, we will let Y denote the Hölder continuous modification $\widetilde{Y}$.

Example 1.1. For our first application of Theorem 1.1 we prove Hölder continuity for the paths of the $(\alpha, d, 1)$ superprocess; see Dawson (1993). This is a continuous Markov process taking values in the space of finite Borel measures on R^d topologized using the Prohorov metric; see Ethier and Kurtz (1986). The process $X(t)$ solves a martingale problem:

If $f : R^d \to R$ is bounded and continuous with two bounded and continuous derivatives, then (denoting $\int f(x)X(t, dx)$ by $X(t, f)$) X satisfies

$$X(t, f) = X(0, f) + \int_0^t X(s, A_\alpha f)ds + M(t, f),$$

where A_α is the generator of a symmetric stable process on R^d of index $\alpha \in (0, 2]$; and if $E[X(0, 1)] < \infty$, $M(\cdot, f)$ is a continuous square integrable martingale with quadratic variation process

$$[M(\cdot, f)](t) = \int_0^t X(s, f^2)ds.$$

Let $e_{-\theta}(x) = e^{-i\theta \cdot x}$ for $\theta, x \in R^d$. Letting $\hat{X}(t, \theta) = X(t, e_{-\theta})$ and $\hat{M}(t, \theta) = M(t, e_{-\theta})$, we obtain

$$\hat{X}(t, \theta) = \hat{X}(0, \theta) - |\theta|^{\alpha} \int_0^t \hat{X}(s, \theta)ds + \hat{M}(t, \theta); \tag{1.1}$$

here $\hat{M}(t, \theta)$ is a complex martingale with

$$[\text{Re } \hat{M}(\cdot, \theta)](t) = \int_0^t X(s, \cos^2[\theta \cdot (\cdot)])ds$$

and

$$[\text{Im } \hat{M}(\cdot, \theta)](t) = \int_0^t X(s, \sin^2[\theta \cdot (\cdot)])ds.$$

Using variation of constants, we rewrite (1.1) as

$$\hat{X}(t,\theta) = e^{-|\theta|^\alpha t}\hat{X}(0,\theta) + \int_0^t e^{-|\theta|^\alpha (t-s)} d\hat{M}(s,\theta). \tag{1.2}$$

Letting $S_\alpha(t)$ denote the Feller semigroup generated by A_α, we write (1.2) as

$$X(t) = S_\alpha(t)X(0) + \int_0^t S_\alpha(t-s)dM(s); \tag{1.3}$$

where we interpret each term in (1.3) as a measure (or signed measure) with Fourier transform as given in (1.2).

We set

$$Y(t) = \int_0^t S_\alpha(t-s)dM(s). \tag{1.4}$$

Let $d\theta$ be Lebesgue measure on R^d. For $\gamma \in R$ and f denoting any tempered distribution having a distributional Fourier transform which can be represented as a function $\hat{f}(\theta)$, define the standard Sobolev spaces

$$H_\gamma = \{f : \|f\|_\gamma^2 = \int_{R^d} |\hat{f}(\theta)|^2(1+|\theta|^2)^\gamma d\theta < \infty\}.$$

Note $H_0 = L^2(d\theta)$ and $H_a \subset H_b$ for $b < a$. In Blount and Bose (2000a,b) it was shown that $Y \in C([0,\infty) : H_\gamma)$ a.s. and $X \in C((0,\infty) : H_\gamma)$ if $\gamma < (\alpha - d)/2$. Now we apply Theorem 1.1 to show that this can be strengthened to obtain Hölder continuity. It also provides a new proof of continuity.

Theorem 1.2. *Assume* X, Y *are as in* (1.3–1.4), $\gamma < (\alpha - d)/2$ *and* $E[X(0,1)^r] < \infty$, *where* $r > 1$ *satisfies* $r > \alpha/(\alpha - 2\gamma - d)$. *Then,* Y *has a modification satisfying*

$$E[\sup_{0\le s<t\le T}(\|Y(t)-Y(s)\|_\gamma/|t-s|^\beta)^{2r}] < \infty$$

for any $\beta \in \left[0, \left(\frac{\alpha-2\gamma-d}{2\alpha}\right) \wedge \frac{1}{2} - \frac{1}{2r}\right)$.

Corollary 1.1. *Assume* $\gamma < (\alpha - d)/2$ *and* $P(X(0,1) < \infty) = 1$. *Then, for any* $\beta \in \left[0, \left(\frac{\alpha-2\gamma-d}{2\alpha}\right) \wedge \frac{1}{2}\right)$ *and* $S, T > 0$,

$$P(\sup_{0\le s<t\le T}(\|Y(t)-Y(s)\|_\gamma/|t-s|^\beta) < \infty) = 1$$

and

$$P(\sup_{S\le s<t\le T}(\|X(t)-X(s)\|_\gamma/|t-s|^\beta) < \infty) = 1.$$

If $\gamma < -(d/2)$, *we may take any* $\beta \in [0,(1/2))$; *and if* $\gamma \le -(\alpha+d)/2$, *we can set* $S = 0$.

Proof of Theorem 1.2. Note $X(t,1)$ is a Feller diffusion process with $X(t,1) = X(0,1) + M(t,1)$, where $M(t,1)$ is a martingale with

$$[M(\cdot,1)](t) = \int_0^t X(s,1)ds.$$

Burkholder's inequality and basic estimates show that, for $r \geq 1$,

$$E[\sup_{0\leq t\leq T} X(t,1)^r] \leq C(r,T) \quad \text{if} \quad E[X(0,1)^r] < \infty. \tag{1.5}$$

Let $y(t,\theta) = \mathrm{Re}\hat{Y}(t,\theta)$ and $m(t,\theta) = \mathrm{Re}\hat{M}(t,\theta)$. Then

$$y(t,\theta) = \int_0^t e^{-|\theta|^\alpha (t-s)} dm(s,\theta)$$

and

$$[m(\cdot,\theta)](t) = \int_0^t X(s, \cos^2[\theta\cdot(\cdot)])ds.$$

Letting $g(t,\theta) = X(t,\cos^2[\theta\cdot(\cdot)])$, we find that $E[g(t,\theta)^r] \leq C(r,T)$ for $t \leq T$ from (1.5).

Let $\nu(d\theta) = (1+|\theta|^2)^\gamma d\theta$ and $\lambda(\theta) = |\theta|^\alpha$. To apply Theorem 1.1, we require

$$\int_{\lambda(\theta)>1} \lambda^{\delta-1}(\theta)\nu(d\theta) < \infty$$

which shows we need

$$\int_{|\theta|>1} |\theta|^{\alpha(\delta-1)+2\gamma} d\theta < \infty;$$

and this holds if $\delta < (\alpha - 2\gamma - d)/\alpha$. The estimates also apply to Im $\hat{Y}(t,\theta)$ and Theorem 1.2 follows from Theorem 1.1 applied to Re $\hat{Y}(t,\theta)$ and Im $\hat{Y}(t,\theta)$ separately. □

Proof of Corollary 1.1. By conditioning on $X(0)$, we may assume $X(0,1)$ has moments of all orders. Then the statement for Y follows from Theorem 1.2. Thus for X we need to consider, after taking the Fourier transform of $(S_\alpha(t) - S_\alpha(s))X(0)$,

$$\int_{R^d} |(e^{-|\theta|^\alpha t} - e^{-|\theta|^\alpha s})\hat{X}(0,\theta)|^2 (1+|\theta|^2)^\gamma d\theta.$$

But

$$|\hat{X}(0,\theta)| \leq X(0,1) \quad \text{and} \quad |e^{-|\theta|^\alpha t} - e^{-|\theta|^\alpha s}| \leq e^{-|\theta|^\alpha s}(1 \wedge |\theta|^\alpha |t-s|).$$

If $0 < S \leq s \leq t$, this implies $\|(S_\alpha(t) - S_\alpha(s))X(0)\|_\gamma^2 \leq C(S,\gamma,\alpha)|t-s|^2$ using the integrability of $e^{-2|\theta|^\alpha S}|\theta|^{2\alpha+2\gamma}$.

To prove the last statement of Corollary 1.1, it follows from our previous discussion that we need only consider

$$\int_{|\theta|\geq 1} (1 - e^{-|\theta|^{\alpha}(t-s)})^2 (1 + |\theta|^2)^{\gamma} d\theta;$$

and it suffices to dominate this by $C|t-s|^{2\beta}$. Letting $\varepsilon \in [0, \frac{1}{2})$ and $\gamma = -(\alpha+d)/2$, we can dominate the integrand by

$$C(t-s)^{2\varepsilon}|\theta|^{2\alpha\varepsilon+2\gamma} = C(t-s)^{2\varepsilon}|\theta|^{-d+\alpha(2\varepsilon-1)}.$$

We can then set $\varepsilon = \beta$ and the integral is finite. The result still holds for any $\gamma \leq -(\alpha + d)/2$. □

Remark 1.2 If $\gamma < -d/2$, then $\int_{R^d} (1 + |\theta|^2)^{\gamma} d\theta < \infty$.
Hence, if ν_1 and ν_2 are finite Borel measures on R^d and we define

$$d_{\gamma}(\nu_1, \nu_2) = \|\nu_1 - \nu_2\|_{\gamma} + |\nu_1(1) - \nu_2(1)|,$$

then, for $\gamma < -d/2$, by subsequent Lemma A.2 $d_{\gamma}(\cdot, \cdot)$ defines the topology of weak convergence on the space of finite Borel measures; that is,

$$\nu_n(f) \longrightarrow \nu(f)$$

for every continuous bounded f if and only if $d_{\gamma}(\nu_n, \nu) \to 0$.

Our calculations show $X(t, 1)$ is Hölder continuous of order β for any $\beta \in (0, (1/2))$; and if we choose $\gamma \leq -(\alpha + d)/2$, Corollary 1.1 shows that X, the $(\alpha, d, 1)$ superprocess, has a.s. β-Hölder continuous paths as a measure-valued process using the metric $d_{\gamma}(\cdot, \cdot)$. In Theorem 7.3.1(a) of Dawson (1993) this was proved for a more general collection of superprocesses using a different, but equivalent, metric on the space of finite measures. Thus, as applied to the $(\alpha, d, 1)$ superprocess, Corollary 1.1 extends Dawson's result. Theorem 1.2 and Corollary 1.1 also provide an interesting tradeoff between space and time regularities. Suppose we consider Y in a more restrictive Hilbert space, H_{γ} with $-d/2 < \gamma < \frac{\alpha-d}{2}$, reflecting more spatial regularity than would be provided by mere finite Borel measure membership. Then, we still obtain Hölder continuity in time but with an exponent restricted to be $\beta < \frac{1}{2} - \frac{2\gamma-d}{2\alpha}$ regardless of the values of r that can be used. □

Example 1.2. We now consider our second example. Let H be a separable Hilbert space with orthonormal basis $\{e_k\}_{k=1}^{\infty}$, and let $S(t)$ be a semigroup on H satisfying $S(t)e_k = e^{-\lambda_k t}e_k$ where there is some $\Lambda \in R$ with $\inf \lambda_k \geq \Lambda$. $G(s)$, $s \geq 0$, is a linear operator valued process with $G^*(s)e_k$ defined and predictable for each k. Let $\|\cdot\|$ and $\langle\cdot, \cdot\rangle$ denote the norm and inner product on H, and let $W(t)$ be a standard cylindrical Brownian motion on H.

Set

$$Y(t) = \int_0^t S(t-s)G(s)dW(s).$$

Note $E[\|Y(t)\|^2] = E\sum_{k=1}^{\infty}\int_0^t e^{-2\lambda_k(t-s)}\|G^*(s)e_k\|^2 ds$ assuming the expectation is finite, and by imposing stronger conditions we can obtain Hölder continuity for Y. Set

$$y_k(t) = \langle Y(t), e_k\rangle$$

and note

$$y_k(t) = \int_0^t e^{-\lambda_k(t-s)} dm_k(s)$$

where $[m_k](t) = \int_0^t g_k(s)ds$ with $g_k(s) = \|G^*(s)e_k\|^2$.

Theorem 1.3. *Assume* $\|G^*(s)e_k\|^2 \le b_k^2 f_k(s)$ *where* $\{b_k\}_1^{\infty}$ *are deterministic constants,* $\sup_k \sup_{0\le s\le t} E[f_k^r(s)] \le C(t,r)$ *for* $0 \le t \le T$*; and for some* $\delta \in (0,1]$, $r\delta > 1$ *and*

$$\sum_{\{k:\lambda_k\le a\}} b_k^2 + \sum_{\{k:\lambda_k > a\}} \frac{b_k^2}{\lambda_k^{1-\delta}} < \infty$$

for some $a \ge 0$*. Then* Y *has a modification satisfying*

$$E[\sup_{0\le s<t\le T}(\|Y(t)-Y(s)\|/|t-s|^{\beta})^{2r}] < \infty$$

for any $\beta \in [0, (\delta/2) - (1/2r))$.

Proof. This follows immediately from Theorem 1.1 by taking $\nu(dk)$ to be counting measure on the positive integers. □

As a particular case, let $\lambda_k = k^{\alpha}$ for $\alpha \in (1,2]$, and set $b_k^2 = k^{2\gamma}$ for $0 \le \gamma < (\alpha-1)/2$. Then, for $\delta < \frac{\alpha-2\gamma-1}{\alpha}$, Theorem 1.3 applies.

2. Additional maximal inequalities and Hölder continuity for the path process

In this section we assume the conditions of Theorem 1.1 are satisfied and we have chosen a continuous modification of Y. We define the path process as follows: For each $t \in [0,T]$,

$$\mathbf{Y}_t(\tau) = \begin{cases} Y(\tau t) & 0 \le \tau \le 1 \\ Y(t) & \tau > 1. \end{cases} \tag{2.1}$$

Then, $\mathbf{Y}_t \in C([0,T]:H)$ which is a Banach space with norm

$$\|h\|_T = \sup_{0\le s\le T}\|h(s)\|. \tag{2.2}$$

We now consider Hölder continuity for $\mathbf{Y}$ using the norm $\|\cdot\|_T$. A key tool is the following maximal inequality from Kouritzin and Heunis (1994).

Theorem 2.1. *Let* $0 \le T_0 < T_1 < \infty$ *and suppose that* $\{Q_t, T_0 \le t \le T_1\}$ *is a process assuming values in some normed vector space* $\mathscr{X}$ *with norm* $\|\cdot\|$ *such that the following hold: (i)* $t \to Q_t(\omega)$ *is continuous on* $[T_0, T_1]$ *for almost all* ω *and (ii) there exist constants* $\gamma > 1$ *and* $\nu > 0$ *such that* $E\|Q_u - Q_t\|^\nu \le [h(t,u)]^\gamma$, *for all* $T_0 \le t \le u \le T_1$, *where* $h(t,u)$ *is a nonnegative function satisfying* $h(t,u) + h(u,v) \le h(t,v)$ *for all* $T_0 \le t < u < v \le T_1$. *Then there exists a constant* $\tilde{A}_{\nu,\gamma}$ *depending only on* ν *and* γ *such that*

$$E\left[\max_{T_0\le t\le u\le T_1} \|Q_u - Q_t\|^\nu\right] \le \tilde{A}_{\nu,\gamma}[h(T_0,T_1)]^\gamma.$$

Before turning to the study of the path process, we can use Theorem 2.1 to strengthen Lemma 1.2 if $r\delta > 1$. We take $h(u,v) = c|v-u|$.

Lemma 2.1. *Assume the conditions of Lemma 1.2 hold with* $r\delta > 1$. *Then, for* $0 \le s \le t \le T$,

$$E[\sup_{s\le u,v\le t} \|Y(v) - Y(u)\|^{2r}] \le C(T,\delta,r,a,\Lambda)|t-s|^{r\delta}.$$

Lemma 2.2. *Assume the conditions of Lemma 1.2 hold with* $r\delta > 1$. *Then, for* $0 \le s \le t \le T$,

$$E[\|\mathbf{Y}_t - \mathbf{Y}_s\|_T^{2r}] \le C(T,\delta,r,a)|t-s|^{r\delta-1}.$$

Proof. Assume $s < t$, choose n so that $1 \le n \le \frac{t}{t-s} < n+1$, and let $I_k = [\frac{k}{n}, \frac{k+1}{n})$ for $0 \le k \le n-1$. Then

$$\|\mathbf{Y}_t - \mathbf{Y}_s\|_T = \sup_{0\le\tau\le1} \|Y(\tau t) - Y(\tau s)\| \le \sum_{k=0}^{n-1} \sup_{\tau\in I_k} \|Y(\tau t) - Y(\tau s)\|.$$

Consider

$$\begin{aligned} &E[\sup_{\tau\in I_k} \|Y(\tau t) - Y(\tau s)\|^{2r}] \\ &\le E[\sup_{\frac{ks}{n}\le u,v\le\frac{(k+1)t}{n}} \|Y(u) - Y(v)\|^{2r}] \\ &\le C(|t-s|(1+\frac{k+1}{n}))^{r\delta} \quad \text{(by Lemma 2.1 and our choice of } n\text{)} \\ &\le C|t-s|^{r\delta}. \end{aligned}$$

Thus, again using our choice of n,

$$\begin{aligned} E[\|\mathbf{Y}_t - \mathbf{Y}_s\|_T^{2r}] &\le C(T,\delta,r,a,\Lambda)\left|\frac{t}{t-s}\right||t-s|^{r\delta} \\ &\le C(T,\delta,r,a,\Lambda)|t-s|^{r\delta-1} \end{aligned}$$

after absorbing a factor of T into the constant. □

We now state the main result of this section.

Theorem 2.2. *Assume the conditions of Lemma 1.2 hold with* $r\delta > 2$*. Then:*

(a) For $0 \le s \le t \le T$*,*

$$E[\sup_{s \le u \le v \le t} \|\mathbf{Y}_v - \mathbf{Y}_u\|_T^{2r}] \le C(T, \delta, r, a, \Lambda)|t-s|^{r\delta - 1}.$$

(b) $E[\sup_{0 \le s < t \le T}(\|\mathbf{Y}_t - \mathbf{Y}_s\|_T / |t-s|^{\beta})^{2r}] < \infty$ *for any* $\beta \in [0, (\delta/2) - (1/r))$.

Proof. (a) follows from Lemma 2.2 and Theorem 2.1 (let $h(u, v)$ be of the form $c(v-u)$). (b) follows from Lemma 2.2 and Theorem I.2.1 of Revuz and Yor (1994). □

Example 2.1. We now consider the $(\alpha, d, 1)$ superprocess and apply our results to the path process $\mathbf{Y}_t$ determined by Y, the convolution term of (1.3). From Lemma 1.2, Theorem 2.2, and the proofs of Theorem 1.2 and Corollary 1.1, we obtain:

Theorem 2.3. *Assume* $\gamma < (\alpha - d)/2$ *and* $E[X(0,1)^r] < \infty$ *where* $r > 1$ *and* $\delta \in (0, 1]$ *satisfy* $\delta \in (\frac{1}{r}, \frac{\alpha - 2\gamma - d}{\alpha})$*. (a) Then, for* $0 \le s \le t \le T$*,*

$$E[\sup_{s \le u, v \le t} \|Y(v) - Y(u)\|_{\gamma}^{2r}] \le C(T, \delta, r)|t-s|^{r\delta}.$$

(b) In addition to the assumptions of (a), assume $r > 2$ *and* $\delta \in (\frac{2}{r}, \frac{\alpha - 2\gamma - d}{\alpha})$*. Then*

$$E[\sup_{s \le u, v \le t} \|\mathbf{Y}_v - \mathbf{Y}_u\|_{\gamma, T}^{2r}] \le C(T, \delta, r)|t-s|^{r\delta - 1}$$

for $0 \le s \le t \le T$*, and*

$$E[\sup_{0 \le s < t \le T}(\|\mathbf{Y}_t - \mathbf{Y}_s\|_{\gamma, T} / |t-s|^{\beta})^{2r}] < \infty$$

for any $\beta \in [0, (\frac{\alpha - 2\gamma - d}{2\alpha}) \wedge \frac{1}{2} - \frac{1}{r})$.

(c) If we replace the moment assumption on $X(0,1)$ *in (b) by* $P(X(0,1) < \infty) = 1$*, then*

$$P(\sup_{0 \le s < t \le T}(\|\mathbf{Y}_t - \mathbf{Y}_s\|_{\gamma, T} / |t-s|^{\beta}) < \infty) = 1$$

for any $\beta \in [0, (\frac{\alpha - 2\gamma - d}{2\alpha}) \wedge \frac{1}{2})$.

We now give additional maximal inequalities for Y that hold when $\sup_{0 \le u < \infty} E[g(u, \theta)^r]^{1/r}$ can be suitably bounded.

Theorem 2.4. *Assume* $r \ge 1$*,* $\delta > 0$*,* $r\delta > 1$ *and* $\sup_{0 \le u < \infty} E[g(u, \theta)^r]^{1/r} \le G(r, \theta)$*, where* G *satisfies the conditions of Lemma 1.2.*

(a) Then

$$E[\sup_{0\le t\le T} \|Y(t)\|^{2r}] \le C(a,r,\delta)(T^r + T^{r\delta}).$$

(b) If $\int_{R^d} G(r,\theta)\lambda^{\delta-1}(\theta)\nu(d\theta) < \infty$ *for* $\delta \in (0,1]$, *then*

$$E[\sup_{0\le t\le T} \|Y(t)\|^{2r}] \le C(r,\delta)T^{r\delta}.$$

Proof. This follows from Lemma 2.1 and a straightforward modification of Lemma 1.1 using the assumptions on $G(r,\theta)$. □

Example 2.2. We note that under the assumptions of Theorem 1.3, the results of Lemma 2.1 and Lemma 2.2 hold for the convolution in Example 1.2. If, in addition, we assume $r\delta > 2$, then we can apply Theorem 2.2.

2.1. Concluding remarks

Dawson (1972) seems to be the first paper to exploit eigenfunction expansions to obtain Hilbert space regularity for stochastic convolutions. A maximal inequality for one-dimensional Ornstein-Uhlenbeck processes was used to obtain sample path continuity, and the drift operator, A, had discrete spectrum. Dawson (1993) is a fundamental reference for results on superprocesses.

Kotelenez (1987) is closest in spirit to our approach in that regularity results are obtained by imposing conditions on the quadratic variation of the driving martingale which need not involve a white noise. However Example 1.2 is also closely related to a result for a stochastic convolution obtained in Theorem 5.2.4 of Kallianpur and Xiong (1995). Furthermore, many results for stochastic convolutions can be found in Da Prato and Zabczyk (1992).

Fourier analysis was used to establish sample path continuity for the $(\alpha, d, 1)$ superprocess in Blount and Bose (2000a,b). However, Hölder continuity was not investigated, and the methods used are very different from this paper.

In Krylov (1997) novel methods are developed for representing superdiffusions as solutions of stochastic partial differential equations. Hilbert space techniques are used, and, as a consequence of his theory (Corollaries 1.9 and 1.10), the following results are obtained for the super-Brownian motion process (the case of $\alpha = 2$ using our notation in Example 1.1):

a) If $X(0) \in H_\gamma$ for some $\gamma < 1 - \frac{d}{2}$, where $X(0,1) < \infty$, then $X(t,\omega) \in H_\gamma$ for almost all (t,ω) with respect to $dt \times P(d\omega)$.
b) If $d = 1$ and $X(0) \in L_2(R) = H_0$, then, for almost every (t,ω), $X(t,\omega) \in H_0$. In particular, for such (t,ω), $X(t,\omega)$ has a square integrable density.

These results follow from our pathwise continuity results, but they can be obtained directly from the following result under the weaker assumption $E[X(0,1)] < \infty$. Before stating our result, we note Krylov also raised the question of including superprocesses within the framework of classical stochastic analysis. Our paper makes a contribution to this goal.

Proposition 2.1. *If* $E[X(0,1)] < \infty$, *then*

$$E\Big[\int_0^T \|X(t)\|_\gamma dt\Big] < \infty \quad \text{if} \quad \gamma < (\alpha - d)/2.$$

Proof. Using (1.3), (1.4), Cauchy-Schwarz, and Fubini's theorem,

$$\begin{aligned}
&E\Big[\int_0^T \|X(t)\|_\gamma dt\Big] \\
&\quad \le E\Big[\Big(T\int_0^T \|S_\alpha(t)X(0)\|_\gamma^2 dt\Big)^{\frac{1}{2}}\Big] + E\Big[\Big(T\int_0^T \|Y(t)\|_\gamma^2 dt\Big)^{\frac{1}{2}}\Big] \\
&\quad \le T^{\frac{1}{2}} E\Big[\Big(\int_0^T \int_{R^d} e^{-2|\theta|^\alpha t}|\widehat{X}(0,\theta)|^2(1+|\theta|^2)^\gamma d\theta dt\Big)^{\frac{1}{2}}\Big] \\
&\qquad + T^{\frac{1}{2}}\Big[\Big(\int_0^T \int_{R^d} E[|\widehat{Y}(t,\theta)|^2](1+|\theta|^2)^\gamma d\theta dt\Big)^{\frac{1}{2}}\Big] \\
&\quad \le T^{\frac{1}{2}} E[X(0,1)]\Big[\int_{R^d}\Big(\frac{1-e^{-2|\theta|^\alpha T}}{2|\theta|^\alpha}\Big)(1+|\theta|^2)^\gamma d\theta\Big]^{\frac{1}{2}} \\
&\qquad + T^{\frac{1}{2}} E^{\frac{1}{2}}[(X(0,1)]\Big[\int_0^T \int_{R^d}\Big(\frac{1-e^{-2|\theta|^\alpha t}}{2|\theta|^\alpha}\Big)(1+|\theta|^2)^\gamma d\theta dt\Big]^{\frac{1}{2}}.
\end{aligned}$$

Note we've used $E[\,|\widehat{Y}(t,\theta)|^2] = E\big[\int_0^t e^{-2|\theta|^\alpha(t-s)}X(s,1)ds\big]$. If $\gamma < (\alpha - d)/2$, the integrals are finite. □

Although Kolmogorov's criterion is used in Kallianpur and Xiong (1995), Da Prato and Zabczyk (1992), and Kotelenez (1987), they do not use the powerful version of it found in Revuz and Yor (1994). In addition, the application of Theorem 2.1 to stochastic evolution equations to obtain Lemma 2.1 with Theorems 2.2 and 2.4 is new.

A. Appendix: Supporting results

We now state and prove the two analytic results utilized in the previous sections. In Lemma A.1 we move a norm inside an integral. In Da Prato and Zabczyk (1992) this is proved for a separable Banach space; but our space, $L^r\big(P(dw)\big)$, may not be separable, and we require a different argument.

Lemma A.1. *If* $f(\theta, w)$ *is measurable in* (θ, w), *and* ν *is a* σ*-finite Borel measure then, for* $r \ge 1$,

$$E\Big[\Big(\int_{R^d}|f(\theta,\cdot)|\nu(d\theta)\Big)^r\Big]^{\frac{1}{r}} \le \int_{R^d} E[|f(\theta,\cdot)|^r]^{\frac{1}{r}}\nu(d\theta).$$

Proof. By standard approximation arguments we may assume ν is a finite measure and f is bounded. By Theorem 11.4 of Billingsley (1986) and the fact that the measurable rectangles form a semiring generating $\mathscr{B}(R^d) \otimes \mathscr{F}$, it suffices to consider functions $f(\theta, w)$ of the form $f(\theta, w) = \sum_{k=1}^n g_k(w)h_k(\theta)$ where, for each θ, $h_k(\theta) \neq 0$ implies $h_j(\theta) = 0$ for $j \neq k$. Letting $\|g\| = E[|g|^r]^{\frac{1}{r}}$, we have $\|f(\theta, \cdot)\| = \sum_{k=1}^n \|g_k\||h_k(\theta)|$. Thus

$$\begin{aligned}\int_{R^d} \|f(\theta, \cdot)\|\nu(d\theta) &= \sum_{k=1}^n \|g_k\| \int_{R^d} |h_k(\theta)|\nu(d\theta) \\ &\geq \left\| \sum_{k=1}^n |g_k| \int_{R^d} |h_k(\theta)|\nu(d\theta) \right\| \\ &= \left\| \int_{R^d} \left(\sum_{k=1}^n |g_k h_k(\theta)| \right) \nu(d\theta) \right\| \\ &= \left\| \int_{R^d} |f(\theta, \cdot)|\nu(d\theta) \right\|. \qquad \square\end{aligned}$$

Lemma A.2. *Let $\gamma < -d/2$ and define $d_\gamma(\nu_1, \nu_2) = \|\nu_1 - \nu_2\|_\gamma + |\nu_1(1) - \nu_2(1)|$ where ν_1, ν_2 are finite Borel measures. Then d_γ defines the topology of weak convergence on the space of finite Borel measures on R^d.*

Proof. If $\nu_n \stackrel{w}{\to} \nu$, then $\hat{\nu}_n(\theta) \to \hat{\nu}(\theta)$ for all θ and $\sup_n |\hat{\nu}_n(\theta)| \leq \sup_n |\nu_n(1)| < \infty$. Thus $d_\gamma(\nu_n, \nu) \to 0$ by the dominated convergence theorem since $(1+|\theta|^2)^\gamma d\theta$ is a finite measure on R^d for $\gamma < -d/2$.

Now assume $d_\gamma(\nu_n, \nu) \to 0$, and let f be a Schwartz function (rapidly decreasing and C^∞). Recall $f(x) = (2\pi)^{-d/2} \int_{R^d} e_\theta(x) \hat{f}(\theta) d\theta$ by the Fourier inversion theorem; and $\hat{f}(\theta)$ is a Schwartz function. Using Fubini's theorem and letting $\bar{a}$ denoting complex conjugation, we find that

$$\begin{aligned}\left| \int_{R^d} f(x)\nu_n(dx) - \int_{R^d} f(x)\nu(dx) \right| &= \left| (2\pi)^{-d/2} \int_{R^d} (\bar{\hat{\nu}}_n(\theta) - \bar{\hat{\nu}}(\theta)) \hat{f}(\theta) d\theta \right| \\ &\leq C \int_{R^d} (1 + |\theta|^2)^{\gamma/2} |\hat{\nu}_n(\theta) \\ &\quad - \hat{\nu}(\theta)| (1 + |\theta|^2)^{-\gamma/2} |\hat{f}(\theta)| d\theta \\ &\leq C \|\nu_n - \nu\|_\gamma \|f\|_{-\gamma} \to 0.\end{aligned}$$

This implies vague convergence, and with the extra condition, $\nu_n(1) \to \nu(1)$, we obtain weak convergence using standard arguments. $\square$

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