The projective plane

If F is a field and V a F-vector space, then the *projective space* $\mathbb{P}(V)$ associated to V is the set of lines through the origin. In other words, an element x in $\mathbb{P}(V)$ is a subset of V the form $\{tv \mid t \in F\}$ for some nonzero $v \in V$.
The *projective line* is defined as $\mathbb{P}(F^2)$ and is denoted as $\mathbb{P}^1$.
The *projective plane* is defined as $\mathbb{P}(F^3)$ and denoted $\mathbb{P}^2$.
More generally *projective n-space* is defined as $\mathbb{P}(F^{n+1})$. We can describe a point in $\mathbb{P}^n$ by giving a nonzero element in the line it represents: we write $[x_0, x_1, \ldots, x_n]$ for the line containing $(x_0, x_1, \ldots, x_n) \in F^{n+1}$: the set of all $(tx_0, tx_1, \ldots, tx_n)$ with $t \in F$. Here, not all of the x_i are allowed to be zero, and $[x_0, x_1, \ldots, x_n] = [\lambda x_0, \lambda x_1, \ldots, \lambda x_n]$ whenever $\lambda \neq 0$ is an element of F. $[x_0, x_1, \ldots, x_n]$ are the *projective coordinates* on $\mathbb{P}^n$.
$\mathbb{P}^1$ may be thought of as $F \cup \{\infty\}$: indeed, to $t \in F$ associate the point $[t, 1] \in \mathbb{P}^1$. Notice that $[t, 1] = [t', 1]$ if and only if $t = t'$. So this defines an injective map $F \to \mathbb{P}^1$. The image contains all points except $[1, 0]$ (playing the role of ∞). This makes sense: suppose $F = \mathbb{R}$. Then for all nonzero $t \in \mathbb{R}$, we have $[t, 1] = [1, t^{-1}]$. So for $|t| \to \infty$, it is "reasonable" to define the limit as $[1, 0]$.
We could have also chosen the map $F \to \mathbb{P}^1$ sending t to $[1, t]$. For nonzero t this is the image of t^{-1} under the map above. It maps 0 to $[1, 0]$ and the "∞" with respect to this map is then $[0, 1]$, the old 0.
Notice that for $[x, y, z] \in \mathbb{P}^2$, the condition that $z \neq 0$ is well defined: if $(x, y) \neq (0, 0)$, then all elements of the line through $(x, y, 0)$ will have last coordinate zero. So all projective coordinates for this line will have $z = 0$.
Let $U \subset \mathbb{P}^2$ be the set where $z \neq 0$. Any $[x, y, z] \in U$ is represented by a unique triple of the form $[x, y, 1]$ (by dividing all coordinates with z^{-1}. This gives a bijection $F^2 \to U$ sending (x, y) to $[x, y, 1]$. On the other hand the complement L of U is the set of points $[x, y, z]$ in $\mathbb{P}^2$ for which $z = 0$. It is naturally in with $\mathbb{P}^1$: $[x, y, 0]$ may be identified with $[x, y] \in \mathbb{P}^1$.
We can therefore think of $\mathbb{P}^2$ as $U \cup L$ as F^2 with a copy of $\mathbb{P}^1$ attached, the "line at infinity", playing a role similar to ∞ in the case of $\mathbb{P}^1$. Any line C in F^2 that contains $(0, 0)$ will then give another copy of $\mathbb{P}^1$ in $\mathbb{P}^2$ by attaching its slope "at infinity": we put $\hat{C}$ to be the set of all points in $C \subset U$ together with $[x, y, 0]$. If C has slope different from infinity with the x-axis in U, say, then the point $x \neq 0$ and so $[x, y, 0] = [1, y/x, 0]$ may be identified with the "slope". Notice that if we identify L with $F \cup \{\infty\}$ where $[1, t, 0]$ corresponds to $t \in F$ and ∞ is $[0, 1, 0]$, then this means exactly that $\hat{C} = C \cup \{\text{slope}\}$. Any line parallel to C in U has the same slope so we add the same point to it. That way any two parallel lines in $F^2 = U$ "intersect" at infinity.
The set of lines in $\mathbb{P}^2$ is then the collection of all these $\hat{C}$ together with L. Compare this to the picture in class, where $F = \mathbb{Z}_2$. F^2 has four elements. $\mathbb{P}^2$ has 7 elements: there are 7 nonzero vectors in F^3 each one corresponding to a unique line through the origin $(0, 0, 0)$ (every such line contains the origin and precisely one nonzero element, because $F = \{0, 1\}$). We could therefore number them through as follows 1: $(1, 0, 0)$, 2: $(1, 0, 1)$, 3: $(1, 1, 0)$, 4: $(0, 0, 1)$, 5: $(0, 1, 1)$, 6: $(1, 1, 1)$, and 7: $(0, 1, 0)$. This enumeration is somewhat arbitrary, the only reason for it to be chosen here is that it coincides with the picture in class.

The set U would then consist of the $[x, y, z]$ with $z \neq 0$. Thus

$$U = \{[0, 0, 1], [1, 0, 1], [0, 1, 1], [1, 1, 1]\}$$

or in the other notation $U = \{2, 4, 5, 6\}$. The "line at infinity", L is then $[1, 0, 0], [1, 1, 0]$ and $[0, 1, 0]$, or $\{1, 3, 7\}$.

You see that we can identify U with $F^2 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ simply by appending a 1 to each vector, so that 4 corresponds to $(0, 0)$, and so on. The line through the origin in F^2, that also contains $(1, 0)$ for example has slope 0 with itself (the x-axis). And so in $\mathbb{P}^2$, we would add $[1, 0, 0]$ to it and obtain the line $\{4, 2, 1\}$ in our picture. The line containing $(0, 1)$ and $(1, 1)$ is parallel to the x-axis and therefore has the same slope (we get $\{5, 6, 1\}$. In this manner we find that the set of all these lines in $\mathbb{P}^2$ (including L) will correspond to the lines in our picture (in a natural way). These lines are all naturally in bijection with a copy of $\mathbb{P}^1$. Indeed, any such line is of the form $\mathbb{P}(W)$ for a uniquely determined two-dimensional linear subspace of F^3 (given by a single linear equation). It is the subspace spanned by any two points (i.e. lines in F^3) contained in it.

For example, the line L is $\mathbb{P}(W)$ for $W = \ker z = \{(x, y, 0) \mid x, y \in F\}$. The line $\{5, 6, 1\}$ mentioned earlier is the projective space of the span of $(0, 1, 1)$ and $(1, 1, 1)$, which is the space satisfying $y + z = 0$.

In general, if a line $C \subset U$ is given by the (affine) linear equation $ax + by + c = 0$, $\hat{C}$ is the projective space of the kernel of $ax + by + cz = 0$ (and so the point at infinity of $\hat{C}$ in L is the solution $[x, y, 0]$ of $ax + by = 0$).