Pairwise Independence versus Mutual Independence

The text defines independence for two events, but never defines independence for three or more sets until problem 2.119 on page 73. This problem and its solution are given below:

Problem 2.119 Three events A, B, and C in the sample space S of a random experiment are said to be *mutually independent* (or *independent*) if

$$P(A \cap B) = P(A) \cdot P(B) \qquad P(B \cap C) = P(B) \cdot P(C)$$
$$P(A \cap C) = P(A) \cdot P(C) \qquad P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C).$$

Suppose a balanced coin is independently tossed two times. Define the following events:
A : Head appears on the first toss.
B : Head appears on the second toss.
C : Both tosses yield the same outcome.
Are the events A, B, and C mutually independent?

Solution. The sample space S consists of the following outcomes: $S = \{HH, HT, TH, TT\}$ and the events listed above are $A = \{HH, HT\}$, $B = \{HH, TH\}$, $C = \{HH, TT\}$.
Since the coin is balanced, all outcomes are assigned the same probability, namely $\frac{1}{4}$, so the probabilities of the events are

$$P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2},$$

and

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4},$$

while $P(A \cap B \cap C) = P(\{HH\}) = \dfrac{1}{4}$. Therefore,

$$P(A \cap B) = P(A) \cdot P(B), \quad P(A \cap C) = P(A) \cdot P(C), \quad P(B \cap C) = P(B) \cdot P(C),$$

and the events A, B, C are *pairwise independent.*

However,

$$P(A \cap B \cap C) = \frac{1}{4} \neq \frac{1}{8} = P(A) \cdot P(B) \cdot P(C),$$

so the events A, B, C are *not* mutually independent. □

But, in example 2.20 on page 58, the text uses the fact that the events B_3 and $B_1 \cap B_2$ are independent, a fact that is not necessarily true, unless B_1, B_2, B_3 are independent, not just pairwise independent. For example, in the above problem, A and $B \cap C$ are not independent, since

$$P(A \cap B \cap C) = \frac{1}{4} \neq \frac{1}{8} = P(A) \cdot P(B \cap C).$$