MATH 421 Winter 2006
Combinatorics
Assignment 5
Due: Wednesday April 12, 2006

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Question 1. [p 469, #1]

Suppose that $D = \{a, b, c\}$ and $R = \{1, 2\}$. Find all colorings in $C(D, R)$.

Question 2. [p 469, #2]

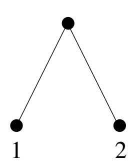
How many colorings (not necessarily distinct) are there for the vertices of a cube if the set of allowable colors is $\{\text{red}, \text{green}, \text{blue}\}$?

Question 3. [p 469, #3]

How many allowable colorings (not necessarily distinct) are there for the vertices of a regular tetrahedron if six colors are available?

Question 4. [p 470, #10]

Suppose that V is the set of all colorings of the binary tree in the figure below



in which each vertex gets one of the colors black or white. Two colorings are considered the same if one can be obtained from the other by interchanging the colors of the vertices labeled 1 and 2. Find:

(a) D (b) R (c) G (d) G^*

(e) The number of distinct colorings.

Question 5. [p 471, #17]

Consider a 2×2 array in which each block is occupied or not. We color the block black if it is occupied and color it white or leave it uncolored otherwise. Let V be the collection of all such colorings. Suppose that we allow rotation of the array by $0°$, $90°$, $180°$, or $270°$. We also allow reflections in either a vertical, a horizontal, or a diagonal line. (The latter would switch the colors assigned to two diagonally opposite cells). We consider colored arrays a and b the same, and write $a\,S\,b$, if b can be obtained from a by one of the rotations or reflections in question.

(a) Find G^*.

(b) Use Burnside's Lemma to compute the number of distinct colorings.

(c) Check your answer by comparing the enumeration of equivalence classes you gave as your answer to Question 1 on Assignment 4.

Question 6. [p 478, #1]

Find the cycle decomposition of each of the permutations below.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 2 & 6 & 3 & 5 & 4 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 7 & 5 & 8 & 3 & 4 & 1 & 2 \end{pmatrix}$

Question 7. [p 478, #4]

Suppose that $A = \{1, 2, 3, 4, 5\}$ and G consists of the following permutations

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \qquad \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix}$$
$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{pmatrix} \qquad \pi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix}.$$

For each of the four permutations above, encode the permutation as $x_1^{b_1} x_2^{b_2} \cdots x_k^{b_k}$.

Question 8. [p 478, #5]

For the permutation group G in Question 7 above, compute the cycle index.

Question 9. [p 479, #22]

Consider a cube in 3-space. There are eight vertices. The following symmetries correspond to permutations of these vertices. Encode each of these symmetries in the form $x_1^{b_1} x_2^{b_2} \cdots x_k^{b_k}$ and compute the cycle index of the group G of all the permutations corresponding to these symmetries.

(a) The identity symmetry.

(b) Rotations by 180° around lines connecting the centers of opposite faces (there are three).

(c) Rotations by 90° or 270° around lines connecting the centers of opposite faces (there are six).

(d) Rotations by 180° around lines connecting the midpoints of opposite edges (there are six).

(e) Rotations by 120° around lines connecting opposite vertices (there are eight).

Question 10.

In Question 9, find the number of distinct ways of coloring the vertices of the cube with two colors, red and blue.