MATH 421 Winter 2006
Combinatorics
Assignment 4
Due: Friday March 31, 2006

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Question 1. [p 447, #9]

Consider a 2×2 array in which each block is occupied or not. We color the block black if it is occupied and color it white or leave it uncolored otherwise. Let V be the collection of all such colorings. Suppose that we allow rotation of the array by $0°$, $90°$, $180°$, or $270°$. We also allow reflections in either a vertical, a horizontal, or a diagonal line. (The latter would switch the colors assigned to two diagonally opposite cells). We consider colored arrays a and b the same, and write $a\,S\,b$, if b can be obtained from a by one of the rotations or reflections in question.

(a) Show that S is an equivalence relation on V.

(b) Identify all equivalence classes of colorings. (Only two colors are used, black and white.)

Question 2. [p 448, #20]

Consider the set $A = \{1, 2, \ldots, n\}$.

(a) How many binary relations are possible on A?

(b) How many reflexive relations are possible on A?

(c) How many symmetric relations are possible on A?

(d) How many transitive relations are possible on A?

(e) How many equivalence relations are possible on A?

Question 3. [p 449, #22]

Let E_n be the be the number of equivalence relations on the set $A = \{1, 2, \ldots, n\}$. Show that E_n satisfies the recurrence:

$$E_n = \sum_{i=0}^{n-1} \binom{n-1}{i} E_i, \quad n \geq 1.$$

Question 4. [p 455, #5(a)]

Let X be the permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

and let $\circ$ be the operation of composition. Check which of the four axioms for a group hold.

Question 5. [p 455, #10(a)]

If A and G are as follows, find the equivalence classes under the equivalence relation S induced by G.

$$A = \{1, 2, 3, 4, 5\}$$
$$G = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \right\}.$$

Question 6. [p 456, #19]

If π_1 and π_2 are permutations, $\pi_1 \circ \pi_2$ may not equal $\pi_2 \circ \pi_1$. (Thus, we say that the product of permutations is not necessarily commutative.)

(a) Demonstrate this with $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$ and $\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$

(b) Find two symmetries π_1 and π_2 of the 2×2 array (π_1 and π_2 can be rotations or reflections) such that $\pi_1 \circ \pi_2 \neq \pi_2 \circ \pi_1$.

Question 7. [p 460, #1]

Suppose that $A = \{1, 2, 3, 4, \}$ and G consists of the following permutations

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$
$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \qquad \pi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

Verify that the four permutations above form a group and that the equivalence classes under the induced equivalence relation are $\{1, 2\}$ and $\{3, 4\}$.

Question 8. [p 461, #4]

Suppose that $A = \{1, 2, 3, 4, 5, 6, 7\}$ and G is the following group of permutations:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} \right\},$$

use Burnside's Lemma to find the number of equivalence classes under S.

Question 9. [p 461, #7]

Let

$$A = \{1, 2, 3, 4, 5\}$$
$$G = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \right\}.$$

If S is the equivalence relation on A induced by G, check that

$$\frac{1}{|C(1)|} + \frac{1}{|C(2)|} + \cdots$$

gives the number of equivalence classes under S.

Question 10.

Use Burnside's Lemma to compute the number of distinct ways to seat 5 negotiators in fixed chairs around a circular table if rotating seat assignments around the circle is not considered to change the seating arrangement.