MATH 421 Winter 2006
Combinatorics
Assignment 3
Due: Friday March 10, 2006

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Question 1.

The generating function for a sequence $\{a_n\}_{n \geq 0}$ is given by

$$G(x) = (1+x)^{\frac{1}{3}}, \quad -1 < x < 1.$$

Find an expression for a_n, valid for $n \geq 0$.

Question 2.

For positive integers n and r, let $p(n, r)$ be the number of partitions of n with exactly r parts.

Note that

$$p(n, 1) = 1 = p(n, n) \qquad \text{and} \qquad p(n, r) = 0 \quad \text{if} \quad r > n,$$

also $p(n) = \sum_{r=1}^{n} p(n, r)$.

(a) Show that the numbers $p(n, r)$ satisfy the following recurrence relations for $1 < r < n$:

(i) $p(n, r) = p(n-1, r-1) + p(n-r, r)$

(ii) $p(n, r) = \sum_{k=1}^{r} p(n-r, k)$

(b) Show that the number $p(n, r)$ of partitions of n with r parts is equal to the number of partitions of n with largest part equal to r.

Question 3.

Given a positive integer n, let a_n be the number of n-digit numbers that can be formed using only the digits 1, 2, 3, 4 and such that 1 and 2 are not adjacent.

(a) Show that a_n satisfies the recurrence relation

$$\begin{aligned} a_{n+1} &= 3a_n + 2a_{n-1} \\ a_1 &= 4 \\ a_2 &= 14 \end{aligned}$$

for $n \geq 2$.

(b) Solve the recurrence relation above to find a_n for all $n \geq 1$.

Question 4.

Code words from the alphabet $\{0, 1, 2, 3\}$ are to be recognized as *legitimate* if and only if they have an even number of 0's. How many legitimate code words of length n are there?

Question 5.

Define

$$a_n = \sum_{k=0}^{n} \binom{n+k}{2k} \qquad \text{for } n \geq 1$$

$$b_n = \sum_{k=0}^{n-1} \binom{n+k}{2k+1} \qquad \text{for } n \geq 1$$

with $a_0 = 1$ and $b_0 = 0$.

(a) Show that a_n and b_n satisfy the recurrence relations

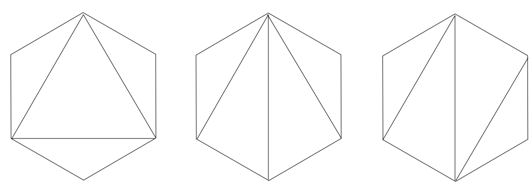
$$a_{n+1} = a_n + b_{n+1} \quad \text{for } n \geq 0$$

$$b_{n+1} = a_n + b_n \qquad \text{for } n \geq 0$$

(b) Express a_n and b_n in terms of the Fibonacci numbers.

Question 6.

Let a_n be the number of ways that a convex polygon with n sides can be divided into triangles by drawing diagonals that do not intersect. (A polygon is *convex* if all its diagonals lie in the interior of the polygon.)



(a) Calculate $a_3,\ a_4,\ a_5,\ a_6$.

(b) Show that no matter how a convex polygon with n sides is triangulated, the number of diagonals is always $n-3$ and the number of triangles is always $n-2$.

(c) Define $a_2 = 1$, and show that for $n \geq 3$,

$$a_n = a_2\, a_{n-1} + a_3\, a_{n-2} + a_4\, a_{n-3} + \cdots + a_{n-1}\, a_2.$$

(d) Solve this recurrence relation to find an explicit formula for $a_n,\ n \geq 3$.

Question 7.

Let $a(n,k)$ be the number of k-element subsets that can be selected from the set $\{1, 2, \ldots, n\}$ and which do not contain two consecutive integers.

(a) Show that

$$a(n,k) = a(n-2, k-1) + a(n-1, k).$$

(b) Show that $a(n,1) = n$ for $n \geq 1$, and $a(n,n) = 0$ for $n \geq 2$.

(c) Use the principle of mathematical induction to show that

$$a(n,k) = \binom{n-k+1}{k}$$

for $1 \leq k \leq n$.

Question 8.

Recall that the *Fibonacci sequence* $\{F_n\}_{n\geq 0}$ is the unique solution to the discrete initial value problem

$$\begin{aligned} F_{n+2} &= F_{n+1} + F_n, \quad n \geq 0 \\ F_0 &= 0 \\ F_1 &= 1. \end{aligned}$$

The first few terms of the sequence are: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

Let $\gamma_n = \left\{ \frac{F_n}{F_{n+1}} \right\}_{n\geq 1}$

(a) Show that $0 < \gamma_{2n-2} < \gamma_{2n} < 1$ for all $n \geq 2$.

(b) Show that $0 < \gamma_{2n+1} < \gamma_{2n-1} < 1$ for all $n \geq 2$.

(c) Explain why the limits $u = \lim_{n\to\infty} \gamma_{2n}$ and $v = \lim_{n\to\infty} \gamma_{2n+1}$ both exist and that $u = v$.

(d) Show that $\lim_{n\to\infty} \gamma_n = \frac{\sqrt{5}-1}{2}$.

Question 9.

Let a and b be positive integers, and let

$$a_n = \left(\frac{a+\sqrt{b}}{2}\right)^n + \left(\frac{a-\sqrt{b}}{2}\right)^n$$

and

$$b_n = \frac{1}{\sqrt{b}} \left[\left(\frac{a+\sqrt{b}}{2}\right)^n - \left(\frac{a-\sqrt{b}}{2}\right)^n \right]$$

for $n \geq 0$.

(a) Find a second order, constant coefficient, homogeneous, linear recurrence relation which is satisfied by both $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$.

(b) Show that if $a^2 - b$ is divisible by 4, then a_n and b_n are both integers for all $n \geq 0$.

Question 10.

The sequence $\{a_n\}_{n\geq 0}$ satisfies the recurrence relation and initial conditions

$$\begin{aligned} a_n &= 2a_{n-1} - 2a_{n-2}, \quad n \geq 2 \\ a_0 &= 0 \\ a_1 &= 1. \end{aligned}$$

(a) Use the method of characteristic roots to solve this recurrence relation.

(b) Use the solution above to prove the identity

$$2^{\frac{n}{2}} \sin \tfrac{n\pi}{4} = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \binom{n}{2k+1}.$$