MATH 421 Winter 2006
Combinatorics
Assignment 2
Due: Friday February 17, 2006

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Question 1. [p 422, #17]

How many permutations of $\{1, 2, 3, 4, 5, 6\}$ have the property that $i+1$ never immediately follows i?

Question 2. [p 423, #29]

Use inclusion and exclusion to find the number of solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 18$$

in which each x_i is a positive integer and $x_i \leq 8$.

Question 3. [p 433, #6]

How many words of length 6 have an even number of vowels?

Question 4. [p 434, #13]

Use Whitworth's Theorem (Theorem 7.4 in the text) to compute the number of ways to get exactly m heads if a coin is tossed n times.

Question 5. [p 294, #2]

For each of the following functions, use known Maclaurin expansions to find the Maclaurin expansion, by adding, composing, differentiating, and so on.

(h) $f(x) = \ln(1-x)$

(k) $f(x) = \ln(1+x^2)$

(ℓ) $f(x) = \dfrac{1}{1-2x}\, e^x$

Question 6. [p 295, #7]

Professor Jones wants to teach Calculus I or Linear Algebra, Professor Smith wants to teach Linear Algebra or Combinatorics, and Professor Green wants to teach Calculus I or Combinatorics. Each professor can be assigned to teach at most one course, with no more than one professor per course, and a professor only gets a course he or she wants to teach. Set up a generating function and use it to answer the following questions.

(a) In how many ways can we assign one professor to a course?

(b) In how many ways can we assign two professors to courses?

(c) In how many ways can we assign three professors to courses?

Question 7. [p 302, #11]

Make use of derivatives to find the ordinary generating function for the following sequences $\{b_k\}$.

(a) $b_k = k^2$

(b) $b_k = k(k+1)$

(c) $b_k = \dfrac{k+1}{k!}$

Question 8. [p 302, #13]

Suppose that

$$\alpha_k = \begin{cases} \sum_{i=0}^{k-2} b_i b_{k-2-i} & \text{if } k \geq 2 \\ 0 & \text{if } k = 0 \text{ or } k = 1. \end{cases}$$

Suppose that $A(x)$ is the ordinary generating function for $\{a_k\}$ and $B(x)$ is the ordinary generating function for $\{b_k\}$. Find an expression for $A(x)$ in terms of $B(x)$.

Question 9. [p 311, #12]

Let $p(k)$ be the number of partitions of the integer k and let

$$G(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots}.$$

Show that $G(x)$ is the ordinary generating function for the sequence $\{p(k)\}$.

Question 10. [p 312, #16]

(a) Show that for $|x| < 1$,

$$(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots(1+x^{2^k})\cdots = 1.$$

(b) Deduce that for $|x| < 1$,

$$1 + x + x^2 + x^3 + \cdots = (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots(1+x^{2^k})\cdots.$$

(c) Conclude that any integer can be written uniquely in binary form, that is, as a sum

$$a_0 2^0 + a_1 2^1 + a_2 2^2 + \cdots,$$

where each a_i is 0 or 1. (This conclusion is a crucial one underlying the binary arithmetic that pervades computing machines.)