MATH 324 Summer 2011
Elementary Number Theory
Assignment 1
Hint for Question 3
Tuesday July 5, 2011

Question 3. **[p 20. #5]**

Find and prove a formula for

$$\sum_{k=1}^{n} \left\lfloor \sqrt{k} \right\rfloor$$

in terms of n and $\lfloor \sqrt{n} \rfloor$.

HINT: Note that if $a_1, a_2, \cdots, a_n$ are positive integers, and we let

$f(1)$ denote the number of them that are greater than or equal to 1,

$f(2)$ denote the number of them that are greater than or equal to 2,

$f(3)$ denote the number of them that are greater than or equal to 3,

$\vdots$

then

$$a_1 + a_2 + \cdots + a_n = f(1) + f(2) + f(3) + \cdots$$

since a_k contributes 1 to each of the numbers $f(1), f(2), \cdots, f(a_k)$.

For this particular problem, we take $a_k = \left\lfloor \sqrt{k} \right\rfloor$ for $1 \leq k \leq n$, and note that:

- $f(1)$ is the number of a_k's such that $\sqrt{k} \geq 1$, that is, the number of k's with $1 \leq k \leq n$ such that $k \geq 1$, so that $f(1) = n$.
- $f(2)$ is the number of a_k's such that $\sqrt{k} \geq 2$, that is, the number of k's with $1 \leq k \leq n$ such that $k \geq 2^2 = 4$, so that $f(2) = n - 3$.
- $f(3)$ is the number of a_k's such that $\sqrt{k} \geq 3$, that is, the number of k's with $1 \leq k \leq n$ such that $k \geq 3^2 = 9$, so that $f(3) = n - 8$.

  $\vdots$

In general, if $1 \leq m \leq \lfloor \sqrt{n} \rfloor$, then

- $f(m)$ is the number of a_k's such that $\sqrt{k} \geq m$, that is, the number of k's with $1 \leq k \leq n$ such that $k \geq m^2$, so that $f(m) = n - (m^2 - 1)$.