MATH 324 Summer 2011
Elementary Number Theory
Assignment 5
Due: Wednesday August 10, 2011

Question 1. [p 246. #21]

Show that if m and n are positive integers and $(m,n) = p$, where p is prime, then

$$\phi(m\,n) = \frac{p\,\phi(m)\,\phi(n)}{p-1}.$$

Question 2. [p 246. #22]

Show that if m and k are positive integers, then

$$\phi(m^k) = m^{k-1}\phi(m).$$

Question 3. [p 246. #23]

Show that if a and b are positive integers and $d = (a,b)$, then

$$\phi(a\,b) = \frac{d\,\phi(a)\,\phi(b)}{\phi(d)}.$$

Conclude that if $d > 1$, then $\phi(a\,b) > \phi(a)\,\phi(b)$.

Question 4. [p 246. #30]

Show that if n is a positive integer with $n \neq 2$ and $n \neq 6$, then $\phi(n) \geq \sqrt{n}$.

Question 5. [p 246. #32]

Show that if m and n are positive integers with $m \mid n$, then $\phi(m) \mid \phi(n)$.

Question 6. [p 253. #4]

For which positive integers n is the sum of divisors of n odd?

Question 7. [p 255. #21, #22, #23]

Let $\sigma_k(n)$ denote the sum of the kth powers of the divisors of n, so that

$$\sigma_k(n) = \sum_{d|n} d^k.$$

(a) Find a formula for $\sigma_k(p)$, where p is a prime.

(b) Find a formula for $\sigma_k(p^\alpha)$, where p is a prime and α is a positive integer.

(c) Show that the arithmetic function σ_k is multiplicative.

Question 8. [p 255. #27]

Show that the number of ordered pairs of positive integers with least common multiple equal to the positive integer n is $\tau(n^2)$.

Question 9. [p 255. #34]

Show that if n is a positive integer, then

$$\left(\sum_{d|n} \tau(d)\right)^2 = \sum_{d|n} \tau(d)^3.$$

Question 10. [p 255. #35]

Show that if n is a positive integer, then

$$\tau(n^2) = \sum_{d|n} 2^{\omega(d)},$$

where $\omega(n)$ equals the number of prime divisors of n.