MATH 324 Summer 2011
Elementary Number Theory
Assignment 2
Due: Wednesday July 20, 2011

Question 1. [p 76. #6]

Show that no integer of the form $n^3 + 1$ is a prime, other than $2 = 1^3 + 1$.

Question 2. [p 76. #7]

Show that if a and n are positive integers with $n > 1$ and $a^n - 1$ is prime, then $a = 2$ and n is prime.
Hint: Use the identity $a^{k\ell} - 1 = (a^k - 1)(a^{k(\ell-1)} + a^{k(\ell-2)} + \cdots + a^k + 1)$.

Question 3. [p 76. #10]

Using Euclid's proof that there are infinitely many primes, show that the n^{th} prime p_n does not exceed $2^{2^{n-1}}$ whenever n is a positive integer. Conclude that when n is a positive integer, there are at least $n + 1$ primes less than 2^{2^n}.

Question 4. [p 76. #12]

Show that if p_k is the k^{th} prime, where k is a positive integer, then $p_n \leq p_1 p_2 \cdots p_{n-1} + 1$ for all integers n with $n \geq 3$.

Question 5. [p 76. #13]

Show that if the smallest prime factor p of the positive integer n exceeds $\sqrt[3]{n}$, then $\dfrac{n}{p}$ must be prime or 1.

Question 6. [p 90. #14]

Show that every integer greater than 11 is the sum of two composite integers.

Question 7. [p 91. #24]

Let n be a positive integer greater than 1 and let $p_1, p_2, \ldots, p_t$ be the primes not exceeding n. Show that $p_1 p_2 \cdots p_t < 4^n$.

Question 8. [p 91. #25]

Let n be a positive integer greater than 3 and let p be a prime such that $2n/3 < p \leq n$. Show that p does not divide the binomial coefficient $\dbinom{2n}{n}$.

Question 9. [p 91. #26]

Use Exercises 24 and 25 to show that if n is a positive integer, then there exists a prime p such that $n < p < 2n$. (This is *Bertrand's conjecture.*)

Question 10. [p 91. #29]

Use Bertrand's postulate to show that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+m}$$

does not equal an integer when n and m are positive integers. In particular,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

is never an integer for $n > 1$.

Question 11.

Use the prime number theorem

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$$

where $\pi(x)$ is the number of primes less than or equal to x, to show that if p_n is the n^{th} prime, then

$$\lim_{n\to\infty} \frac{p_n}{n\log n} = 1,$$

so that $p_n \sim n\log n$ for large n.

Question 12.

Let p be a prime and n a positive integer. Show that the largest exponent $\nu(n)$ such that $p^{\nu(n)} \mid n!$ is given by **dePolignac's formula**:

$$\nu(n) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots.$$