MATH 324 Summer 2006
Elementary Number Theory

Notes on the Riemann Zeta Function

Department of Mathematical and Statistical Sciences
University of Alberta

Riemann Zeta Function

In this note we give an introduction to the Riemann zeta function, which connects the ideas of real analysis with the arithmetic of the integers.

Define a function of a real variable s as follows,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This function is called the *Riemann zeta function*, after the German mathematician Bernhard Riemann who systematically studied the deeper properties of this function beginning in 1859.

The infinite series for $\zeta(s)$ was actually first introduced by Euler nearly 100 years before Riemann's work.

Note that from the integral test, the series for $\zeta(s)$ converges for $s > 1$, and therefore the function $\zeta(s)$ is defined for all real numbers $s > 1$. Also, since

$$\frac{1}{n^s} \geq \int_n^{n+1} \frac{dx}{x^s},$$

for all $n \geq 1$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \geq \int_1^{\infty} \frac{dx}{x^s} = \frac{1}{s-1}$$

for all $s > 1$. Now we let $s \to 1^+$, and we see that

$$\lim_{s \to 1^+} \zeta(s) = \infty.$$

Infinite Products

In order to discuss the connection of $\zeta(s)$ with the primes, we need the concept of an infinite product, which is very similar to the notion of an infinite sum.

Definition. Given a sequence of real numbers $\{a_n\}_{n\geq 1}$, let

$$\begin{aligned} p_1 &= a_1, \\ p_2 &= a_1 \cdot a_2, \\ p_3 &= a_1 \cdot a_2 \cdot a_3, \\ &\vdots \\ p_n &= a_1 \cdot a_2 \cdots a_n = \prod_{k=1}^{n} a_k. \end{aligned}$$

The ordered pair of sequences $\{\{a_n\}, \{p_n\}\}$ is called an *infinite product*. The real number p_n is called the n^{th} *partial product* and a_n is called the n^{th} *factor* of the product, and the following symbols are used to denote the infinite product defined above:

$$a_1 \cdot a_2 \cdots a_n \cdots \qquad \text{or} \qquad \prod_{n=1}^{\infty} a_n.$$

Using the the analogy with infinite series, where we say that the series converges if and only if the sequence of partial sums converges, it is tempting to say that the infinite product converges if and only if the sequence of partial products converges. However, if we do this, then every product which has *one* factor equal to zero would converge, regardless of the behavior of the remaining factors. The definition below is more useful:

Definition. Given an infinite product $\prod_{n=1}^{\infty} a_n$, let $p_n = \prod_{k=1}^{n} a_k$ be the n^{th} partial product.

(a) If infinitely many factors a_n are zero, then we say the product *diverges to zero.*

(b) If no factor a_n is zero, then we say that the product *converges* if and only if there exists a $p \neq 0$, such that $p_n \to p$ as $n \to \infty$. In this case, p is called the *value of the product*, and we write $p = \prod_{n=1}^{\infty} a_n$. If $p_n \to 0$ as $n \to \infty$, then we say the product *diverges to zero.*

(c) If there exists an integer N such that $n > N$ implies that $a_n \neq 0$, then we say that $\prod_{n=1}^{\infty} a_n$ *converges*, provided that $\prod_{n=N+1}^{\infty} a_n$ converges as described in part (b). In this case, the value of the product is

$$a_1 \cdot a_2 \cdots a_N \cdot \prod_{n=N+1}^{\infty} a_n.$$

(d) $\prod_{n=1}^{\infty} a_n$ is said to be *divergent* if it does not converge as described in parts (b) and (c) above.

Note: The value of a convergent infinite product can be zero, but this is the case if and only if a finite number of the factors are zero.

Also, the convergence of an infinite product is not affected by inserting or removing a finite number of factors, zero or not.

Now we give a criterion for the convergence of infinite products, completely analogous to the corresponding criterion for convergence of infinite series.

Theorem (Cauchy Criteria for Convergence of an Infinite Product).

The infinite product $\prod_{n=1}^{\infty} a_n$ converges if and only if given any $\epsilon > 0$, there exists an integer n_0 such that

$$\left| \prod_{k=n+1}^{m} a_k - 1 \right| < \epsilon$$

whenever $m > n \geq n_0$.

Proof. Assume first that the product $\prod_{n=1}^{\infty} a_n$ converges, we may also assume that no a_n is zero (discarding a finite number of terms if necessary). Let

$$p_n = a_1 \cdot a_2 \cdots a_n \qquad \text{and} \qquad p = \lim_{n\to\infty} p_n,$$

so that $p \neq 0$, and there exists an $M > 0$ such that $|p_n| > M$ for all $n \geq 1$. Now, since $\{p_n\}$ satisfies the Cauchy criteria for sequences, given an $\epsilon > 0$, there exists an integer n_0 such that

$$|p_m - p_n| < \epsilon M$$

whenever $m > n \geq n_0$. Dividing by $|p_n|$ we get $\left| \prod_{k=n+1}^{m} a_k - 1 \right| < \epsilon$ whenever $m > n \geq n_0$.

Conversly, suppose that given any $\epsilon > 0$, there exists an integer n_0 such that

$$\left| \prod_{k=n+1}^{m} a_k - 1 \right| < \epsilon \qquad (*)$$

whenever $m > n \geq n_0$.

First note that if $m > n_0$ this implies that $a_m \neq 0$, since (assuming that $0 < \epsilon < 1$), taking $n = m - 1$, we have

$$\big||a_m| - 1\big| \leq |a_m - 1| < \epsilon,$$

that is,

$$0 < 1 - \epsilon < |a_m| < 1 + \epsilon$$

for $m \geq n_0$.

Now take $\epsilon = \frac{1}{2}$, and let

$$q_m = a_{n_0+1}a_{n_0+2}\cdots a_m$$

for $m > n_0$, then

$$\tfrac{1}{2} < |q_m| < \tfrac{3}{2},$$

for all $m > n_0$. Therefore, if $\{q_m\}$ converges, it cannot converge to zero.

To see that the sequence $\{q_n\}$ does actually converge, let $0 < \epsilon < \frac{1}{2}$ be arbitrary, then there exists an n_0 such that

$$\left|\frac{q_m}{q_n} - 1\right| < \tfrac{2}{3}\epsilon$$

whenever $m > n \geq n_0$, that is,

$$|q_m - q_n| < \tfrac{2}{3}\epsilon|q_n| < \tfrac{3}{2}\cdot\tfrac{2}{3}\epsilon = \epsilon$$

whenever $m > n \geq n_0$. Thus, the sequence $\{q_n\}$ is a Cauchy sequence of real numbers, and hence converges, and so the infinite product $\prod_{n=1}^{\infty} a_n$ converges. □

Note: Taking $n = m - 1$ in $(*)$, we see that if the infinite product $\prod_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} a_n = 1.$$

Because of this, the factors of a product are often written as $1 + a_n$, so that convergence of a product $\prod_{n=1}^{\infty}(1 + a_n)$ implies that $\lim_{n\to\infty} a_n = 0$.

Theorem. If $a_n > 0$ for all $n \geq 1$, then the infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if the infinite series $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Let the partial sums and partial products be given by

$$s_n = a_1 + a_2 + \cdots + a_n \qquad \text{and} \qquad p_n = (1 + a_1)(1 + a_2)\cdots(1 + a_n)$$

for $n \geq 1$. Since $a_n > 0$ for all $n \geq 1$, then both of the sequences $\{s_n\}$ and $\{p_n\}$ are monotone increasing, and to prove the theorem, we only have to show that $\{s_n\}$ is bounded above if and only if $\{p_n\}$ is bounded above.

First we note that $s_n < p_n$ for all $n \geq 1$. Now using the fact that $1 + x \leq e^x$ for all real numbers x, we have

$$p_n = (1 + a_1)(1 + a_2)\cdots(1 + a_n) \leq e^{a_1}e^{a_2}\cdots e^{a_n} = e^{s_n}$$

for all $n \geq 1$. Therefore the sequence $\{p_n\}$ is bounded above if and only if the sequence $\{s_n\}$ is bounded above.

Also, note that the sequence of partial products $\{p_n\}$ cannot converge to zero, since $p_n \geq 1$ for $n \geq 1$. Finally, note that

$$p_n \to +\infty \quad \text{if and only if} \quad s_n \to +\infty.$$

Definition. The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is said to *converge absolutely* if and only if the infinite product $\prod_{n=1}^{\infty}(1+|a_n|)$ converges.

Theorem. If the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely, then it converges.

Proof. Use the Cauchy criterion together with the inequality

$$|(1+a_{n+1})(1+a_{n+2})\cdots(1+a_m)-1| \le (1+|a_{n+1}|)(1+|a_{n+2}|)\cdots(1+|a_m|)-1.$$

□

Note: For positive terms $a_n > 0$ for all $n \ge 1$, we know that $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges, so that $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely if and only if $\sum_{n=1}^{\infty} a_n$ converges absolutely. However, if the terms are not positive, we have the following examples to show the results need not be valid.

Exercise 1. Let $a_n = \dfrac{(-1)^n}{\sqrt{n}}$ for $n \ge 1$, show that $\prod_{n=1}^{\infty}(1+a_n)$ diverges, but $\sum_{n=1}^{\infty} a_n$ converges.

Exercise 2. Let

$$a_{2n-1} = \frac{-1}{\sqrt{n}} \qquad \text{and} \qquad a_{2n} = \frac{1}{\sqrt{n}} + \frac{1}{n}$$

for $n = 1, 2, \ldots$, show that $\prod_{n=1}^{\infty}(1+a_n)$ converges, but $\sum_{n=1}^{\infty} a_n$ diverges.

We do have a theorem analogous to the positive term result:

Theorem. If $a_n \ge 0$ for all $n \ge 1$, then the infinite product $\prod_{n=1}^{\infty}(1-a_n)$ converges if and only if the infinite series $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Note that convergence of $\sum_{n=1}^{\infty} a_n$ implies the absolute convergence of $\prod_{n=1}^{\infty}(1-a_n)$, and hence the convergence of $\prod_{n=1}^{\infty}(1-a_n)$.

Conversely, suppose that $\sum\limits_{n=1}^{\infty} a_n$ diverges, if $\{a_n\}$ does not converge to zero, then $\prod\limits_{n=1}^{\infty}(1-a_n)$ also diverges. Therefore, we may assume that $a_n \to 0$ as $n \to \infty$, and by discarding finitely many terms if necessary, we may assume that $a_n \leq \frac{1}{2}$ for all $n \geq 1$. But then, $1 - a_n \geq \frac{1}{2}$ for all $n \geq 1$, and so $a_n \neq 0$ for $n \geq 1$.

Let
$$p_n = (1-a_1)(1-a_2)\cdots(1-a_n) \qquad \text{and} \qquad q_n = (1+a_1)(1+a_2)\cdots(1+a_n)$$
for all $n \geq 1$, then since
$$(1-a_k)(1+a_k) = 1 - a_k^2 \leq 1$$
we have $p_n \leq \dfrac{1}{q_n}$ for all $n \geq 1$.

Now since $\sum\limits_{n=1}^{\infty} a_n$ diverges, then $\prod\limits_{n=1}^{\infty}(1+a_n)$ diverges also, and $q_n \to +\infty$ as $n \to \infty$. Therefore, $p_n \to 0$ as $n \to \infty$ and so by part (b) of the definition of convergence, it follows that $\prod\limits_{n=1}^{\infty}(1-a_n)$ diverges to 0. $\square$

Euler's Product Formula for the Riemann Zeta Function

We now have enough information to prove Euler's theorem and make the connection between infinite products and prime numbers.

Theorem (Euler's Product Formula)

Let p_k denote the k^{th} prime number, if $s > 1$, then
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-s}},$$
and the infinite product converges absolutely.

Proof. For $m \geq 1$, let
$$P_m = \prod_{k=1}^{m} \frac{1}{1 - p_k^{-s}}$$
be the m^{th} partial product of the infinite product. If we write each factor as a convergent geomteric series, we have
$$P_m = \prod_{k=1}^{m} \left(1 + \frac{1}{p_k^s} + \frac{1}{p_k^{2s}} + \cdots\right),$$
which is a product of a finite number of absolutely convergent infinite series, and multiplying these series together and rearranging the terms according to increasing denominators, we get another absolutely convergent infinite series, and a typical term looks like
$$\frac{1}{p_1^{a_1 s} p_2^{a_2 s} \cdots p_m^{a_m s}} = \frac{1}{n^s}$$
where $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, and $a_i \geq 0$ for $i = 1, 2, \ldots, m$. Therefore,
$$P_m = \sum \left\{ \frac{1}{n^s} \ : \ \text{all prime factors of } n \text{ are } \leq p_m \right\}$$
and by the Fundamental Theorem of Arithmetic, each such n occurs exactly once in the summation.

Therefore,

$$\zeta(s) - P_m = \sum \left\{ \frac{1}{n^s} \ : \ \text{at least one prime factor of } n \text{ is } \ > p_m \right\}$$

and since these prime factors occur among the integers $n > p_m$, we have

$$|\zeta(s) - P_m| \leq \sum_{n > p_m} \frac{1}{n^s}.$$

As $m \to \infty$, the sum on the right tends to 0, since $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges, and therefore

$$\lim_{m \to \infty} P_m = \zeta(s).$$

Finally, the product has the form $\prod_{n=1}^{\infty} (1 + a_n)$ where

$$a_n = \frac{1}{p_n^s} + \frac{1}{p_n^{2s}} + \cdots ,$$

and the series $\sum_{n=1}^{\infty} a_n$ converges absolutely since it is dominated by $\sum_{n=1}^{\infty} \frac{1}{n^s}$, and therefore the infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ also converges absolutely.

□

Evaluating the Zeta Function

In 1734, Euler found that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

and he later found closed formulas for

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6}$$

$$\zeta(8) = \sum_{n=1}^{\infty} \frac{1}{n^8}$$

$$\vdots$$

but he failed to find a closed formula for $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$. In fact, it was not even known if $\zeta(3)$ was irrational or not until 1979, when R. Apery proved that $\zeta(3)$ is irrational. It is still unknown whether or not $\zeta(3)$ is transcendental.

An elementary proof that $\zeta(2) = \dfrac{\pi^2}{6}$ uses the theory of Fourier series to sum the series. There is another elementary proof given below that uses only the binomial theorem and some elementary facts about the roots of polynomials with integer coefficients.

Theorem.

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof. Note that for $0 < x < \dfrac{\pi}{2}$, we have $0 < \sin x < x < \tan x$, so that $0 < \cot x < \dfrac{1}{x} < \csc x$, and therefore

$$\cot^2 x < \frac{1}{x^2} < \csc^2 x \tag{1}$$

for $0 < x < \dfrac{\pi}{2}$.

Now let $m \in \mathbb{N}$, then for $k = 1, \ldots, m$ we have $0 < \dfrac{k\pi}{2m+1} < \dfrac{\pi}{2}$, so that

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} < \sum_{k=1}^{m} \frac{(2m+1)^2}{k^2\pi^2} < \sum_{k=1}^{m} \csc^2 \frac{k\pi}{2m+1}$$

for all $m \geq 1$; therefore,

$$\frac{\pi^2}{(2m+1)^2} \cdot \sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} < \sum_{k=1}^{m} \frac{1}{k^2} < \frac{\pi^2}{(2m+1)^2} \cdot \sum_{k=1}^{m} \csc^2 \frac{k\pi}{2m+1} \tag{2}$$

for all $m \geq 1$.

Now we find explicit formulas for $\sum\limits_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1}$ and $\sum\limits_{k=1}^{m} \csc^2 \frac{k\pi}{2m+1}$. From DeMoivre's formula and the binomial theorem, we have, for $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\cos nx + i \sin nx &= \big(\cos x + i \sin x\big)^n \\
&= \sum_{j=0}^{n} \binom{n}{j} \big(\cos x\big)^{n-j} \big(i \sin x\big)^j \\
&= \sum_{j=0}^{[\frac{n}{2}]} \binom{n}{2j} \cos^{n-2j} x \, (i)^{2j} \sin^{2j} x + \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n}{2j+1} \cos^{n-2j-1} x \, (i)^{2j+1} \sin^{2j+1} x \\
&= \sum_{j=0}^{[\frac{n}{2}]} \big(-1\big)^j \binom{n}{2j} \cos^{n-2j} x \, \sin^{2j} x + i \sum_{j=0}^{[\frac{n-1}{2}]} \big(-1\big)^j \binom{n}{2j+1} \cos^{n-2j-1} x \, \sin^{2j+1} x,
\end{aligned}$$

and equating real and imaginary parts in this expression, we have

$$\cos nx = \sum_{j=0}^{[\frac{n}{2}]} \big(-1\big)^j \binom{n}{2j} \cos^{n-2j} x \, \sin^{2j} x \tag{3}$$

and

$$\sin nx = \sum_{j=0}^{[\frac{n-1}{2}]} \big(-1\big)^j \binom{n}{2j+1} \cos^{n-2j-1} x \, \sin^{2j+1} x. \tag{4}$$

Now, if $\sin x \neq 0$, and $m \in \mathbb{N}$, then taking $n = 2m+1$ in (4), we have

$$\begin{aligned}
\sin(2m+1)x &= \sum_{j=0}^{m} \big(-1\big)^j \binom{2m+1}{2j+1} \cos^{2m-2j} x \, \sin^{2j+1} x \\
&= \sin^{2m+1} x \cdot \sum_{j=0}^{m} \big(-1\big)^j \binom{2m+1}{2j+1} \big(\cos^2 x\big)^{m-j} \big(\sin^2 x\big)^{j-m} \\
&= \sin^{2m+1} x \cdot \sum_{j=0}^{m} \big(-1\big)^j \binom{2m+1}{2j+1} \big(\cot^2 x\big)^{m-j},
\end{aligned}$$

that is,

$$\sin(2m+1)x = \sin^{2m+1} x \cdot \sum_{j=0}^{m} \big(-1\big)^j \binom{2m+1}{2j+1} \big(\cot^2 x\big)^{m-j} \tag{5}$$

for all $m \in \mathbb{N}$, provided that $\sin x \neq 0$.

Taking $x = \frac{k\pi}{2m+1}$, for $k = 1, 2, \ldots, m$; then

$$\sin \tfrac{k\pi}{2m+1} \neq 0 \qquad \text{but} \qquad \sin\big((2m+1)\tfrac{k\pi}{2m+1}\big) = \sin k\pi = 0$$

for $k = 1, 2, \ldots, m$, and (5) yields

$$\sum_{j=0}^{m} \big(-1\big)^j \binom{2m+1}{2j+1} \big(\cot^2 \tfrac{k\pi}{2m+1}\big)^{m-j} = 0 \tag{6}$$

for $k = 1, 2, \ldots, m$. Therefore, the m roots of the equation

$$\sum_{j=0}^{m} \big(-1\big)^j \binom{2m+1}{2j+1} t^{m-j} = 0$$

are $t = \cot^2 \frac{k\pi}{2m+1}, \quad k = 1, 2, \ldots, m.$

Now, if $p(t) = a_0 + a_1 t + \cdots + a_m t^m$ is a polynomial in t of degree m, and if $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the roots of p, then

$$\begin{aligned}
p(t) &= a_m(t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_m) \\
&= a_m t^m - a_m(\lambda_1 + \cdots + \lambda_m)t^{m-1} + \cdots
\end{aligned}$$

so the coefficient of t^{m-1} is $-a_m(\lambda_1 + \cdots + \lambda_m)$. Now, for $p(t) = \sum_{j=0}^{m} (-1)^j \binom{2m+1}{2j+1} t^{m-j}$, the coefficient of t^{m-1} is $-\binom{2m+1}{3}$ and $a_m = \binom{2m+1}{1}$, and therefore from (6) we have

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} = \frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} = \frac{m(2m-1)}{3} \tag{7}$$

for all $m \geq 1$. and since $\csc^2 x = 1 + \cot^2 x$, then

$$\sum_{k=1}^{m} \csc^2 \frac{k\pi}{2m+1} = \frac{2m(m+1)}{3} \tag{8}$$

for all $m \geq 1$.

Combining (2), (7) and (8), we have

$$\frac{2m(2m-1)}{(2m+1)^2}\cdot\frac{\pi^2}{6} < \sum_{k=1}^{m}\frac{1}{k^2} < \frac{2m(2m+2)}{(2m+1)^2}\cdot\frac{\pi^2}{6}, \tag{9}$$

for all $m \geq 1$, and letting $m \to \infty$, we have

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

as required.

□

Example. The usual Fourier series evaluation of $\zeta(2)$ looks at the function

$$f(x) = x^2, \quad 0 < x < \pi,$$

and then finds the Fourier cosine series of $\widehat{f}$, the even 2π-periodic extension of f, to be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

for $n = 0, 1, 2, \ldots$.

For $n = 0$, we have

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\cdot\frac{x^3}{3}\bigg|_0^{\pi} = \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

For $n \geq 1$, we have

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \frac{2}{\pi}\left[\frac{x^2\sin nx}{n}\bigg|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nx\,dx\right] = -\frac{4}{n\pi}\int_0^{\pi} x\sin nx\,dx \\ &= -\frac{4}{n\pi}\left[-\frac{x\cos nx}{n}\bigg|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right] = \frac{4}{n^2}(-1)^n - \frac{4}{n^3\pi}\sin nx\bigg|_0^{\pi} \\ &= \frac{4(-1)^n}{n^2}. \end{aligned}$$

Therefore, the Fourier cosine series for $\widehat{f}$ is given by

$$\widehat{f}(x) \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

for all $x \in \mathbb{R}$.

From Dirichlet's theorem, since the even 2π-periodic extension of f is continuous at each $x \in \mathbb{R}$, then the Fourier cosine series converges to $\widehat{f}(x)$ for all $x \in \mathbb{R}$, and we can write

$$\widehat{f}(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

for all $x \in \mathbb{R}$, in particular, when $x = \pi$, the series converges to π^2, so that

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2},$$

and therefore

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exercise 3. Show that $\zeta(2) = \dfrac{\pi^2}{6}$ by evaluating the double integral

$$I = \int_0^1 \int_0^1 \frac{dx\, dy}{1 - xy}$$

in two different ways.

(a) Obtain $I = \zeta(2)$ from the expansion

$$\frac{1}{1 - xy} = 1 + xy + x^2y^2 + x^3y^3 + \cdots,$$

which is valid for $|xy| < 1$.

(b) Evaluate the integral directly by rotating the coordinate system about the origin through an angle of $\dfrac{\pi}{4}$ to obtain

$$I = 4\int_0^{1/\sqrt{2}} \left(\int_0^u \frac{dv}{2 - u^2 + v^2} \right) du + 4\int_{1/\sqrt{2}}^{\sqrt{2}} \left(\int_0^{\sqrt{2}-u} \frac{dv}{2 - u^2 + v^2} \right) du,$$

integrating with respect to v and then making the substitution $u = \sqrt{2}\cos\theta$.