MATH 324 Summer 2006
Elementary Number Theory
Solutions to Assignment 2
Due: Thursday July 27, 2006

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Question 1. [p 74. #6]

Show that no integer of the form $n^3 + 1$ is a prime, other than $2 = 1^3 + 1$.

Solution: If $n^3 + 1$ is a prime, since

$$n^3 + 1 = (n + 1)(n^2 - n + 1),$$

then either $n + 1 = 1$ or $n^2 - n + 1 = 1$. The $n + 1 = 1$ is impossible, since $n \geq 1$, and therefore we must have $n^2 - n + 1 = 1$, that is, $n(n - 1) = 0$, so that $n = 1$.

Question 2. [p 74. #7]

Show that if a and n are positive integers with $n > 1$ and $a^n - 1$ is prime, then $a = 2$ and n is prime.

Hint: Use the identity $a^{k\ell} - 1 = (a^k - 1)(a^{k(\ell-1)} + a^{k(\ell-2)} + \cdots + a^k + 1)$.

Solution: Suppose that $a^n - 1$ is prime, where $n > 1$. Since

$$a^n - 1 = (a - 1)(a^{n-1} + a^{n-2} + \cdots + a + 1)$$

and the second factor is clearly greater than 1, it follows that $a - 1 = 1$, that is, $a = 2$. Otherwise, the first factor would also be greater than 1 and $a^n - 1$ would be composite.

Also, if n is composite, so that $n = k \cdot \ell$, with $k > 1$ and $\ell > 1$, then we can factor $2^n - 1$ as in the hint:

$$2^{k\ell} - 1 = (2^k - 1)(2^{k(\ell-1)} + 2^{k(\ell-2)} + \cdots + 2^k + 1).$$

and each factor on the right is clearly greater than 1. which is a contradiction, so n must be prime.

Question 3. [p 74. #10]

Using Euclid's proof that there are infinitely many primes, show that the n^{th} prime p_n does not exceed $2^{2^{n-1}}$ whenever n is a positive integer. Conclude that when n is a positive integer, there are at least $n + 1$ primes less than 2^{2^n}.

Solution: The proof is by strong induction.

Base Case: If $n = 1$, then $p_1 = 2 \leq 2^{2^0} = 2$.

Inductive Step: Now assume that $p_k \leq 2^{2^{k-1}}$ for $k = 1, 2, \ldots, n$. If $M = p_1 p_2 \cdots p_n + 1$, since M has a prime divisor p which is different from each p_i, with $1 \leq i \leq n$, then

$$p_{n+1} \leq p \leq M = p_1 p_2 \cdots p_n + 1 \leq 2^{2^0} 2^{2^1} \cdots 2^{2^{n-1}} + 1 = 2^{2^0 + 2^1 + \cdots + 2^{n-1}} + 1 = 2^{2^n - 1} + 1 < 2^{2^n - 1} + 2^{2^n - 1} = 2^{2^n}.$$

By the principle of mathematical induction $p_n \leq 2^{2^{n-1}}$ for all $n \geq 1$.

From the above, $p_{n+1} \leq 2^{2^n}$, and since 2^{2^n} cannot be prime if $n > 0$, there must be $n + 1$ primes which are strictly less than 2^{2^n}.

Question 4. [p 74. #12]

Show that if p_k is the k^{th} prime, where k is a positive integer, then $p_n \leq p_1 p_2 \cdots p_{n-1} + 1$ for all integers n with $n \geq 3$.

SOLUTION: Let $M = p_1 p_2 \cdots p_{n-1} + 1$, where p_k is the k^{th} prime, from Euler's proof, some prime p different from $p_1, p_2, \ldots, p_{n-1}$ divides M, so that

$$p_n \leq p \leq M = p_1 p_2 \cdots p_{n-1} + 1$$

for all $n \geq 3$.

Question 5. [p 74. #13]

Show that if the smallest prime factor p of the positive integer n exceeds $\sqrt[3]{n}$, then $\dfrac{n}{p}$ must be prime or 1.

SOLUTION: Let p be the smallest prime factor of n, and assume that $p > \sqrt[3]{n}$.

Case 1: If n is prime, then the smallest prime factor of n is $p = n$, and in this case $\dfrac{n}{p} = 1$.

Case 2: If $n > 1$ is not prime, then n must be composite, so that

$$n = p \cdot \frac{n}{p},$$

and since $p > \sqrt[3]{n}$, then

$$\frac{n}{p} < \frac{n}{\sqrt[3]{n}} = \left(\sqrt[3]{n}\right)^2.$$

Now, if $\dfrac{n}{p}$ is not prime then $\dfrac{n}{p}$ has a prime factor q with

$$q < \sqrt{\frac{n}{p}} < \sqrt[3]{n} < p$$

and this prime factor q is also a divisor of n, which contradicts the definition of p. Therefore, $\dfrac{n}{p}$ must be prime.

Question 6. [p 87. #12]

Show that every integer greater than 11 is the sum of two composite integers.

SOLUTION: If $n > 11$ and n is even, then $n - 4$ is even and $n - 4 > 7$, so that $n - 4 \geq 8$. Therefore,

$$n = (n - 4) + 4$$

is the sum of two composite integers

If $n > 11$ and n is odd, then $n - 9$ is even and $n - 9 > 2$, so that $n - 9 \geq 4$. Therefore,

$$n = (n - 9) + 9$$

is the sum of two composite integers.

Question 7. [p 87. #22]

Let n be a positive integer greater than 1 and let $p_1, p_2, \ldots, p_t$ be the primes not exceeding n. Show that

$$p_1 p_2 \cdots p_t < 4^n. \qquad (*)$$

SOLUTION: The proof is by strong induction.

Base Case: if $n = 2$, then $p_1 = 2$ is the only prime less than or equal to 2, and

$$2 < 4^2 = 16$$

so that $(*)$ is true for $n = 2$.

Inductive Step: Now suppose that $(*)$ is true for 2, 3, ..., $n-1$ where $n \geq 3$.

Note that we can restrict our attention to odd n, since if n is even, then

$$\prod_{p \leq n} p = \prod_{p \leq n-1} p < 4^{n-1} < 4^n.$$

Setting $n = 2m+1$, we have

$$\binom{2m+1}{m} = \frac{(2m+1)!}{m!(m+1)!}$$

and this is divisible by every prime p with $m+2 \leq p \leq 2m+1$, so that

$$\prod_{p \leq 2m+1} p \leq \binom{2m+1}{m} \prod_{p \leq m+1} p < \binom{2m+1}{m} 4^{m+1}$$

by the inductive hypothesis.

Now, the binomial coefficients

$$\binom{2m+1}{m} \qquad \text{and} \qquad \binom{2m+1}{m+1}$$

are equal and both occur in the expansion of the binomial $(1+1)^{2m+1}$, so that

$$\binom{2m+1}{m} \leq \frac{1}{2} \cdot 2^{2m+1} = 4^m,$$

and therefore

$$\prod_{p \leq 2m+1} p < 4^m \cdot 4^{m+1} = 4^{2m+1}$$

and $(*)$ is true for $n = 2m+1$ also.

By the Principle of Mathematical Induction

$$\prod_{p \leq n} p < 4^n$$

for all positive integers $n \geq 2$.

Question 8. [p 87. #23]

Let n be a positive integer greater than 3 and let p be a prime such that $2n/3 < p \leq n$. Show that p does not divide the binomial coefficient $\binom{2n}{n}$.

Solution: Note that the restrictions on n are such that

(a) $p > 2$

(b) p and $2p$ are the only multiples of p which are less than or equal to $2n$, since $3p > 2n$

(c) p itself is the only multiple of p which is less than or equal to n

From (a) and (b) we have

$$p^2 \mid (2n)! \qquad \text{but} \qquad p^3 \nmid (2n)!,$$

while from (c),

$$p^2 \mid (n!)^2 \qquad \text{but} \qquad p^3 \nmid (n!)^2.$$

Therefore,

$$p \nmid \frac{(2n)!}{(n!)^2},$$

that is,

$$p \nmid \binom{2n}{n}.$$

Question 9. [p 87. #24]

Use Exercises 22 and 23 to show that if n is a positive integer, then there exists a prime p such that $n < p < 2n$. (This is *Bertrand's conjecture.*)

Solution: You can easily check using the Sieve of Eratosthenes that the result holds for $2 \leq n \leq 127$.

Now let $n \geq 128$, and suppose that there is no prime between n and $2n$. Let

$$\binom{2n}{n} = \prod_{p \leq 2n} p^{r_p}$$

be the prime power decomposition of $\binom{2n}{n}$. By assumption there are no primes between n and $2n$, so that

$$\binom{2n}{n} = \prod_{p \leq n} p^{r_p}.$$

If p is a prime with $\frac{2n}{3} < p \leq n$, then

$$n < \frac{4n}{3} < 2p < 2n \qquad \text{and} \qquad 2n < 3p < 3n,$$

so that p divides $n!$ exactly once and p divides (2n)! exactly twice, and so

$$p \nmid \binom{2n}{n}.$$

Therefore,

$$\binom{2n}{n} = \prod_{p\le\sqrt{2n}} p^{r_p} \prod_{\sqrt{2n}<p\le 2n/3} p^{r_p} \le \prod_{p\le\sqrt{2n}} 2n \prod_{p\le 2n/3} p$$

since if $\sqrt{2n} < p \le 2n/3$, then p divides $\binom{2n}{n}$ exactly once.

Now, the number of primes less than $\sqrt{2n}$ is less than the number of odd integers less than $\sqrt{2n}$, that is, less than $\sqrt{2n}/2 - 1 = \sqrt{n/2} - 1$, therefore

$$\prod_{p\le\sqrt{2n}} 2n \le (2n)^{\sqrt{n/2}-1}.$$

From Question 7, we have (replace n by $\lfloor 2n/3 \rfloor$)

$$\prod_{p\le\lfloor 2n/3\rfloor} p < 4^{\lfloor 2n/3\rfloor},$$

which implies that

$$\prod_{p\le 2n/3} p < 4^{2n/3},$$

so that

$$\binom{2n}{n} < (2n)^{\sqrt{n/2}-1} 4^{2n/3}.$$

Since $\binom{2n}{n}$ is the largest of the $2n+1$ terms in the binomial expansion of $(1+1)^{2n}$, we have

$$(2n+1)\binom{2n}{n} > (2n)\binom{2n}{n} > 2^{2n},$$

so that

$$\frac{1}{2n} 2^{2n} < \binom{2n}{n} < (2n)^{\sqrt{n/2}-1} 4^{2n/3},$$

which implies that

$$2^{2n/3} < (2n)^{\sqrt{n/2}}.$$

Taking logarithms and dividing by $\sqrt{2n}/6$, we get

$$\sqrt{8n}\log 2 - 3\log(2n) < 0. \qquad (**)$$

Now define the function $f : \mathbb{N} \to \mathbb{R}$ by

$$f(n) = \sqrt{8n}\log 2 - 3\log(2n),$$

and differentiate to get

$$f'(n) = \frac{\sqrt{2n}\log 2 - 3}{n}.$$

Note that $f(128) = 8\log 2 > 0$, and $f'(n) > 0$ for $n \ge 128$, so that $f(n)$ is increasing and therefore positive for $n \ge 128$. However, this contradicts the inequality $(**)$.

Therefore, for any positive integer $n > 1$, there is a prime p satisfying $n < p < 2n$.

Question 10. **[p 87. #26]** *(Extra Credit)*

Use Bertrand's postulate to show that every positive integer n with $n \geq 7$ is the sum of distinct primes.

SOLUTION: First note that the result is true for all positive integers n with $7 \leq n \leq 16$:

$$\begin{aligned} 7 &= 2+5 & 12 &= 5+7 \\ 8 &= 3+5 & 13 &= 2+11 \\ 9 &= 2+7 & 14 &= 3+11 \\ 10 &= 3+7 & 15 &= 3+5+7 \\ 11 &= 11 & 16 &= 3+13 \end{aligned}$$

Now let n be a positive integer with $n \geq 17$, and let $n_1 = n$, from Bertrand's postulate, there is a prime p_1 with

$$\left\lfloor \frac{n_1 - 7}{2} \right\rfloor < p_1 \leq 2 \left\lfloor \frac{n_1 - 7}{2} \right\rfloor,$$

or since $\left\lfloor \dfrac{n_1 - 5}{2} \right\rfloor = \left\lfloor \dfrac{n_1 - 7}{2} \right\rfloor + 1,$

$$\left\lfloor \frac{n_1 - 5}{2} \right\rfloor \leq p_1 \leq 2 \left\lfloor \frac{n_1 - 7}{2} \right\rfloor.$$

Now let $n_2 = n_1 - p_1$, and if $n_2 \geq 17$, we repeat the above procedure. By Bertrand's postulate there exists a prime p_2 with

$$\left\lfloor \frac{n_2 - 5}{2} \right\rfloor \leq p_2 \leq 2 \left\lfloor \frac{n_2 - 7}{2} \right\rfloor.$$

Now let $n_3 = n_2 - p_2$, and if $n_3 \geq 17$, we repeat the above procedure. By Bertrand's postulate there exists a prime p_3 with

$$\left\lfloor \frac{n_3 - 5}{2} \right\rfloor \leq p_3 \leq 2 \left\lfloor \frac{n_3 - 7}{2} \right\rfloor.$$

Continuing in this way, at each stage if $n_j \geq 17$, Bertrand's postulate guarantees the existence of a prime p_j such that

$$\left\lfloor \frac{n_j - 5}{2} \right\rfloor \leq p_j \leq 2 \left\lfloor \frac{n_j - 7}{2} \right\rfloor.$$

This process will stop when $j = k$ and $n_{k+1} \leq 16$.

Now,

$$n_{k+1} = n_k - p_k = n_{k-1} - p_{k-1} - p_k = \cdots = n_j - \sum_{i=j}^{k} p_i < n_j - p_j,$$

and $\left\lfloor \dfrac{n_j - 5}{2} \right\rfloor \leq p_j$ implies that

$$\frac{n_j - 5}{2} < \left\lfloor \frac{n_j - 5}{2} \right\rfloor + 1 \leq p_j + 1,$$

so that

$$n_j - p_j - 1 < n_j - \frac{n_j - 5}{2} = \frac{n_j + 5}{2}.$$

Therefore,

$$n_{k+1} \le n_j - p_j - 1 \le \frac{n_j+5}{2},$$

so that

$$n_{k+1} \le \left\lfloor \frac{n_j+5}{2} \right\rfloor < \left\lfloor \frac{n_j+6}{2} \right\rfloor.$$

Also, since

$$p_j \le 2\left\lfloor \frac{n_j-7}{2} \right\rfloor \le 2\frac{n_j-7}{2} = n_j - 7,$$

for $1 \le j \le k$, then

$$n_{k+1} = n_k - p_k \ge 7.$$

Therefore, the final value n_{k+1} will satisfy

$$7 \le n_{k+1} \le 16.$$

Note that the above argument has shown that for each j with $1 \le j \le k$, we have

$$p_{j+1} \le 2\left\lfloor \frac{n_j-7}{2} \right\rfloor \le 2\left\lfloor \left(\left\lfloor \frac{n_j+6}{2} \right\rfloor - 7 \right)/2 \right\rfloor \le \left\lfloor \frac{n_j-8}{2} \right\rfloor < \left\lfloor \frac{n_j-5}{2} \right\rfloor \le p_j,$$

and the sequence $\{p_j\}$ will be decreasing with no duplicates.

Now we note that

$$n_j \le 2p_j + 6.$$

In fact, since

$$\left\lfloor \frac{n_j-5}{2} \right\rfloor \le p_j$$

then there exists a real number θ with $0 \le \theta < 1$ such that

$$\frac{n_j-5}{2} = \left\lfloor \frac{n_j-5}{2} \right\rfloor + \theta,$$

so that

$$\frac{n_j-5}{2} - \theta \le p_j,$$

that is,

$$n_j \le 2p_j + 5 + 2\theta < 2p_j + 7,$$

and since n_j is an integer, then

$$n_j \le 2p_j + 6.$$

Therefore, since $n_j > 16$ for $j \le k$, then $p_j > 5$, so that p_k, the smallest of the p_j's will be at least 7.

Also note that:

if $p_j = 7$, then $n_j \le 20$ and $n_{j+1} \le 13$

if $p_j = 11$, then $n_j \le 28$ and $n_{j+1} \le 17$

if $p_j = 13$, then $n_j \le 32$ and $n_{j+1} \le 19$.

Now all that is left is to show that n can be written as a sum of distinct primes when

$$n_{k+1} = 16, 15, \ldots, 9, 8, 7$$

and we do this by considering each of these cases in turn.

case 1: If $n_{k+1} = 16$, then $16 = n_k - p_k \leq 2p_k + 6 - p_k = p_k + 6$, so that $p_k \geq 10$, and since p_k is prime, this implies $p_k \geq 11$.

Since $16 = 13 + 3 = 11 + 5$, we only need to be concerned with the case that $p_k = 11$ and $p_{k-1} = 13$. If this happens, then

$$n_{k-1} \leq 2p_{k-1} + 6 = 26 + 6 = 32,$$

and

$$n_{k+1} = n_{k-1} - p_{k-1} - p_k \leq 32 - 13 - 11 = 8,$$

which contradicts the fact that $n_{k+1} = 16$, so we cannot have both $p_k = 11$ and $p_{k-1} = 13$. Therefore, using either $16 = 13 + 3$ or $16 = 11 + 5$ we have a partition of n into distinct primes.

case 2: If $n_{k+1} = 15$, then again $15 = n_k - p_k \leq 2p_k + 6 - p_k$, so that $p_k \geq 9$, and since p_k is prime, this implies that $p_k \geq 11$. Since $15 = 7 + 5 + 3$, we have a partition of n into distinct primes.

case 3: If $n_{k+1} = 14$, then again $14 = n_k - p_k \leq 2p_k + 2 - p_k$, so that $p_k \geq 8$, and since p_k is prime, this implies that $p_k \geq 11$. Since $14 = 7 + 5 + 2$, we have a partition of n into distinct primes.

case 4: If $n_{k+1} = 13$, then $13 = n_k - p_k \leq 2p_k + 6 - p_k$, so that $p_k \geq 7$. We cannot have both $p_{k-1} = 13$ and $p_k = 11$, since this implies

$$n_{k-1} \leq 2p_{k-1} + 6 = 26 + 6 = 32,$$

so that

$$n_{k+1} = n_{k-1} - p_{k-1} - p_k \leq 32 - 13 - 11 = 8$$

which is a contradiction. Using either $13 = 13$ or $13 = 11 + 2$ we have a partition of n into distinct primes.

case 5: If $n_{k+1} = 12$, then $12 = n_k - p_k \leq 2p_k + 6 - p_k$, so that $p_k \geq 6$, and so $p_k \geq 7$.

if $p_k > 7$, since $12 = 7 + 5$, we have a partition of n into distinct primes.

if $p_k = 7$, then $n_{k+1} = n_k - p_k$ implies that $n_k = 12 + 7 = 19$. We cannot also have $p_{k-1} = 11$, since this implies that

$$n_{k-1} \leq 2 \cdot 11 + 6 = 28,$$

so that $n_{k+1} \leq 28 - 11 - 7 = 10$, which is a contradiction.

Therefore, $19 = 11 + 5 + 3$, and we have a partition of n into distinct primes.

case 6: If $n_{k+1} = 11$, then $11 = n_k - p_k \leq 2p_k + 6 - p_k$ implies that $p_k \geq 5$, so in fact we must have $p_k \geq 7$. We cannot have both $p_k = 7$ and $p_{k-1} = 11$, otherwise, $n_{k-1} \leq 2p_{k-1} + 6 = 28$, so that $n_{k+1} = n_{k-1} - p_k - p_{k-1} \leq 28 - 7 - 11 = 10$, which is a contradiction. So if $p_k = 7$ or $p_k > 11$, we have a partition of n into distinct primes.

If $p_k = 11$, then $n_{k+1} = n_k - p_k$ imples that $n_k = 11 + 11 = 22$, and we cannot also have $p_{k-1} = 13$, since then $11 = n_{k-1} - p_{k-1} - p_k \leq 32 - 11 - 13 = 8$, which is a contradiction. Thus, with $22 = 13 + 7 + 2$ we have a partition of n into distinct primes.

case 7: If $n_{k+1} = 10$, since $p_k \geq 7$ and $10 = 5 + 3 + 2$, we have a partition of n into distinct primes.

case 8: If $n_{k+1} \leq 9$, and $p_k = 7$, then $n_k = n_{k+1} + p_k \leq 16$, which is a contradiction, since we constructed the sequence so that $n_k \geq 17$. Therefore, we must have $p_k > 7$, and with $9 = 7 + 2,\ 8 = 5 + 3,$ or $7 = 7$, we have a partition of n into distinct primes.

Note: This result was first proven by H. E. Richert in 1950, a proof by induction can be found in the book *Elementary Theory of Numbers* by Sierpiński.

Question 11. [p 87. #27]

Use Bertrand's postulate to show that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+m}$$

does not equal an integer when n and m are positive integers. In particular,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

is never an integer for $n > 1$.

SOLUTION:

Case 1: If $m < n$, then $m \le n-1$, so that $n + m \le 2n - 1$, and

$$\begin{aligned}
\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+m} &\le \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+m} + \cdots + \frac{1}{2n-1} \\
&< \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ times}} \\
&= n \cdot \frac{1}{n} = 1,
\end{aligned}$$

and the sum cannot be an integer in this case.

Case 2: If $m \ge n$, from Bertrand's postulate, there is a prime p such that $n < p < 2n \le n + m$.

Let p be the largest prime such that

$$n < p < n + m,$$

then $n + m < 2p$, since if not, there would be a prime q with $p < q < 2p \le n + m$, which contradicts the choice of p.

Now suppose that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+m} = N \tag{†}$$

where N is an integer. Since $n < p < n+m < 2p$, then p occurs as a factor in only one of the denominators.

Define

$$M = \prod_{k=n}^{n+m} k, \qquad \text{and} \qquad M_k = \frac{M}{k} \;\text{ for }\; n \le k \le n+m,$$

and multipy (†) by M to get

$$M_n + M_{n+1} + \cdots + M_{n+m} = M \cdot N, \tag{††}$$

and solving for M_p, we have

$$M_p = M \cdot N - \sum_{\substack{k=1 \\ k \ne p}}^{m+n} M_k.$$

Now note that every term on the right is divisible by p, which implies that $p \,\big|\, M_p$, which is a contradiction.

Therefore,

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+m}$$

is never an integer for any positive integers m and n.

Question 12.

Use the prime number theorem

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$$

where $\pi(x)$ is the number of primes less than or equal to x, to show that if p_n is the n^{th} prime, then

$$\lim_{n\to\infty} \frac{p_n}{n\log n} = 1,$$

so that $p_n \sim n\log n$ for large n.

SOLUTION: Suppose that the prime number theorem is true, so that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1,$$

if we let $x = p_n$ for $n \geq 1$, then $\pi(p_n) = n$, so that

$$\lim_{n\to\infty} \frac{n\log p_n}{p_n} = 1. \qquad (+)$$

The natural logarithm is a continuous function on $\mathbb{R}+$, so that

$$\lim_{n\to\infty} \{\log n + \log\log p_n - \log p_n\} = 0,$$

and

$$\lim_{n\to\infty} \log p_n \left\{ \frac{\log n}{\log p_n} + \frac{\log\log p_n}{\log p_n} - 1 \right\} = 0.$$

Now, since $\lim\limits_{n\to\infty} \log p_n = +\infty$, this implies that

$$\lim_{n\to\infty} \left\{ \frac{\log n}{\log p_n} + \frac{\log\log p_n}{\log p_n} - 1 \right\} = 0,$$

and since $\lim\limits_{n\to\infty} \dfrac{\log\log p_n}{\log p_n} = 0$, we have

$$\lim_{n\to\infty} \left\{ \frac{\log n}{\log p_n} - 1 \right\} = 0,$$

that is,

$$\lim_{n\to\infty} \frac{\log n}{\log p_n} = 1.$$

From (+), this implies that

$$\lim_{n\to\infty} \frac{n\log n}{p_n} = \lim_{n\to\infty} \frac{n\log p_n}{p_n} \cdot \frac{\log n}{\log p_n} = \lim_{n\to\infty} \frac{n\log p_n}{p_n} = 1,$$

and therefore $p_n \sim n\log n$ when n is large.