MATHEMATICS 324

SOLUTIONS TO MIDTERM EXAMINATION

TIME: 70 Minutes DATE: July 31, 2006

INSTRUCTIONS:

1. No calculators, books, tables or notes are allowed.
2. Show your work. All answers must be justified to receive full credit.
3. Attempt all 5 problems.

Question 1.

(a) Show that the number of positive integers less than or equal to x, where x is a positive real number, that are divisible by the positive integer d equals $\lfloor x/d \rfloor$.

(b) If a and b are positive integers, how many multiples of b are there in the sequence a, $2a$, $3a$, $\ldots$, ba?

Hint: Note that any such multiple of b in the sequence must be a common multiple of a and b.

SOLUTION:

(a) We want the largest integer k such that $1 \leq k \cdot d \leq x$, that is, the greatest integer less than or equal to x/d, that is, $\lfloor x/d \rfloor$.

(b) Note that ka is a multiple of b, if and only if it is a common multiple of a and b, that is, if and only if it is divisible by $[a, b]$, the least common multiple of a and b. Therefore, the number of multiples of b in the sequence is just the number of multiples of $[a, b]$, and by part (a), this is

$$\left\lfloor \frac{ab}{[a,b]} \right\rfloor = (a, b),$$

since $ab = [a, b] \cdot (a, b)$.

Question 2.

Using mathematical induction, show that for $n \geq 2$ the last decimal digit of 2^{2^n} is a 6.

SOLUTION: We give a proof by induction.

Base Case: For $n = 2$, $2^{2^n} = 2^{2^2} = 2^4 = 16$, and the last decimal digit is a 6.

Inductive Step: Now suppose that the decimal expansion of 2^{2^n} ends in a 6 for some $n \geq 2$, then since $6^2 = 36 = 3 \cdot 10 + 6$, we have

$$\begin{aligned} 2^{2^{n+1}} &= \left(2^{2^n}\right)^2 = \left(a_k \cdot 10^k + \cdots + a_1 \cdot 10 + 6\right)^2 \\ &= b_{2k+1} \cdot 10^{2k+1} + b_{2k} \cdot 10^{2k} + \cdots + b_1 \cdot 10 + 6, \end{aligned}$$

where $a_1, \cdots, a_k, b_1, \cdots, b_{2k+1} \in \{0, 1, 2, \cdots, 9\}$. This implies that the decimal expansion of $2^{2^{n+1}}$ ends in a 6 also.

By the principle of mathematical induction, the last digit in the decimal expansion of 2^{2^n} is a 6 for all $n \geq 2$.

Question 3.

Show that the positive integer n has exactly 4 positive divisors if and only if either $n = p^3$ for some prime p, or $n = p \cdot q$ where p and q are distinct primes.

Hint: If the unique prime factorization of n is $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then the number of divisors of n is

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1).$$

Solution: Note first that if $n = p^3$ where p is a prime, then the positive divisors of n are

$$1,\ p,\ p^2,\ p^3,$$

while if $n = p \cdot q$ where p and q are distinct primes, then the positive divisors of n are

$$1,\ p,\ q,\ p \cdot q,$$

and in both cases n has exactly 4 positive divisors.

Conversely, suppose that n as exactly 4 positive divisors, since $\tau(n) = 4$, we must have

$$\tau(n) = (3 + 1) \qquad \text{or} \qquad \tau(n) = (1 + 1)(1 + 1),$$

that is, in the prime factorization of n, either $n = p^3$ or $n = p \cdot q$ where p and q are distinct primes.

Question 4.

Let $a > 1$ and $b > 1$ be positive integers with $(a, b) = 1$, and let d be a positive divisor of $a \cdot b$. Show that $d = d_1 \cdot d_2$, where $d_1 \,|\, a$ and $d_2 \,|\, b$.

Hint: Use the Fundamental Theorem of Arithmetic.

Solution: Suppose the prime power decompositions of a and b are given by

$$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n} \qquad \text{and} \qquad b = q_1^{b_1} q_2^{b_2} \cdots q_m^{b_m}$$

where $p_1 < p_2 < \cdots < p_n$ and $q_1 < q_2 < \cdots < q_m$ are distinct primes. Since $(a, b) = 1$, then $p_i \neq q_j$ for each i and j, and if d is any positive divisor of $a \cdot b$, then the prime power decomposition of d must have the form

$$d = p_1^{c_1} p_2^{c_2} \cdots p_n^{c_n} \cdot q_1^{d_1} q_2^{d_2} \cdots q_m^{d_m}$$

where $0 \leq c_i \leq a_i$ and $0 \leq d_j \leq b_j$ for all i and j, so that $d = d_1 \cdot d_2$ where $d_1 \,\big|\, a$ and $d_2 \,\big|\, b$.

Question 5.

When Mr. Smith cashed a cheque at his bank, the teller mistook the number of cents for the number of dollars, and vice versa.

Unaware of this, Mr. Smith spent 68 cents and then noticed to his surprise that he had twice the amount of the original cheque.

Determine the smallest value for which the cheque could have been written.

Hint: If x is the number of dollars and y the number of cents in the cheque, then

$$100y + x - 68 = 2(100x + y).$$

SOLUTION: From the hint, the linear diophantine equation to be solved is

$$-199x + 98y = 68.$$

Using the Euclidean algorithm to find the gcd(199, 98) we have

$$\begin{aligned} 199 &= 2 \cdot 98 + 3 \\ 98 &= 32 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0, \end{aligned}$$

and the last nonzero remainder is $\gcd(199, 98) = 1$. Working backwards, we have

$$\begin{aligned} 1 &= 3 - 2 \\ &= 3 - (98 - 32 \cdot 3) \\ &= 33 \cdot 3 - 98 \\ &= 33(199 - 2 \cdot 98) - 98 \\ &= 33 \cdot 199 - 67 \cdot 98. \end{aligned}$$

Therefore, $1 = 33 \cdot 199 - 67 \cdot 98$, and multipying this by 68, we have

$$(68 \cdot 33)199 - (68 \cdot 67)98 = 68,$$

and a particular solution to the diophantine equation is given by

$$x_0 = -68 \cdot 33, \qquad y_0 = -67 \cdot 68.$$

The general solution is then

$$\begin{aligned} x &= -68 \cdot 33 + 98k \\ y &= -68 \cdot 67 + 199k \end{aligned}$$

where k is an integer.

The smallest nonnegative integer solutions occur for $k = 23$, and then $x = 10$ and $y = 21$, and the smallest value for which the cheque could have been written is \$10.21.