# MATH 324 Summer 2006
# Elementary Number Theory
# Assignment 2
# Due: Thursday July 27, 2006

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**Question 1.** [p 74. #6]

Show that no integer of the form $n^3 + 1$ is a prime, other than $2 = 1^3 + 1$.

**Question 2.** [p 74. #7]

Show that if $a$ and $n$ are positive integers with $n > 1$ and $a^n - 1$ is prime, then $a = 2$ and $n$ is prime.

*Hint*: Use the identity $a^{k\ell} - 1 = (a^k - 1)(a^{k(\ell-1)} + a^{k(\ell-2)} + \cdots + a^k + 1)$.

**Question 3.** [p 74. #10]

Using Euclid's proof that there are infinitely many primes, show that the $n^{\text{th}}$ prime $p_n$ does not exceed $2^{2^{n-1}}$ whenever $n$ is a positive integer. Conclude that when $n$ is a positive integer, there are at least $n+1$ primes less than $2^{2^n}$.

**Question 4.** [p 74. #12]

Show that if $p_k$ is the $k^{\text{th}}$ prime, where $k$ is a positive integer, then $p_n \leq p_1 p_2 \cdots p_{n-1} + 1$ for all integers $n$ with $n \geq 3$.

**Question 5.** [p 74. #13]

Show that if the smallest prime factor $p$ of the positive integer $n$ exceeds $\sqrt[3]{n}$, then $\dfrac{n}{p}$ must be prime or 1.

**Question 6.** [p 87. #12]

Show that every integer greater than 11 is the sum of two composite integers.

**Question 7.** [p 87. #22]

Let $n$ be a positive integer greater than 1 and let $p_1, p_2, \ldots, p_t$ be the primes not exceeding $n$. Show that $p_1 p_2 \cdots p_t < 4^n$.

**Question 8.** [p 87. #23]

Let $n$ be a positive integer greater than 3 and let $p$ be a prime such that $2n/3 < p \leq n$. Show that $p$ does not divide the binomial coefficient $\dbinom{2n}{n}$.

**Question 9.** [p 87. #24]

Use Exercises 22 and 23 to show that if $n$ is a positive integer, then there exists a prime $p$ such that $n < p < 2n$. (This is *Bertrand's conjecture.*)

**Question 10.** [p 87. #26]

Use Bertrand's postulate to show that every positive integer $n$ with $n \geq 7$ is the sum of distinct primes.

**Question 11. [p 87. #27]**

Use Bertrand's postulate to show that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+m}$$

does not equal an integer when $n$ and $m$ are positive integers. In particular,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

is never an integer for $n > 1$.

**Question 12.**

Use the prime number theorem

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$$

where $\pi(x)$ is the number of primes less than or equal to $x$, to show that if $p_n$ is the $n^{\text{th}}$ prime, then

$$\lim_{n\to\infty} \frac{p_n}{n\log n} = 1,$$

so that $p_n \sim n\log n$ for large $n$.