MATH 324 Summer 2006
Elementary Number Theory
Assignment 1
Due: Wednesday July 19, 2006

Department of Mathematical and Statistical Sciences
University of Alberta

Question 1. [p 13. #5]

Use the well–ordering property to show that $\sqrt{3}$ is irrational.

Question 2. [p 13. #19]

Show that

$$\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$$

whenever x is a nonnegative real number.

Question 3. [p 21. #5]

Find and prove a formula for

$$\sum_{k=1}^{n} \left\lfloor \sqrt{k} \right\rfloor$$

in terms of n and $\lfloor \sqrt{n} \rfloor$.

Question 4. [p 27. #5]

Conjecture a formula for A^n where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Prove your conjecture using mathematical induction.

Question 5. [p 27. #8]

Use mathematical induction to prove that

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

for every positive integer n.

Question 6. [p 28. #20]

Use mathematical induction to prove that $2^n < n!$ for $n \geq 4$.

Question 7. [p 34. #16]

Prove that

$$f_1 f_2 + f_2 f_3 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$$

if n is a positive integer.

Question 8. [p 35. #31]

Show that $f_n \leq \alpha^{n-1}$ for every integer n with $n \geq 2$, where $\alpha = (1 + \sqrt{5})/2$.

Question 9. [p 35. #33]

Prove that whenever n is a nonnegative integer,

$$\sum_{k=1}^{n} \binom{n}{k} f_k = f_{2n},$$

where f_k is the kth Fibonacci number.

Question 10. [p 40. #21]

Show that the number of positive integers less than or equal to x, where x is a positive real number, that are divisible by the positive integer d equals $\lfloor x/d \rfloor$.

Question 11. [p 41. #34]

Use mathematical induction to show that $n^7 - n$ is divisible by 7 for every positive integer n.

Question 12. [p 41. #36]

Let f_n denote the nth Fibonacci number. Show that f_n is even if and only if n is divisible by 3.

Question 13. [p 41 #40]

Show that

$$f_{n+m} = f_m f_{n+1} + f_{m-1} f_n$$

whenever m and n are positive integers with $m > 1$. Use this result to show that $f_n \mid f_m$ when m and n are positive integers with $n \mid m$.

Question 14. [p 41. #45]

Show that $\lfloor (2+\sqrt{3})^n \rfloor$ is odd whenever n is a nonnegative integer.

Question 15. [p 50. #29]

> A **Cantor expansion** of a positive integer n is a sum
>
> $$n = a_m m! + a_{m-1}(m-1)! + \cdots + a_2 2! + a_1 1!,$$
>
> where each a_k is an integer with $0 \le a_k \le k$ and $a_m \ne 0$.

Show that every positive integer has a unique Cantor expansion. (*Hint*: For each positive integer n there is a positive integer m such that $m! \le n < (m+1)!$. For a_m, take the quotient from the division algorithm when n is divided by $m!$, then iterate.)