Math 311 - Spring 2014
Solutions to Assignment # 8
Completion Date: Friday May 30, 2014

Question 1. [p 149, #2]

By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

(a) \( \int_{i}^{2} e^{xz} \, dz \); (b) \( \int_{0}^{\pi + 2i} \cos \left( \frac{z}{2} \right) \, dz \); (c) \( \int_{1}^{3} (z - 2)^{3} \, dz \).

Ans: (a) \( (1 + i)/\pi \); (b) \( e + (1/e) \); (c) 0.

Solution:

(a) Since \( \frac{d}{dz} \left( \frac{1}{\pi} e^{\pi z} \right) = e^{\pi z} \) for all \( z \in \mathbb{C} \), then \( F(z) = \frac{1}{\pi} e^{\pi z} \) is an antiderivative of \( f(z) = e^{\pi z} \), so that

\[
\int_{i}^{i/2} e^{\pi z} \, dz = \left. \frac{1}{\pi} e^{\pi z} \right|_{i}^{i/2} = \frac{1}{\pi} \left( e^{i\pi/2} - e^{i\pi} \right) = \frac{1}{\pi} (1 + i).
\]

(b) Since \( \frac{d}{dz} \left( 2\sin(z/2) \right) = \cos(z/2) \) for all \( z \in \mathbb{C} \), then the function \( F(z) = 2\sin(z/2) \) is an antiderivative of \( f(z) = \cos(z/2) \), and

\[
\int_{0}^{\pi + 2i} \cos \left( \frac{z}{2} \right) \, dz = 2 \sin \left( \frac{z}{2} \right) \bigg|_{0}^{\pi + 2i} = 2 \left[ \sin \left( \pi/2 + i \right) - \sin(0) \right] = 2 \sin \left( \pi/2 + i \right) = 2 \cos(i) = e + \frac{1}{e}.
\]

(c) Since \( \frac{d}{dz} \left( \frac{1}{4} (z - 2)^{4} \right) = (z - 2)^{3} \) for all \( z \in \mathbb{C} \), then the function \( F(z) = \frac{1}{4} (z - 2)^{4} \) is an antiderivative of \( f(z) = (z - 2)^{3} \), and

\[
\int_{1}^{3} (z - 2)^{3} \, dz = \left. \frac{1}{4} (z - 2)^{4} \right|_{1}^{3} = \frac{1}{4} \left[ (3 - 2)^{4} - (1 - 2)^{4} \right] = 0.
\]

Question 2. [p 149, #5]

Show that

\[
\int_{-1}^{1} z^{i} \, dz = \frac{1 + e^{-\pi}}{2} (1 - i),
\]

where \( z^{i} \) denotes the principal branch

\[ z^{i} = \exp(i \log z) \quad (|z| > 0, \ -\pi < \text{Arg} \ z < \pi) \]

and where the path of integration is any contour from \( z = -1 \) to \( z = 1 \) that, except for its end points, lies above the real axis. (Compare with Exercise 7, Sec. 42.)
Suggestion: Use an antiderivative of the branch
\[ z^i = \exp(i \log z) \quad \left( |z| > 0, \quad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2} \right) \]
of the same power function.

Solution: Let
\[ f(z) = z^i, \quad |z| > 0, \quad -\pi < \text{Arg} \ z < \pi, \]
then \( f(z) = z^i = \exp(i \log z) \) is not defined at \( z = -1 \), but if we consider the branch
\[ f_1(z) = z^i = \exp(i \log z), \]
where \( \log z = \ln |z| + i\theta, \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \), then \( f_1(z) \) is defined and analytic at each point of \( C \) and its values coincide with the values of \( f(z) \) except at \( z = -1 \).

An antiderivative of \( f_1(z) \) is given by \( F_1(z) = \frac{1}{1+i} z^{i+1} \), that is,
\[ F_1(z) = \frac{1}{1+i} \exp((i+1) \log z), \quad |z| > 0, \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2}. \]
Therefore,
\[ \int_{-1}^{1} z^i \, dz = \int_{-1}^{1} f_1(z) \, dz = \int_{-1}^{1} F_1'(z) \, dz = F_1(1) - F_1(-1), \]
and
\[ \int_{-1}^{1} z^i \, dz = \frac{1}{1+i} [1 - (-1)^{i+1}] = \frac{1}{1+i} \left[ 1 - e^{((1+i)\pi \mid \ln |z| + \theta}) \right], \]
that is,
\[ \int_{-1}^{1} z^i \, dz = \frac{1}{1+i} \left( 1 - e^{(1+i)\pi i} \right) = \frac{1-i}{2} \left( 1 - e^{i\pi} \right) = \frac{1-i}{2} (1 + e^{-\pi}). \]

Question 3. [p 160, #1 (a)]

Apply the Cauchy-Goursat theorem to show that \( \int_C f(z) \, dz = 0 \) when the contour \( C \) is the circle \( |z| = 1 \), in either direction, and when \( f(z) = \frac{z^2}{z-3} \).
SOLUTION: Since \( f(z) = \frac{z^2}{z-3} \) is analytic inside and on the contour \(|z| = 1\), then
\[
\int_{|z|=1} \frac{z^2}{z-3} \, dz = 0
\]
if the contour is traversed in either direction.

Question 4. [p 160, #1 (c)]

Apply the Cauchy-Goursat theorem to show that \( \int_C f(z) \, dz = 0 \) when the contour \( C \) is the circle \(|z| = 1\), in either direction, and when \( f(z) = \frac{1}{z^2 + 2z + 2} \).

SOLUTION: Since \( f(z) = \frac{1}{z^2 + 2z + 2} = \frac{1}{(z+1-i)(z+1+i)} \) is analytic inside and on the contour \(|z| = 1\), then
\[
\int_{|z|=1} \frac{1}{z^2 + 2z + 2} \, dz = 0
\]
if the contour is traversed in either direction.

Question 5. [p 160, #1 (f)]

Apply the Cauchy-Goursat theorem to show that \( \int_C f(z) \, dz = 0 \) when the contour \( C \) is the circle \(|z| = 1\), in either direction, and when \( f(z) = \log(z+2) \).

SOLUTION: Since the branch cut for \( f(z) = \log(z+2) \) extends from the point \( z = -2 \) along the negative real axis, then \( f(z) \) is analytic inside and on the contour \(|z| = 1\), so that
\[
\int_{|z|=1} \log(z+2) \, dz = 0
\]
if the contour is traversed in either direction.

Question 6. [p 170, #1 (a)]

Let \( C \) denote the positively oriented boundary of the square whose sides lie along the lines \( x = \pm 2 \) and \( y = \pm 2 \). Evaluate the integral
\[
\oint_C \frac{e^{-z}}{z-(\pi i/2)} \, dz.
\]

Ans: \( 2\pi \).

SOLUTION: Let \( f(z) = e^{-z} \), then \( f \) is analytic inside and on \( C \), and since \( z_0 = \frac{\pi i}{2} \) is interior to \( C \), then by Cauchy’s integral formula
\[
\frac{1}{2\pi i} \oint_C \frac{e^{-z}}{z-(\pi i/2)} \, dz = f(\pi i/2) = e^{-\pi i/2} = -i,
\]
so that
\[
\oint_C \frac{e^{-z}}{z-(\pi i/2)} \, dz = 2\pi i(-i) = 2\pi.
\]
Question 7. [p 170, #1 (b)]

Let $C$ denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate the integral

$$\int_C \frac{\cos z}{z(z^2 + 8)} \, dz.$$

**Ans:** $\pi i/4$.

**Solution:** Let $f(z) = \frac{\cos z}{z(z^2 + 8)}$, then $f$ is analytic inside and on $C$, and since $z_0 = 0$ is interior to $C$, then by Cauchy’s integral formula

$$\frac{1}{2\pi i} \int_C \frac{\cos z}{z(z^2 + 8)} \, dz = f(0) = \frac{\cos(0)}{8} = \frac{1}{8},$$

so that

$$\int_C \frac{\cos z}{z(z^2 + 8)} \, dz = \frac{\pi i}{4}.$$ 

Question 8. [p 170, #1 (e)]

Let $C$ denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate the integral

$$\int_C \tan\left(\frac{z}{2}\right) \left(\frac{z}{z-x_0}\right)^2 \, dz \quad (-2 < x_0 < 2).$$

**Ans:** $i\pi \sec^2(x_0/2)$.

**Solution:** Let $f(z) = \tan(z/2)$, then the singularities of $f(z)$ are the zeros of $\cos(z/2)$ and these occur at the points $z = 2\left(n + \frac{1}{2}\right)\pi$, $n = 0, \pm 1, \pm 2, \ldots$, all of which are outside the square. Therefore, $f(z)$ is analytic inside and on $C$, and since $x_0$ is interior to $C$, from Cauchy’s integral formula we have

$$f(x_0) = \frac{1}{2\pi i} \int_C \frac{\tan(z/2)}{(z-x_0)^2} \, dz, \quad \text{and} \quad f'(x_0) = \frac{1}{2\pi i} \int_C \frac{\tan(z/2)}{(z-x_0)^2} \, dz.$$

Now, $f'(x_0) = \frac{1}{2} \sec^2(x_0/2)$, so that

$$\frac{1}{2\pi i} \int_C \frac{\tan(z/2)}{(z-x_0)^2} \, dz = \frac{1}{2} \sec^2(x_0/2),$$

that is,

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2} \, dz = i\pi \sec^2(x_0/2).$$

Question 9. [p 170, #2]

Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(a) $g(z) = \frac{1}{z^2 + 4}$; \hspace{1cm} (b) $g(z) = \frac{1}{(z^2 + 4)^2}$.

**Ans:** (a) $\pi/2$; \hspace{1cm} (b) $\pi/16$. 
Solution:

(a) Let \( g(z) = \frac{1}{z^2 + 4} \) and \( f(z) = \frac{1}{z + 2i} \), then \( f \) is analytic inside and on \( C \), and since \( z_0 = 2i \) is interior to \( |z - i| = 2 \), then from Cauchy’s integral formula we have
\[
\frac{1}{2\pi i} \oint_{|z-i|=2} \frac{f(z)}{z^2 + 4} \, dz = \frac{1}{2\pi i} \oint_{|z-i|=2} \frac{f(z)}{z - 2i} \, dz = f(2i) = \frac{1}{4i},
\]
so that
\[
\oint_{|z-i|=2} \frac{1}{z^2 + 4} \, dz = \frac{\pi}{2}.
\]

(b) Let \( g(z) = \frac{1}{(z^2 + 4)^2} \) and let \( f(z) = \frac{1}{(z + 2i)^2} \), then \( f \) is analytic inside and on \( C \), and since \( z_0 = 2i \) is interior to \( |z - i| = 2 \), then from Cauchy’s integral formula we have
\[
\frac{1}{2\pi i} \oint_{|z-i|=2} g(z) \, dz = \frac{1}{2\pi i} \oint_{|z-i|=2} f(z) \, dz = f'(2i) = -\frac{2}{(2i + 2i)^3} = -\frac{2}{64(-i)},
\]
so that
\[
\oint_{|z-i|=2} \frac{1}{(z^2 + 4)^2} \, dz = \frac{2\pi i}{32i} = \frac{\pi}{16}.
\]

Question 10. [p 171, #3]

Let \( C \) be the circle \( |z| = 3 \), described in the positive sense. Show that if
\[ g(w) = \int_C \frac{2z^2 - z - 2}{z - w} \, dz \quad (|w| \neq 3), \]
then \( g(2) = 8\pi i \). What is the value of \( g(w) \) when \( |w| > 3 \)?

Solution: Let \( f(z) = 2z^2 - z - 2 \), then \( f \) is analytic inside and on \( C \), and from the Cauchy integral formula we have
\[
g(2) = \oint_{|z|=3} \frac{2z^2 - z - 2}{z - 2} \, dz = 2\pi i f(2) = 2\pi i(8 - 2 - 2) = 8\pi i.
\]
If \( |w| > 3 \), then \( h(z) = \frac{2z^2 - z - 2}{z - w} \) is analytic inside and on \( C \), and from the Cauchy-Gourat theorem,
\[
\oint_{|z|=3} \frac{2z^2 - z - 2}{z - w} \, dz = 0
\]
for \( |w| > 3 \).

Question 11. [p 171, #7]

Let \( C \) be the unit circle \( z = e^{i\theta} \) \((-\pi \leq \theta \leq \pi)\). First show that, for any real constant \( a \),
\[ \oint_C \frac{e^{az}}{z} \, dz = 2\pi i. \]
Then write this integral in terms of \( \theta \) to derive the integration formula
\[ \int_0^\pi e^{a\cos \theta} \cos(a \sin \theta) \, d\theta = \pi. \]
SOLUTION: Let \( f(z) = e^{az} \), then \( f \) is analytic inside and on \( C \), and from the Cauchy integral formula, we have
\[
\frac{1}{2\pi i} \oint_C \frac{e^{az}}{z} \, dz = e^0 = 1,
\]
that is,
\[
\oint_C \frac{e^{az}}{z} \, dz = 2\pi i.
\]
Now, on \( C \), we have \( z = e^{i\theta} \) and \( dz = ie^{i\theta} \, d\theta \), so that
\[
\oint_C \frac{e^{az}}{z} \, dz = \int_{-\pi}^{\pi} \frac{e^{a(cos \theta + i sin \theta)}}{e^{i\theta}} \, i e^{i\theta} \, d\theta = i \int_{-\pi}^{\pi} e^{a \cos \theta} \, e^{ai \sin \theta} \, d\theta,
\]
and therefore
\[
2\pi i = \oint_C \frac{e^{az}}{z} \, dz = i \int_{-\pi}^{\pi} e^{a \cos \theta} [\cos(a \sin \theta) + i \sin(a \sin \theta)] \, d\theta,
\]
that is,
\[
\int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) \, d\theta + i \int_{-\pi}^{\pi} e^{a \cos \theta} \sin(a \sin \theta) \, d\theta = 2\pi.
\]
Equating real and imaginary parts, we have
\[
\int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) \, d\theta = 2\pi \quad \text{and} \quad \int_{-\pi}^{\pi} e^{a \cos \theta} \sin(a \sin \theta) \, d\theta = 0,
\]
and since the function \( h(\theta) = e^{a \cos \theta} \cos(a \sin \theta) \) is even, then
\[
\int_{0}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) \, d\theta = \pi.
\]