Question 1. [p 56, #10 (a)]

Use the theorem of Sec. 17 to show that \( \lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = 4 \).

**Solution:** We have
\[
\lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = \lim_{z \to 0} \frac{4(1/z)^2}{1/(z-1)^2} = \lim_{z \to 0} \frac{4}{(1-z)^2} = 4.
\]

Question 2. [p 56, #13]

Show that a set is unbounded (Sec. 11) if and only if every neighborhood of the point at infinity contains at least one point in \( S \).

**Solution:** If \( S \subseteq \mathbb{C} \) is unbounded, then for each \( n \geq 1 \), there is a point \( z_n \in S \) with \( |z_n| \geq n \).

Now, given \( \epsilon > 0 \), choose \( n_0 \geq 1 \) with \( 0 < \frac{1}{n_0} < \epsilon \), then
\[
|z_{n_0}| \geq n_0 > \frac{1}{\epsilon},
\]
and \( z_{n_0} \) is in the \( \epsilon \)-neighborhood \( \{ z : |z| > 1/\epsilon \} \) of the point of infinity. Thus, every neighborhood of the point at infinity contains at least one point in \( S \).

Conversely, if every neighborhood of the point at infinity contains at least one point in \( S \), then for each \( n \geq 1 \), we can choose a point \( z_n \in S \) with \( |z_n| \geq n \) (that is, \( z_n \) is in the \( 1/n \)-neighborhood of the point at infinity). Then \( S \) cannot be bounded, since \( z_n \in S \) for all \( n \geq 1 \), and \( \lim_{n \to \infty} |z_n| = +\infty \), and there is no \( M > 0 \) such that \( |z| \leq M \) for all \( z \in S \).

Question 3. [p 62, #1]

Use the results in Sec. 20 to find \( f'(z) \) when

(a) \( f(z) = 3z^2 - 2z + 4 \); \hspace{1cm} (b) \( f(z) = (1 - 4z^2)^3 \);

(c) \( f(z) = \frac{z - 1}{2z + 1} (z \neq -1/2) \); \hspace{1cm} (d) \( f(z) = \frac{(1 + z^2)^4}{z^2} (z \neq 0) \).

**Solution:**

(a) \( f'(z) = 6z - 2 \).
(b) \( f'(z) = 3(1 - 4z^2)^2(-8z) = -24z(1 - 4z^2)^2. \)

(c) \( f'(z) = \frac{1 \cdot (2z + 1) - (z - 1) \cdot 2}{(2z + 1)^2} = \frac{3}{(2z + 1)^2}, \) for \( z \neq -1/2. \)

(d) \( f'(z) = \frac{4(1 + z^2)^2 \cdot 2z \cdot z^2 - (1 + z^2)^4 \cdot 2z}{z^4} = \frac{2(1 + z^2)^3}{z^3} \cdot (3z^2 - 1), \) for \( z \neq 0. \)

Question 4. [p 62, #3]

Apply definition (3), Sec. 19, of derivative to give a direct proof that

\[ f'(z) = -\frac{1}{z^2} \quad \text{when} \quad f(z) = \frac{1}{z} \quad (z \neq 0). \]

Solution: If \( f(z) = \frac{1}{z} \) for \( z \neq 0, \) then

\[ f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = \lim_{h \to 0} \frac{\frac{1}{z + h} - \frac{1}{z}}{h} = \lim_{h \to 0} \frac{z - (z + h)}{hz(z + h)}, \]

that is,

\[ f'(z) = \lim_{h \to 0} \frac{-h}{hz(z + h)} = \lim_{h \to 0} \frac{-1}{z(z + h)} = -\frac{1}{z^2} \]

for \( z \neq 0. \)

Question 5. [p 63, #8 (b)]

Use the method in Example 2, Sec. 19, to show that \( f'(z) \) does not exist at any point \( z \) when \( f(z) = \text{Im} \ z. \)

Solution: Let \( f(z) = \text{Im}(z), \) then for \( z, h \in \mathbb{C}, \) with \( h \neq 0, \) we have

\[ \frac{\text{Im}(z + h) - \text{Im}(z)}{h} = \frac{\text{Im}(z) + \text{Im}(h) - \text{Im}(z)}{h} = \frac{\text{Im}(h)}{h}. \]

Now, if \( h \to 0 \) through real values, \( \text{Im}(h) = 0, \) and

\[ \lim_{h \to 0 \atop h \text{ real}} \frac{\text{Im}(z + h) - \text{Im}(z)}{h} = \lim_{h \to 0 \atop h \text{ real}} \frac{0}{h} = 0. \]

However, if \( h \to 0 \) through imaginary values, say \( h = it \) where \( t \in \mathbb{R} \) and \( t \to 0, \) then

\[ \frac{\text{Im}(h)}{h} = \frac{t}{it} = -i, \]

and

\[ \lim_{h \to 0 \atop h \text{ imag}} \frac{\text{Im}(z + h) - \text{Im}(z)}{h} = \lim_{h \to 0 \atop h \text{ imag}} (-i) = -i. \]

Therefore, (1) and (2) imply that

\[ \lim_{h \to 0} \frac{\text{Im}(z + h) - \text{Im}(z)}{h} \]

doesn’t exist, that is, \( f'(z) \) does not exist for any \( z \in \mathbb{C}. \)
Question 6. [p 71, #1]

Use the theorem in Sec. 21 to show that \( f'(z) \) does not exist at any point if

(a) \( f(z) = \overline{z} \);  
(b) \( f(z) = z - \overline{z} \);
(c) \( f(z) = 2x + ixy^2 \);
(d) \( f(z) = e^x e^{-iy} \).

Solution:

(a) If \( f(z) = \overline{z} = x - iy \), then \( u(x, y) = x \) and \( v(x, y) = -y \), so that

\[
\frac{\partial u}{\partial x} = 1 \neq 1 = \frac{\partial v}{\partial y},
\]

and the Cauchy-Riemann equations do not hold at any point \( z \in \mathbb{C} \). Therefore \( f'(z) \) does not exist for any \( z \in \mathbb{C} \).

(b) If \( f(z) = z - \overline{z} = 2iy \), then \( u(x, y) = 0 \) and \( v(x, y) = 2y \), so that

\[
\frac{\partial u}{\partial x} = 0 \neq 2 = \frac{\partial v}{\partial y},
\]

and again the Cauchy-Riemann equations do not hold at any point \( z \in \mathbb{C} \). Therefore \( f'(z) \) does not exist for any \( z \in \mathbb{C} \).

(c) If \( f(z) = 2x + ixy^2 \), then \( u(x, y) = 2x \) and \( v(x, y) = xy^2 \), so that

\[
\frac{\partial u}{\partial x} = 2, \quad \frac{\partial v}{\partial y} = 2xy
\]
\[
\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = y^2.
\]

Now the Cauchy-Riemann equations are

\[
2 = 2xy
\]
\[
0 = -y^2
\]

and these equations have no solutions \((x, y)\). Therefore, \( f'(z) \) does not exist at any point \( z \in \mathbb{C} \).

(d) If \( f(z) = e^x e^{-iy} = e^x \cos y - ie^x \sin y \), then

\[
\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial v}{\partial y} = -e^x \cos y
\]
\[
\frac{\partial u}{\partial y} = -e^x \sin y, \quad \frac{\partial v}{\partial x} = -e^x \sin y.
\]

Now the Cauchy-Riemann equations are

\[
2e^x \cos y = 0
\]
\[
2e^x \sin y = 0
\]

and since \( e^x \neq 0 \) for all \( x \in \mathbb{R} \), these equations are

\[
\cos y = 0 \quad \text{and} \quad \sin y = 0,
\]

but this is impossible since \( \cos^2 y + \sin^2 y = 1 \). Therefore, there are no solutions to the Cauchy-Riemann equations, and \( f'(z) \) does not exist for any \( z \in \mathbb{C} \).
Question 7. [p 71, #3]

From the results obtained in Secs. 21 and 22, determine where \( f'(z) \) exists and find its value when

(a) \( f(z) = \frac{1}{z} \);  
(b) \( f(z) = x^2 + iy^2 \);  
(c) \( f(z) = z \text{Im} z \);

Ans: (a) \( f'(z) = \frac{-1}{z^2} (z \neq 0) \); (b) \( f'(x + ix) = 2x \); (c) \( f'(0) = 0 \).

Solution:

(a) If \( f(z) = \frac{1}{z} \), then
\[
 f(z) = \frac{\overline{z}}{|z|^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2},
\]
so that
\[
 u(x, y) = \frac{x}{x^2 + y^2} \quad \text{and} \quad v(x, y) = -\frac{y}{x^2 + y^2}
\]
for \( z \neq 0 \). Now,
\[
 \frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y}
\]
and
\[
 \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2} = \frac{\partial v}{\partial x}.
\]
Since the partial derivatives are all continuous at each \( z \in \mathbb{C}, z \neq 0 \), and the Cauchy-Riemann equations hold at each \( z \in \mathbb{C}, z \neq 0 \), then \( f'(z) \) exists for all \( z \neq 0 \), and
\[
 f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{2ixy}{(x^2 + y^2)^2},
\]
that is,
\[
 f'(z) = \frac{x^2 - y^2 - 2ixy}{(x^2 + y^2)^2} = -\frac{|z|^2}{|z|^4} = -\frac{1}{z^2}
\]
for \( z \neq 0 \).

(b) If \( f(z) = x^2 + iy^2 \), then \( u(x, y) = x^2 \) and \( v(x, y) = y^2 \), and
\[
 \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2y \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0,
\]
so the Cauchy-Riemann equations hold only for those points \( z = x + iy \) with \( x = y \). Since all partial derivatives are continuous everywhere, \( f'(z) \) exists only for the points \( z = x + ix = x(1 + i), x \in \mathbb{R} \), and
\[
 f'(x + ix) = \frac{\partial u}{\partial x}(x, x) + i \frac{\partial v}{\partial x}(x, x) = 2x.
\]

(c) If \( f(z) = z \text{Im}(z) = (x + iy) \cdot y = xy + iy^2 \), then \( u(x, y) = xy \) and \( v(x, y) = y^2 \), so that
\[
 \frac{\partial u}{\partial x} = y, \quad \frac{\partial v}{\partial y} = 2y \quad \frac{\partial u}{\partial y} = x, \quad \frac{\partial v}{\partial x} = 0,
\]
and the Cauchy-Riemann equations hold if and only if
\[
 y = 2y \quad \text{and} \quad x = 0,
\]
that is, if and only if $x = y = 0$. Since all partial derivatives are continuous, then $f'(z)$ exists only for $z = 0$, and

$$f'(0) = \frac{\partial u}{\partial x}(0,0) + i \frac{\partial v}{\partial x}(0,0) = 0.$$

**Question 8.** [p 71, #4 (b)]

Use the theorem in Sec. 23 to show that the function

$$f(z) = \sqrt{r} e^{i\theta/2} \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

is differentiable in the indicated domain of definition, and then use expression (7) in that section to find $f'(z)$.

*Ans:* $f'(z) = \frac{1}{2f(z)}$.

**Solution:** If $f(z) = \sqrt{r} e^{i\theta/2}$, $r > 0$, $\alpha < \theta < \alpha + 2\pi$, then

$$f(z) = \sqrt{r} (\cos \theta/2 + i \sin \theta/2),$$

so that $u(r, \theta) = \sqrt{r} \cos \theta/2$ and $v(r, \theta) = \sqrt{r} \sin \theta/2$.

Now,

$$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}} \cos \theta/2 \quad \text{and} \quad \frac{\partial v}{\partial \theta} = \frac{\sqrt{r}}{2} \cos \theta/2,$$

so that

$$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}} \cos \theta/2 = \frac{1}{r} \frac{\partial v}{\partial \theta},$$

for $r > 0$, $\alpha < \theta < \alpha + 2\pi$.

Also,

$$\frac{\partial u}{\partial \theta} = -\frac{\sqrt{r}}{2} \sin \theta/2 \quad \text{and} \quad \frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}} \sin \theta/2,$$

so that

$$\frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{1}{2\sqrt{r}} \sin \theta/2 = -\frac{\partial v}{\partial r},$$

for $r > 0$, $\alpha < \theta < \alpha + 2\pi$.

The partial derivatives are all continuous for each $(r, \theta)$ with $r > 0$, $\alpha < \theta < \alpha + 2\pi$, and the Cauchy-Riemann equations hold for each such point, so that $f'(z)$ exists for all such points, and

$$f'(z) = e^{-i\theta} \left( \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right) = \frac{1}{2\sqrt{r}} (\cos \theta/2 + i \sin \theta/2) e^{-i\theta} = \frac{1}{2\sqrt{r}} e^{-i\theta/2} = \frac{1}{2f(z)},$$

for $r > 0$, $\alpha < \theta < \alpha + 2\pi$.

**Question 9.** [p 72, #5]

Show that when $f(z) = x^3 + i (1 - y)^3$, it is legitimate to write

$$f'(z) = u_x + i v_x = 3x^2$$

only when $z = i$.  

Solution: If \( f(z) = x^3 + i(1 - y)^3 \), then \( u(x,y) = x^3 \) and \( v(x,y) = (1 - y)^3 \), so that
\[
\frac{\partial u}{\partial x} = 3x^2, \quad \frac{\partial v}{\partial y} = -3(1 - y)^2 \\
\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0.
\]
The Cauchy-Riemann equations become
\[
3x^2 + 3(1 - y)^2 = 0
\]
and these hold only at the point \( z = (0,1) = i \). Since all partial derivatives are continuous everywhere, then \( f'(i) \) exists, and
\[
f'(i) = \frac{\partial u}{\partial x}(0,1) + i \frac{\partial v}{\partial x}(0,1) = 0.
\]

Question 10. [p 72, #10]

(a) Recall (Sec. 5) that if \( z = x + iy \) then
\[
x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}.
\]
By formally applying the chain rule in calculus to a function \( F(x,y) \) of two real variables, derive the expression
\[
\frac{\partial F}{\partial z} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial z} = \frac{1}{2} \left( \frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right).
\]
(b) Define the operator
\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),
\]
suggested by part (a), to show that if the first-order partial derivatives of the real and imaginary parts of a function \( f(z) = u(x,y) + iv(x,y) \) satisfy the Cauchy-Riemann equations, then
\[
\frac{\partial f}{\partial z} = \frac{1}{2} \left[ (u_x - v_y) + i (v_x + u_y) \right] = 0.
\]
Thus derive the complex form \( \frac{\partial f}{\partial z} = 0 \) of the Cauchy-Riemann equations.

Solution:

(a) Since \( z = x + iy \) and \( \overline{z} = x - iy \), then
\[
x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i},
\]
and
\[
\frac{\partial F}{\partial z} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial z} = \frac{1}{2} \left( \frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right).
\]
(b) Now, if \( f(z) = u(x,y) + iv(x,y) \) and the real-valued functions \( u \) and \( v \) satisfy the Cauchy-Riemann equations, then from part (a), we have
\[
\frac{\partial f}{\partial z} = \frac{1}{2} \frac{\partial}{\partial x} (u + iv) = \frac{1}{2} \frac{\partial u}{\partial x} + \frac{i}{2} \frac{\partial v}{\partial y} + i \frac{\partial u}{\partial y} + \frac{1}{2} \frac{\partial v}{\partial x},
\]
that is,
\[
\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0,
\]
since \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \) and \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \).