Question 1. [p 219, #1]

By differentiating the Maclaurin series representation

\[ \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1), \]

obtain the expansions

\[ \frac{1}{(1 - z)^2} = \sum_{n=0}^{\infty} (n+1)z^n \quad (|z| < 1) \]

and

\[ \frac{2}{(1 - z)^3} = \sum_{n=0}^{\infty} (n+1)(n+2)z^n \quad (|z| < 1). \]

**Solution:** Since \( \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \) for \( |z| < 1 \), differentiating the right-hand side term by term, we have

\[ \frac{1}{(1 - z)^2} = \frac{d}{dz} \left( \frac{1}{1 - z} \right) = \sum_{n=0}^{\infty} nz^{n-1} = \sum_{n=1}^{\infty} nz^{n-1} = \sum_{m=0}^{\infty} (m+1)z^m \]

for \( |z| < 1 \).

Differentiating again, we have

\[ \frac{2}{(1 - z)^3} = \frac{d}{dz} \left( \frac{1}{(1 - z)^2} \right) = \sum_{m=0}^{\infty} (m + 1) \cdot m z^{m-1} = \sum_{m=1}^{\infty} (m + 1) \cdot m z^{m-1} = \sum_{n=0}^{\infty} (n+2)(n+1)z^n \]

for \( |z| < 1 \).

Question 2. [p 219, #2]

By substituting \( 1/(1 - z) \) for \( z \) in the expansion

\[ \frac{1}{(1 - z)^2} = \sum_{n=0}^{\infty} (n+1)z^n \quad (|z| < 1), \]

found in Exercise 1, derive the Laurent series representation

\[ \frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n(n-1)}{(z-1)^n} \quad (1 < |z - 1| < \infty). \]

(Compare Example 2, Sec. 65.)
Solution: Since
\[
\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)z^n
\]
for \(|z| < 1\), replacing \(z\) by \(\frac{1}{1-z}\) in this expression, we have
\[
\frac{1}{\left(1 - \frac{1}{1-z}\right)^2} = \sum_{n=0}^{\infty} (n+1) \left(\frac{1}{1-z}\right)^n,
\]
that is,
\[
\frac{(1-z)^2}{z^2} = \sum_{n=0}^{\infty} (n+1) (1-z)^n
\]
for \(\left|\frac{1}{z-1}\right| < 1\), that is, for \(|z-1| > 1\).

Therefore,
\[
\frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(n+1)}{(1-z)^{n+2}} = \sum_{m=2}^{\infty} \frac{(m-1)}{(1-z)^m} = \sum_{m=2}^{\infty} \frac{(-1)^m(m-1)}{(z-1)^m}
\]
for \(1 < |z-1| < \infty\).

Question 3. [p 220, #3]

Find the Taylor series for the function
\[
\frac{1}{z} = \frac{1}{2 + (z-2)} = \frac{1}{2} \cdot \frac{1}{1 + (z-2)/2}
\]
about the point \(z_0 = 2\). Then by differentiating that series term by term, show that
\[
\frac{1}{z^2} = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n(n+1) \left(\frac{z-2}{2}\right)^n \quad (|z-2| < 2).
\]

Solution: We have
\[
\frac{1}{z} = \frac{1}{2 + (z-2)} = \frac{1}{2} \cdot \frac{1}{1 + (z-2)/2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n(z-2)^n}{2^n}
\]
for \(|z-2| < 2\).

Differentiating this expression term by term, we have
\[
-\frac{1}{z^2} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n n(z-2)^{n-1} \frac{2^n}{2^n} = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n n(z-2)^{n-1} \frac{2^n}{2^n} = \frac{1}{2} \sum_{m=0}^{\infty} (-1)^{m+1}(m+1)(z-2)^m,\]
for \(|z-2| < 2\), that is,
\[
\frac{1}{z^2} = \frac{1}{4} \sum_{m=0}^{\infty} (-1)^m(m+1)(z-2)^m
\]
for \(|z-2| < 2\).
Question 4. [p 215, Example 1.]

With the aid of series, prove that the function $f$ defined by means of the equations

$$f(z) = \begin{cases} \frac{e^z - 1}{z} & \text{when } z \neq 0, \\ 1 & \text{when } z = 0 \end{cases}$$

is entire.

**Solution:**

The Maclaurin series for $e^z$ is

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

and this series converges to $e^z$ for all $z$, $|z| < \infty$.

If $z \neq 0$, then

$$\frac{e^z - 1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots + \frac{z^n}{(n+1)!} + \cdots,$$

that is, the series converges to $\frac{e^z - 1}{z}$ for all $z \neq 0$, while if $z = 0$, the series converges to 1.

Therefore, if

$$f(z) = \begin{cases} \frac{e^z - 1}{z} & \text{when } z \neq 0, \\ 1 & \text{when } z = 0 \end{cases},$$

then

$$f(z) = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots + \frac{z^n}{(n+1)!} + \cdots$$

for all $z \in \mathbb{C}$, and $f$ is analytic at each $z \in \mathbb{C}$, that is, $f$ is entire.

Question 5. [p 225, #1]

Use multiplication of series to show that

$$\frac{e^z}{z(z^2 + 1)} = \frac{1}{z} + 1 - \frac{1}{2} z - \frac{5}{6} z^2 + \cdots \quad (0 < |z| < 1).$$

**Solution:** We have

$$e^z \cdot \frac{1}{z^2 + 1} = (1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots) \left(1 - z^2 + z^4 - z^6 + \cdots \right)$$

for $|z| < 1$, that is,

$$e^z \cdot \frac{1}{z^2 + 1} = 1 + z - \frac{1}{2} z^2 + \left(1 - \frac{1}{6} - 1 \right) z^3 + \cdots$$

for $|z| < 1$, that is,

$$e^z \cdot \frac{1}{z^2 + 1} = 1 + z - \frac{1}{2} z^2 - \frac{5}{6} z^3 + \cdots$$

for $|z| < 1$, and therefore,

$$\frac{e^z}{z(z^2 + 1)} = \frac{1}{z} + 1 - \frac{1}{2} z - \frac{5}{6} z^2 + \cdots$$

for $0 < |z| < 1$. 


Question 6. [p 225, #2]

By writing \( \csc z = 1/\sin z \) and then using division, show that

\[
\csc z = \frac{1}{z} + \frac{1}{3!} z + \left[ \frac{1}{(3!)^2} - \frac{1}{5!} \right] z^3 + \cdots \quad (0 < |z| < \pi).
\]

**Solution:** Since \( \sin z = 0 \) for \( z = 0, \pm \pi, \pm 2\pi, \ldots \), then \( \csc z \) is analytic for \( 0 < |z| < \pi \).

Now, for \( 0 < |z| < \pi \)

\[
z \csc z = \frac{z}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots} = \frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots}
\]

is analytic since the denominator doesn’t vanish for \( 0 < |z| < \pi \); and for \( z = 0 \), the series converges to 1.

Therefore the function

\[
g(z) = \begin{cases} 
  z \csc z & 0 < |z| < \pi \\
  1 & z = 0
\end{cases}
\]

is analytic in the entire disk \( |z| < \pi \), and so has a Maclaurin series expansion

\[
g(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots
\]

for \( |z| < \pi \).

Now, for \( 0 < |z| < \pi \),

\[
z = g(z) \cdot \sin z = (a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots)(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots),
\]

that is,

\[
z = a_0 z + a_1 z^2 + \left( a_2 - \frac{a_0}{3!} \right) z^3 + \left( a_3 - \frac{a_1}{3!} \right) z^4 + \left( a_4 - \frac{a_2}{3!} + \frac{a_0}{5!} \right) z^5 + \cdots
\]

for \( |z| < \pi \).

So we must have

\[
a_0 = 1 \\
a_1 = 0 \\
a_2 = \frac{1}{3!} \\
a_3 = 0 \\
a_4 = \frac{1}{(3!)^2} - \frac{1}{5!}
\]

and

\[
g(z) = 1 + \frac{1}{3!} z^2 + \left[ \frac{1}{(3!)^2} - \frac{1}{5!} \right] z^4 + \cdots
\]

for \( |z| < \pi \).

Therefore,

\[
\csc z = \frac{g(z)}{z} = \frac{1}{z} + \frac{1}{3!} z + \left[ \frac{1}{(3!)^2} - \frac{1}{5!} \right] z^3 + \cdots
\]

for \( 0 < |z| < \pi \).