Math 311 Spring 2014

Theory of Functions of a Complex Variable

Identity for Product of Sines

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In this note, as an application of the derivative and roots of unity, we give a proof of the following identity.

Theorem. For each positive integer $n \geq 2$, we have

$$\sin\tfrac{\pi}{n} \cdot \sin\tfrac{2\pi}{n} \cdot \sin\tfrac{3\pi}{n} \cdot \cdots \cdot \sin\tfrac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}.$$

Proof. For $n \geq 2$, the n^{th} roots of unity are solutions to the equation $z^n - 1 = 0$, and are given by

$$z_0 = 1,\ z_1 = e^{\frac{2\pi i}{n}},\ z_2 = e^{\frac{4\pi i}{n}},\ \cdots,\ z_{n-1} = e^{\frac{2(n-1)\pi i}{n}},$$

so that

$$z^n - 1 = (z-1)\left(z - e^{\frac{2\pi i}{n}}\right)\left(z - e^{\frac{4\pi i}{n}}\right)\cdots\left(z - e^{\frac{2(n-1)\pi i}{n}}\right).$$

Therefore,

$$\frac{z^n - 1}{z - 1} = \left(z - e^{\frac{2\pi i}{n}}\right)\left(z - e^{\frac{4\pi i}{n}}\right)\cdots\left(z - e^{\frac{2(n-1)\pi i}{n}}\right),$$

and letting $z \to 1$, we get

$$\frac{d}{dz}\left(z^n\right)\Big|_{z=1} = \left(1 - e^{\frac{2\pi i}{n}}\right)\left(1 - e^{\frac{4\pi i}{n}}\right)\cdots\left(1 - e^{\frac{2(n-1)\pi i}{n}}\right).$$

Thus,

$$n = \left(1 - e^{\frac{2\pi i}{n}}\right)\left(1 - e^{\frac{4\pi i}{n}}\right)\cdots\left(1 - e^{\frac{2(n-1)\pi i}{n}}\right), \qquad (*)$$

and taking the complex conjugate of $(*)$, we have

$$n = \left(1 - e^{-\frac{2\pi i}{n}}\right)\left(1 - e^{-\frac{4\pi i}{n}}\right)\cdots\left(1 - e^{-\frac{2(n-1)\pi i}{n}}\right). \qquad (**)$$

Now, for each $1 \leq k \leq n-1$, from Euler's formula and the double angle formula we have

$$\left(1 - e^{\frac{2k\pi i}{n}}\right)\left(1 - e^{-\frac{2k\pi i}{n}}\right) = 2 - \left(e^{\frac{2k\pi i}{n}} + e^{-\frac{2k\pi i}{n}}\right) = 2\left(1 - \cos\tfrac{2k\pi}{n}\right) = 2 \cdot 2\sin^2\tfrac{k\pi}{n},$$

and multiplying $(*)$ and $(**)$, we have

$$\begin{aligned} n^2 &= 2^{n-1}\left(1 - \cos\tfrac{2\pi}{n}\right)\left(1 - \cos\tfrac{4\pi}{n}\right)\cdots\left(1 - \cos\frac{2(n-1)\pi}{n}\right) \\ &= 2^{n-1} \cdot 2^{n-1} \cdot \sin^2\tfrac{\pi}{n} \cdot \sin^2\tfrac{2\pi}{n} \cdot \cdots \cdot \sin^2\tfrac{(n-1)\pi}{n}, \end{aligned}$$

taking the nonnegative square root of both sides of this equation, we get the desired result.

$\square$