Math 311 Spring 2014

Theory of Functions of a Complex Variable

Techniques for Finding Residues

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In the table below g and h are analytic at z_0 and f has an isolated singularity at z_0.

Function	Test	Type of Singularity	Residue at z_0
1. $f(z)$	$\lim_{z\to z_0}(z-z_0)f(z)=0$	removable	0
2. $\dfrac{g(z)}{h(z)}$	g and h have zeros of same order	removable	0
3. $f(z)$	$\lim_{z\to z_0}(z-z_0)f(z)$ exists and is $\neq 0$	simple pole	$\lim_{z\to z_0}(z-z_0)f(z)$
4. $\dfrac{g(z)}{h(z)}$	$g(z_0)\neq 0,\ h(z_0)=0,$ $h'(z_0)\neq 0$	simple pole	$\dfrac{g(z_0)}{h'(z_0)}$
5. $\dfrac{g(z)}{h(z)}$	g has zero of order k, h has zero of order $k+1$	simple pole	$(k+1)\dfrac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)}$
6. $\dfrac{g(z)}{h(z)}$	$g(z_0)\neq 0$ $h(z_0)=0=h'(z_0)$ $h''(z_0)\neq 0$	second-order pole	$2\,\dfrac{g'(z_0)}{h''(z_0)}-\dfrac{2}{3}\,\dfrac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}$
7. $\dfrac{g(z)}{(z-z_0)^2}$	$g(z_0)\neq 0$	second-order pole	$g'(z_0)$
8. $\dfrac{g(z)}{h(z)}$	$g(z_0)=0,\ g'(z_0)\neq 0,$ $h(z_0)=0,\ h'(z_0)=0,$ $h''(z_0)=0,\ h'''(z_0)\neq 0$	second-order pole	$3\,\dfrac{g''(z_0)}{h'''(z_0)}-\dfrac{3}{2}\,\dfrac{g'(z_0)h^{(4)}(z_0)}{[h'''(z_0)]^2}$
9. $f(z)$	k is the smallest integer such that $\lim_{z\to z_0}\phi(z)$ exists where $\phi(z)=(z-z_0)^k f(z)$	pole of order k	$\lim_{z\to z_0}\dfrac{\phi^{k-1}(z)}{(k-1)!}$
10. $\dfrac{g(z)}{h(z)}$	g has zero of order ℓ, h has zero of order $k+\ell$	pole of order k	$\lim_{z\to z_0}\dfrac{\phi^{(k-1)}(z)}{(k-1)!}$ where $\phi(z)=(z-z_0)^k\,\dfrac{g(z)}{h(z)}$

In the case of an essential singularity there are no simple formulas like the preceding ones, so we must rely on our ability to find the Laurent expansion. This is the case, for example, for the function

$$f(z) = e^{(z+1/z)} = e^z \cdot e^{1/z} = \left(1 + z + \frac{z^2}{2!} + \cdots\right)\left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots\right).$$

Multiplying the series and collecting the terms which multiply $1/z$, we get

$$\frac{1}{z}\left(1 + \frac{1}{2!} + \frac{1}{2!\,3!} + \frac{1}{3!\,4!} + \cdots\right)$$

and the residue at $z = 0$ is

$$\operatorname*{Res}_{z=0}(f) = 1 + \frac{1}{2!} + \frac{1}{2!\,3!} + \frac{1}{3!\,4!} + \cdots$$

Exercise. Try to sum the series explicitly.