Math 311 Spring 2014
Theory of Functions of a Complex Variable

Complex Variable Evaluation of Dirichlet's Integral

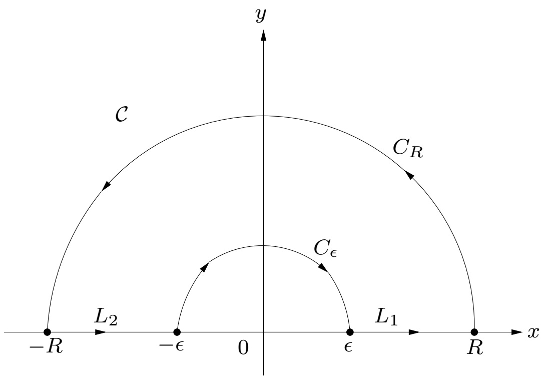
Department of Mathematical and Statistical Sciences
University of Alberta

In this note we use the theory of residues to evaluate Dirichlet's integral.

Theorem.

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Proof. We evaluate the integral using the Cauchy-Goursat theorem and integrating the function e^{iz}/z around the indented contour $\mathcal{C}$ shown below, where $0 < \epsilon < 1 < R$.



Since the function e^{iz}/z is analytic inside and on the contour $\mathcal{C}$, by the Cauchy-Goursat theorem

$$0 = \int_{\mathcal{C}} \frac{e^{iz}}{z}\,dz,$$

that is,

$$\int_{L_1} \frac{e^{iz}}{z}\,dz + \int_{C_R} \frac{e^{iz}}{z}\,dz + \int_{L_2} \frac{e^{iz}}{z}\,dz + \int_{C_\epsilon} \frac{e^{iz}}{z}\,dz = 0. \qquad (*)$$

- On L_1: $z = x$, where $\epsilon \le x \le R$, and

$$\int_{L_1} \frac{e^{iz}}{z}\,dz = \int_\epsilon^R \frac{e^{ix}}{x}\,dx.$$

- On L_2: $z = x$, where $-R \le x \le -\epsilon$, and

$$\int_{L_2} \frac{e^{iz}}{z}\,dz = \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\,dx = -\int_\epsilon^R \frac{e^{-ix}}{x}\,dx.$$

Therefore,

$$\int_{L_1} \frac{e^{iz}}{z}\,dz + \int_{L_2} \frac{e^{iz}}{z}\,dz = 2i\int_\epsilon^R \frac{\sin x}{x}\,dx,$$

and

$$\lim_{\substack{R\to\infty\\ \epsilon\to 0}} \int_{L_1} \frac{e^{iz}}{z}\,dz + \lim_{\substack{R\to\infty\\ \epsilon\to 0}} \int_{L_2} \frac{e^{iz}}{z}\,dz = 2i\int_0^\infty \frac{\sin x}{x}\,dx.$$

• On $C_R:\ z = Re^{i\theta}$, where $0 \le \theta \le \pi$, and

$$\int_{C_R} \frac{e^{iz}}{z}\,dz = \int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta}\,d\theta = i\int_0^\pi e^{iR\cos\theta}\,e^{-R\sin\theta}\,d\theta.$$

From Jordan's inequality, we have

$$\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| \le \int_0^\pi e^{-R\sin\theta}\,d\theta \le \frac{\pi}{R}\left(1 - e^{-R}\right).$$

and therefore,

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z}\,dz = 0.$$

• On $C_\epsilon:\ z = \epsilon e^{i\theta}$, where $0 \le \theta \le \pi$, and the Laurent series expansion of e^{iz}/z about $z = 0$ is

$$\frac{e^{iz}}{z} = \frac{1}{z} + \sum_{n=1}^\infty \frac{(iz)^{n-1}}{n!}$$

valid for $0 < |z| < \infty$, and e^{iz}/z has a simple pole at $z = 0$ with residue 1. The function

$$g(z) = \sum_{n=1}^\infty \frac{(iz)^{n-1}}{n!}$$

for $z \in \mathbb{C}$ is an entire function and is continuous and hence bounded on the disk $|z| \le 1$, so there is an $M > 0$ such that $|g(z)| \le M$ for $|z| \le 1$. Therefore,

$$\int_{C_\epsilon} \frac{e^{iz}}{z}\,dz = \int_{C_\epsilon} \frac{1}{z}\,dz + \int_{C_\epsilon} g(z)\,dz,$$

and since $0 < \epsilon < 1$, then

$$\left|\int_{C_\epsilon} g(z)\,dz\right| \le M\,\pi\,\epsilon$$

while

$$\int_{C_\epsilon} \frac{1}{z}\,dz = \int_\pi^0 \frac{i\epsilon\,e^{i\theta}}{\epsilon\,e^{i\theta}}\,d\theta = -\int_0^\pi i\,d\theta = -\pi i.$$

Therefore,

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{iz}}{z}\,dz = \lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{1}{z}\,dz + \lim_{\epsilon\to 0} \int_{C_\epsilon} g(z)\,dz = -\pi i.$$

Letting $\epsilon \to 0$ and $R \to \infty$ in $(*)$, we get

$$2i \int_0^\infty \frac{\sin x}{x}\,dx - \pi i = 0,$$

that is,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

It is instructive to compare the complex variable proof of this theorem with a proof using real variable techniques.