



Math 311 Spring 2014  
Theory of Functions of a Complex Variable  
Real Variable Evaluation of Dirichlet's Integral

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In this note we will study Dirichlet's integral, and give an elementary real variable method for evaluating it.

**Theorem.**

- (a)  $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ .  
(b)  $\int_0^\infty \left| \frac{\sin x}{x} \right| dx = +\infty$ .

**Proof.** The following proof is outlined on page 397, Miscellaneous Exercise 39, in G. H. Hardy's *A Course of Pure Mathematics*.

(a) For each  $n \geq 1$ , define

$$u_n = \int_0^{\pi/2} \sin 2nx \cot x dx \quad \text{and} \quad v_n = \int_0^{\pi/2} \frac{\sin 2nx}{x} dx,$$

and consider the difference  $u_{n+1} - u_n$ ,

$$\begin{aligned} u_{n+1} - u_n &= \int_0^{\pi/2} [\sin 2(n+1)x - \sin 2nx] \cot x dx \\ &= 2 \int_0^{\pi/2} \sin x \cos(2n+1)x \cot x dx \\ &= 2 \int_0^{\pi/2} \cos x \cos(2n+1)x dx \\ &= \int_0^{\pi/2} [\cos(2n+2)x + \cos 2nx] dx \\ &= \frac{1}{2(n+1)} \sin 2(n+1)x \Big|_0^{\pi/2} + \frac{1}{2n} \sin 2nx \Big|_0^{\pi/2} \\ &= 0; \end{aligned}$$

therefore,  $u_{n+1} - u_n = 0$  for all  $n \geq 1$ . Thus,  $u_n$  is constant for  $n \geq 1$ , and

$$\begin{aligned} u_1 &= \int_0^{\pi/2} \sin 2x \cot x dx = 2 \int_0^{\pi/2} \cos^2 x dx \\ &= \int_0^{\pi/2} (1 + \cos 2x) dx = \frac{\pi}{2} + \frac{\sin 2x}{2} \Big|_0^{\pi/2} \\ &= \frac{\pi}{2}, \end{aligned}$$

and therefore  $u_n = u_1 = \pi/2$  for all  $n \geq 1$ .

Making the substitution  $t = 2nx$  in the integral  $\int_0^{\pi/2} (\sin 2nx/x) dx$ , then

$$v_n = \int_0^{\pi/2} \frac{\sin 2nx}{x} dx = \int_0^{n\pi} \frac{\sin t}{t} dt, \quad (1)$$

and integrating by parts, we have

$$\begin{aligned} u_n - v_n &= \int_0^{\pi/2} \left( \cot x - \frac{1}{x} \right) \sin 2nx dx \\ &= -\frac{\cos 2nx}{2n} \left( \cot x - \frac{1}{x} \right) \Big|_0^{\pi/2} \\ &\quad - \frac{1}{2n} \int_0^{\pi/2} \left( \csc^2 x - \frac{1}{x^2} \right) \cos 2nx dx, \end{aligned}$$

and therefore

$$\begin{aligned} u_n - v_n &= \frac{(-1)^n}{n\pi} + \lim_{p \rightarrow 0^+} \frac{\cos 2np}{2n} \left( \cot p - \frac{1}{p} \right) \\ &\quad - \frac{1}{2n} \int_0^{\pi/2} \left( \csc^2 x - \frac{1}{x^2} \right) \cos 2nx dx. \end{aligned} \quad (2)$$

Using L'Hospital's rule, we have

$$\lim_{p \rightarrow 0^+} \left( \cot p - \frac{1}{p} \right) = 0, \quad \text{and} \quad \lim_{x \rightarrow 0^+} \left( \csc^2 x - \frac{1}{x^2} \right) = \frac{1}{3}.$$

Therefore, we may redefine the integrand on the right-hand side of (2) to be continuous on the interval  $[0, \pi/2]$ , and hence bounded there. Thus, there exists a constant  $M > 0$  such that

$$\begin{aligned} \left| \int_0^{\pi/2} \left( \csc^2 x - \frac{1}{x^2} \right) \cos 2nx dx \right| &\leq \int_0^{\pi/2} \left| \left( \csc^2 x - \frac{1}{x^2} \right) \cos 2nx \right| dx \\ &\leq \frac{\pi M}{2}. \end{aligned}$$

Letting  $n \rightarrow \infty$  in (1) and (2), we obtain

$$\lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} (v_n - u_n + u_n) = \lim_{n \rightarrow \infty} (v_n - u_n) + \lim_{n \rightarrow \infty} u_n,$$

so that

$$\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}.$$

(b) For each positive integer  $n$ , we have

$$\begin{aligned} \int_0^{(n+1)\pi} \left| \frac{\sin x}{x} \right| dx &= \sum_{k=0}^n \int_{k\pi}^{(k+1)\pi} \left| \frac{\sin x}{x} \right| dx \\ &= \sum_{k=0}^n \int_0^{\pi} \left| \frac{\sin(t+k\pi)}{t+k\pi} \right| dt = \sum_{k=0}^n \int_0^{\pi} \left| \frac{(-1)^k \sin t}{t+k\pi} \right| dt \\ &= \sum_{k=0}^n \int_0^{\pi} \frac{|\sin t|}{t+k\pi} dt = \sum_{k=0}^n \int_0^{\pi} \frac{\sin t}{t+k\pi} dt, \end{aligned}$$

where the last equality follows since  $\sin t \geq 0$  for  $0 \leq t \leq \pi$ . For  $0 < t < \pi$ , we have  $1/(t + k\pi) > 1/[\pi(k + 1)]$ , so that

$$\int_0^{(n+1)\pi} \left| \frac{\sin x}{x} \right| dx > \sum_{k=0}^n \frac{1}{\pi(k+1)} \int_0^\pi \sin t dt = \frac{2}{\pi} \sum_{k=0}^n \frac{1}{k+1}.$$

Now, for  $x \geq 1$  we have

$$\log x = \int_1^x \frac{1}{t} dt \leq \int_1^x 1 dt = x - 1,$$

and replacing  $x$  by  $(k + 2)/(k + 1)$ , we have

$$\log \left( \frac{k+2}{k+1} \right) \leq \frac{1}{k+1}$$

for all  $k \geq 0$ . Therefore,

$$\sum_{k=0}^n \frac{1}{k+1} \geq \sum_{k=0}^n (\log(k+2) - \log(k+1)) = \log(n+2),$$

so

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k+1} \geq \lim_{n \rightarrow \infty} \log(n+2) = +\infty,$$

and this implies that

$$\int_0^\infty \left| \frac{\sin x}{x} \right| dx = +\infty.$$

□

It is instructive to compare the real variable proof of part (a) of this theorem with a proof using Cauchy's residue theorem.