Math 311 Spring 2014
Theory of Functions of a Complex Variable

Manipulating of Power Series

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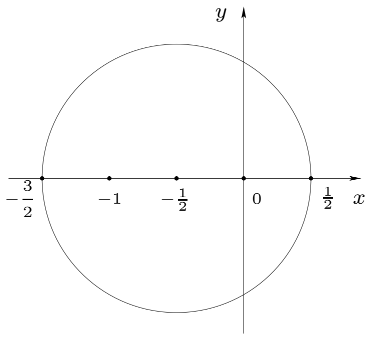
Manipulating Power Series

Just as with real power series, some care has to be used when manipulating complex power series, as the following example shows.

The power series

$$S(z) = \sum_{n=0}^{\infty} \left(z + \tfrac{1}{2}\right)^n$$

converges for $\left|z + \frac{1}{2}\right| < 1$.
In paricular, it **converges** for $z = -1$.



If we expand the powers of $\left(z + \frac{1}{2}\right)$ by the binomial theorem, we have

$$\left(z + \tfrac{1}{2}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{z^k}{2^{n-k}},$$

so that

$$\sum_{n=0}^{\infty} \left(z + \tfrac{1}{2}\right)^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{z^k}{2^{n-k}}. \qquad (*)$$

If we could legitimately interchange the order of summation, we would have

$$S(z) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} \frac{z^k}{2^{n-k}},$$

that is,

$$S(z) = \sum_{k=0}^{\infty} \left\{ \sum_{n=k}^{\infty} \binom{n}{k} \frac{1}{2^{n-k}} \right\} z^k,$$

and reindexing the inner sum by letting $m = n - k$, then

$$S(z) = \sum_{k=0}^{\infty} \left\{ \sum_{m=0}^{\infty} \binom{k+m}{k} \frac{1}{2^m} \right\} z^k.$$

Now, with $\alpha = -k-1$, we have

$$\begin{aligned}\binom{\alpha}{m} &= \frac{(-k-1)(-k-2)\cdots(-k-m)}{m!} \\ &= (-1)^m \frac{(k+m)(k+m-1)\cdots(k+1)}{m!} \\ &= (-1)^m \frac{(k+m)!}{k!\,m!} = (-1)^m \binom{k+m}{k},\end{aligned}$$

so that

$$(-1)^m \binom{-k-1}{m} = \binom{k+m}{k},$$

and the inner sum is just the binomial series for

$$\frac{1}{\left(1-\frac{1}{2}\right)^{k+1}} = \sum_{m=0}^{\infty} \binom{-k-1}{k} \left(-\frac{1}{2}\right)^m = \sum_{m=0}^{\infty} \binom{k+m}{k} \frac{1}{2^m}.$$

Therefore,

$$S(z) = \sum_{k=0}^{\infty} 2^{k+1} z^k, \tag{$**$}$$

and the series on the right in $(**)$ converges for $|2z| < 1$, that is, for $|z| < \frac{1}{2}$, and diverges for $|z| > \frac{1}{2}$.

In particular, the series on the right **diverges** for $z = -1$.