Math 311 Spring 2014

Theory of Functions of a Complex Variable

Limits in the Euclidean Plane and in the Complex Plane

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In this note we show that limits in the complex plane $\mathbb{C}$ are exactly the same as limits in the plane $\mathbb{R}^2$ equipped with the Euclidean norm, and we prove the following theorem.

Theorem. Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$, and $w_0 = u_0 + iv_0$, then

$$\lim_{z\to z_0} f(z) = w_0$$

if and only if

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y) = u_0 \qquad \text{and} \qquad \lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y) = v_0.$$

Proof. Suppose that $\lim\limits_{z\to z_0} f(z) = w_0$, then given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon$$

whenever $0 < |z - z_0| < \delta$, and therefore

$$|u(x,y) - u_0| \le \sqrt{(u(x,y)-u_0)^2 + (v(x,y)-v_0)^2} = |f(z) - w_0| < \epsilon$$

and

$$|v(x,y) - v_0| \le \sqrt{(u(x,y)-u_0)^2 + (v(x,y)-v_0)^2} = |f(z) - w_0| < \epsilon$$

whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} = |z - z_0| < \delta$, that is,

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y) = u_0 \qquad \text{and} \qquad \lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y) = v_0.$$

Conversely, suppose that

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y) = u_0 \qquad \text{and} \qquad \lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y) = v_0,$$

and let $\epsilon > 0$. Choose $\delta > 0$ so that

$$|u(x,y) - u_0| < \frac{\epsilon}{\sqrt{2}} \qquad \text{and} \qquad |v(x,y) - v_0| < \frac{\epsilon}{\sqrt{2}}$$

whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

If $|z - z_0| < \delta$, then

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = |z - z_0| < \delta,$$

implies that

$$|f(z) - w_0| = \sqrt{(u(x, y) - u_0)^2 + v(x, y) - v_0)^2} < \sqrt{\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}} = \epsilon,$$

and therefore

$$\lim_{z \to z_0} f(z) = w_0.$$

□