Math 311 Spring 2014
Theory of Functions of a Complex Variable

Lacunary Series

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In this note we will study the power series

$$f(z) = \sum_{n=0}^{\infty} z^{n!}$$

called a **lacunary series** or a **series with gaps**. We will show that the series is unbounded in every neighborhood of every point on the boundary of its circle of convergence (so the boundary is called a **natural boundary** of the series).

Lemma 1. The series

$$f(z) = \sum_{n=0}^{\infty} z^{n!}$$

is absolutely convergent for any $z \in \mathbb{C}$ with $|z| < 1$.

Proof. To see this, we compare it to the geometric series, $\sum_{n=0}^{\infty} z^n$.

If we let $a_n = z^{n!}$ and $b_n = z_n$, then

$$\frac{a_n}{b_n} = \frac{z^{n!}}{z^n} = z^{n!-n} = z^{n[(n-1)!-1]},$$

and if $|z| < 1$, then

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|z^{n[(n-1)!-1]}\right| = 0.$$

Thus, given any $\epsilon > 0$, there is an integer N_0 such that

$$\frac{|a_n|}{|b_n|} < \epsilon$$

for all $n \geq N_0$, and therefore,

$$|a_n| < \epsilon |b_n|$$

for all $n \geq N_0$. Hence,

$$\sum_{n=N_0}^{\infty} \left|z^{n!}\right| \leq \epsilon \sum_{n=N_0}^{\infty} |z^n| < \infty,$$

and the series $\sum_{n=0}^{\infty} z^{n!}$ converges absolutely if $|z| < 1$.

$\square$

Lemma 2. The series

$$f(z) = \sum_{n=0}^{\infty} z^{n!}$$

diverges if $z = 1$.

Proof. For $z = 1$, the Nth partial sum of the series is

$$S_N = \sum_{n=0}^{N} 1 = \underbrace{1 + 1 + \cdots + 1}_{N+1} = N + 1,$$

and

$$\lim_{N\to\infty} S_N = +\infty,$$

so the series diverges for $z = 1$.

$\square$

Lemma 3. The circle of convergence for the series

$$f(z) = \sum_{n=0}^{\infty} z^{n!}$$

is $|z| = 1$.

Proof. The series converges absolutely if $|z| < 1$ and diverges if $|z| > 1$, so the radius of convergence is $R = 1$.

$\square$

Lemma 4. Let ω be a point on the unit circle, $\omega = \cos\frac{2p\pi}{q} + i\sin\frac{2p\pi}{q}$ where p and q are positive integers, then

$$f(z) = \sum_{n=0}^{\infty} z^{n!}$$

is unbounded in a neighborhood of ω.

Proof. Let ω be a point on the unit circle, and let $z = r\omega$, where $0 < r < 1$, then

$$\sum_{n=0}^{\infty} z^{n!} = \sum_{n=0}^{q-1} z^{n!} + \sum_{n=q}^{\infty} r^{n!}$$

since $\omega^q = 1$ so that $\omega^{n!} = 1$ for $n \geq q$.

Therefore,

$$\begin{aligned}\left|\sum_{n=0}^{\infty} z^{n!}\right| &\geq \sum_{n=q}^{\infty} r^{n!} - \sum_{n=0}^{q-1} |z|^{n!} \\ &= \sum_{n=q}^{\infty} r^{n!} - \sum_{n=0}^{q-1} r^{n!},\end{aligned}$$

since $|z| = |r\omega| = r|\omega| = r$, so that

$$\left|\sum_{n=0}^{\infty} z^{n!}\right| \geq \sum_{n=q}^{\infty} r^{n!} - \sum_{n=0}^{q-1} r^{n!}.$$

Now let N be an arbitrary positive integer, and let $k = 2q + N$, then

$$\left|\sum_{n=0}^{\infty} z^{n!}\right| > \sum_{n=q}^{k} r^{n!} - \sum_{n=0}^{q-1} r^{n!} > (k - q + 1)r^{k!} - (q - 1)$$

since $0 < r < 1$ and $q \leq n \leq k$ imply that $r^{n!} > r^{k!}$, and so

$$\left|\sum_{n=0}^{\infty} z^{n!}\right| > (k - q + 1)r^{k!} - (q - 1).$$

Now,

$$(k - q + 1)r^{k!} - (q - 1) \longrightarrow k - 2(q - 1) = N + 2$$

as $r \to 1^-$. Therefore,

$$\left|\sum_{n=0}^{\infty} z^{n!}\right| > N$$

if r is close enough to 1, and so $\left|\sum_{n=0}^{\infty} z^{n!}\right|$ is unbounded in a neighborhood of ω.

□

Theorem. The function

$$f(z) = \sum_{n=0}^{\infty} z^{n!}$$

is unbounded in every neighborhood of every point on the boundary of its circle of convergence; that is, on the **natural boundary** of the series.

Proof. The unit circle is the set of points

$$z = \cos 2\pi t + i \sin 2\pi t$$

for $0 \leq t \leq 1$, and since any real number t has a rational approximation p/q as close as we please, if p is the largest integer contained in qt, then

$$p \leq qt < p + 1,$$

so that

$$\frac{p}{q} \leq t < \frac{p}{q} + \frac{1}{q}.$$

Therefore, any neighborhood of the point

$$z_t = \cos 2\pi t + i \sin 2\pi t$$

contains a point

$$\omega = \cos \tfrac{2\pi p}{q} + i \sin \tfrac{2\pi p}{q},$$

and therefore, given any $N > 0$ it contains a point z where

$$\left|\sum_{n=0}^{\infty} z^{n!}\right| > N.$$

Thus, $\left|\sum_{n=0}^{\infty} z^{n!}\right|$ is unbounded in every neighborhood of every point on the boundary of its circle of convergence, so the power series canot be continued across *any* point of this circumference.

□