Math 311 Spring 2014
Theory of Functions of a Complex Variable

Jordan's Lemma

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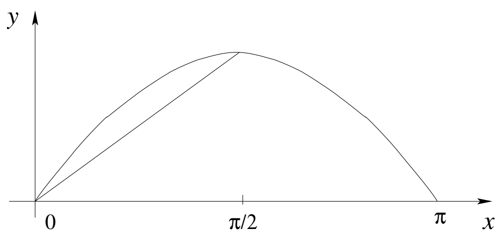
In this note we will prove *Jordan's Inequality*:

Lemma. If $R > 0$, then

$$\int_0^{\pi} e^{-R\sin\theta}\, d\theta \le \frac{\pi}{R}\left(1 - e^{-R}\right).$$

Proof. From the figure



it is clear that

$$0 < \frac{2}{\pi}\,\theta < \sin\theta$$

for $0 < \theta < \frac{\pi}{2}$.

If $R > 0$, then

$$\frac{2R}{\pi}\,\theta < R\sin\theta,$$

so that

$$-R\sin\theta < -\frac{2R}{\pi}\,\theta,$$

and since the exponential function is monotone increasing, we have

$$e^{-R\sin\theta} < e^{-\frac{2R\theta}{\pi}}$$

for $0 < \theta < \frac{\pi}{2}$.

Therefore,

$$\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta \le \int_0^{\frac{\pi}{2}} e^{-\frac{2R\theta}{\pi}}\, d\theta = -\frac{\pi}{2R} e^{-\frac{2R\theta}{\pi}}\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2R}\left(1 - e^{-R}\right),$$

and

$$\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta \le \frac{\pi}{2R}\left(1 - e^{-R}\right). \qquad (*)$$

Making the substitution $\phi = \pi - \theta$ in the integral of $e^{-R\sin\theta}$ over the interval $\frac{\pi}{2} < \theta < \pi$, we have

$$\int_{\frac{\pi}{2}}^{\pi} e^{-R\sin\theta}\, d\theta = -\int_{\frac{\pi}{2}}^{0} e^{-R\sin(\pi-\phi)}\, d\phi = \int_{0}^{\frac{\pi}{2}} e^{-R\sin\phi}\, d\phi,$$

so that

$$\int_{\frac{\pi}{2}}^{\pi} e^{-R\sin\theta}\, d\theta = \int_{0}^{\frac{\pi}{2}} e^{-R\sin\phi}\, d\phi,$$

and

$$\int_{\frac{\pi}{2}}^{\pi} e^{-R\sin\theta}\, d\theta \le \frac{\pi}{2R}\left(1 - e^{-R}\right). \qquad (**)$$

From $(*)$ and $(**)$ we have

$$\int_{0}^{\pi} e^{-R\sin\theta}\, d\theta = \int_{0}^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta + \int_{\frac{\pi}{2}}^{\pi} e^{-R\sin\theta}\, d\theta \le \frac{\pi}{2R}\left(1 - e^{-R}\right) + \frac{\pi}{2R}\left(1 - e^{-R}\right),$$

so that

$$\int_{0}^{\pi} e^{-R\sin\theta}\, d\theta \le \frac{\pi}{R}\left(1 - e^{-R}\right).$$

□

Now we can state:

Jordan's Lemma:

(a) Let $f(z)$ be analytic at all points in the upper half plane $y \ge 0$ that are exterior to a circle $|z| = R_0$.

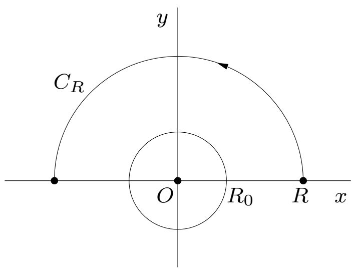
(b) Let C_R denote the semicircle $z = R\, e^{i\theta}$, $0 \le \theta \le \pi$, where $R > R_0$.

(c) Suppose that for all points z on C_R, there is a positive constant M_R such that

$$|f(z)| \le M_R \qquad \text{and} \qquad \lim_{R\to\infty} M_R = 0.$$

Then, for every positive constant a,

$$\lim_{R\to\infty} \int_{C_R} f(z)\, e^{iaz}\, dz = 0.$$



Proof. If statements (a), (b), and (c) are true, then using the parametrization of the semicircle, we have

$$\int_{C_R} f(z)\, e^{iaz}\, dz = \int_{0}^{\pi} f\left(Re^{i\theta}\right) e^{iaRe^{i\theta}} Rie^{i\theta}\, d\theta.$$

Now,

$$\left|f\left(Re^{i\theta}\right)\right| \le M_R \qquad \text{and} \qquad \left|e^{iaRe^{i\theta}}\right| \le e^{-aR\sin\theta},$$

and from Jordan's inequality we have

$$\left|\int_{C_R} f(z)\, e^{iaz}\, dz\right| \leq M_R R \int_0^\pi e^{-aR\sin\theta}\, d\theta < \frac{M_R \pi}{a}\left(1 - e^{-aR}\right),$$

so that

$$\lim_{R\to\infty} \int_{C_R} f(z)\, e^{iaz}\, dz = 0.$$

since $M_R \to 0$ as $R \to \infty$.

$\square$