Math 311 Spring 2014
Theory of Functions of a Complex Variable
The Extended Complex Plane

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The Extended Complex Plane

To discuss the situation where a function $f(z)$ becomes infinite as the variable z approaches a given point z_0, we introduce the **extended complex plane**

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

Also, to discuss continuity properties of functions assuming the value infinity, we introduce a distance function or metric on $\overline{\mathbb{C}}$.

To get a concrete realization of $\overline{\mathbb{C}}$, we represent it as the unit sphere in $\mathbb{R}^3$,

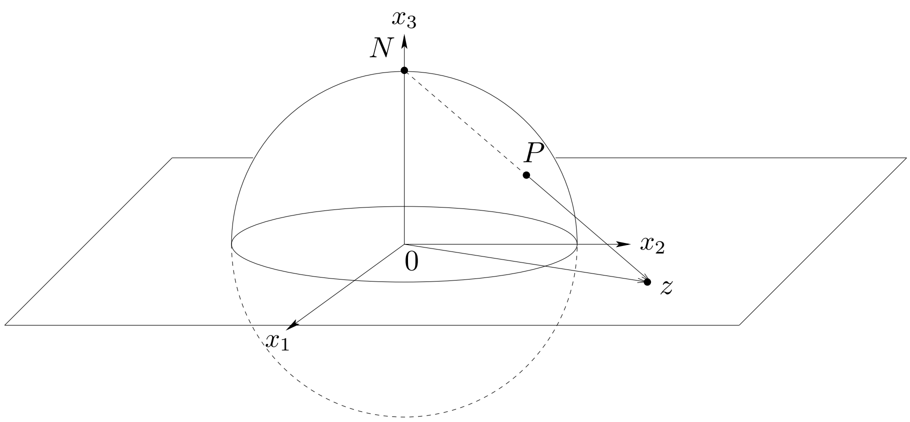
$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,\big|\, x_1^2 + x_2^2 + x_3^2 = 1\}$$

called the **Riemann sphere**.

Let $N = (0, 0, 1)$ be the north pole on S^2, and identify $\mathbb{C}$ with the subset of $\mathbb{R}^3$ given by

$$\{(x_1, x_2, 0) \in \mathbb{R}^3 \,\big|\, x_1, x_2 \in \mathbb{R}\},$$

and note that $\mathbb{C}$ intersects S^2 along the equator, as in the figure below.



For each point z in $\mathbb{C}$, we consider the vectors $\overrightarrow{z}$ and $\overrightarrow{Nz}$ and take $P \neq N$ to be the unique point of intersection of $\overrightarrow{Nz}$ with the sphere $\overrightarrow{x} \cdot \overrightarrow{x} = 1$ as shown.

The directed line segment $\overrightarrow{Nz}$ has equation

$$(x_1, x_2, x_3) = (0, 0, 1) + t(x, y, -1)$$

for $0 \le t \le 1$; that is, $x_1 = tx$, $x_2 = ty$, and $x_3 = 1 - t$, for $0 \le t \le 1$, and intersects the sphere when

$$x_1^2 + x_2^2 + x_3^2 = t^2(x^2 + y^2) + (1 - t)^2 = 1.$$

Setting $r^2 = |z|^2 = x^2 + y^2$ and solving for t, we have

$$t = 0 \qquad \text{and} \qquad t = \frac{2}{|z|^2 + 1}.$$

The solution $t = 0$ corresponds to $N = (0, 0, 1)$, while the solution $t = \dfrac{2}{|z|^2 + 1}$ corresponds to the point $P = (x_1, x_2, x_3)$ on the sphere, where

$$x_1 = \frac{2x}{|z|^2 + 1}, \qquad x_2 = \frac{2y}{|z|^2 + 1}, \qquad x_3 = 1 - \frac{2}{|z|^2 + 1} = \frac{|z|^2 - 1}{|z|^2 + 1};$$

that is,

$$x_1 = \frac{z + \overline{z}}{|z|^2 + 1}, \qquad x_2 = \frac{-i(z - \overline{z})}{|z|^2 + 1}, \qquad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}. \tag{$*$}$$

Note that if $|z| < 1$, then the point P on the sphere is in the southern hemisphere, while if $|z| > 1$, then the point P on the sphere is in the northern hemisphere. Also, for $|z| = 1$, we have $P = z$.

It is clear from the figure that P approaches N as $|z| \to \infty$, and so we identify N and the point at infinity in $\overline{\mathbb{C}}$. Therefore, $\overline{\mathbb{C}}$ is represented as the sphere S^2, this correspondence between points of S^2 and $\overline{\mathbb{C}}$ is called the **stereographic projection**.

Note that if we are given the point $P = (x_1, x_2, x_3)$, $P \neq N$, and we want to find the corresponding value of z, the we can set $x_3 = 1 - t$ in the parametric equations of the segment $\overrightarrow{Nz}$ and solve for x and y to get

$$x = \frac{x_1}{1 - x_3} \qquad \text{and} \qquad y = \frac{x_2}{1 - x_3};$$

that is,

$$z = \frac{x_1 + ix_2}{1 - x_3}.$$

We define a distance function or metric on $\overline{\mathbb{C}}$ as follows: for points z and z' in the extended plane, we define the distance from z to z', denoted by $d(z, z')$, to be the distance between the corresponding points P and P' in $\mathbb{R}^3$, this is called the **chordal metric**.

If $P = (x_1, x_2, x_3)$ and $P' = (x_1', x_2', x_3')$, then

$$d(z, z') = \left[(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2\right]^{1/2},$$

and since P and P' are on the sphere S^2, then

$$d(z, z')^2 = 2 - 2(x_1x_1' + x_2x_2' + x_3x_3').$$

From $(*)$ we have

$$d(z, z') = \frac{2|z - z'|}{\left[(1 + |z|^2)(1 + |z'|^2)\right]^{1/2}}$$

for $z, z' \in \mathbb{C}$, while

$$d(z, \infty) = \frac{2}{(1 + |z|^2)^{1/2}}$$

for $z \in \mathbb{C}$.

Now that we have a metric defined on $\overline{\mathbb{C}}$, we can use this to define an ϵ-neighborhood of ∞, the point at infinity.

Given $\epsilon > 0$, an **epsilon neighborhood** of ∞ is the set

$$B_\epsilon(\infty) = \{z \in \overline{\mathbb{C}} \,|\, d(z, \infty) < \epsilon\}$$

and since $d(\infty, \infty) = 0$, then

$$B_\epsilon(\infty) = \left\{ z \in \mathbb{C} \,\middle|\, \frac{2}{(|z|^2+1)^{1/2}} < \epsilon \right\},$$

and this is then the same as a deleted ϵ-neighborhood of ∞.

Now note that

$$\frac{2}{(|z|^2+1)^{1/2}} < \epsilon$$

if and only if

$$|z|^2 > \frac{4}{\epsilon^2} - 1,$$

that is,

$$|z| > \sqrt{\frac{4}{\epsilon^2} - 1},$$

and thus,

$$B_\epsilon(\infty) = \left\{ z \in \mathbb{C} \,\middle|\, |z| > \sqrt{\frac{4}{\epsilon^2} - 1} \right\}.$$

Therefore, when we talk about limits involving the point at infinity, we are not too far off the mark by replacing this by

$$B_\epsilon(\infty) = \left\{ z \in \mathbb{C} \,\middle|\, |z| > \frac{1}{\epsilon} \right\}.$$