Math 311 Spring 2014

Theory of Functions of a Complex Variable

The Exponential Function as a Limit

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In this note, using a series of lemma, we show that

$$e^z = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n$$

for each $z \in \mathbb{C}$. We assume that the Maclaurin series expansion $e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!}$, valid for all $z \in \mathbb{C}$, is known.

Lemma 1. If $0 \le x_i \le 1$ for $1 \le i \le n$, then

$$1 - \sum_{i=1}^{n} x_i \le \prod_{i=1}^{n}(1-x_i) \qquad (*)$$

Proof. The proof is by mathematical induction.

Base Case: For $n = 1$, if $0 \le x_1 \le 1$, then

$$1 - \sum_{i=1}^{n} x_i = 1 - x_1 = \prod_{i=1}^{n}(1-x_i),$$

and $(*)$ holds for $n = 1$.

Inductive Step: Assume that $(*)$ is true for some integer $n \ge 1$, and suppose that $0 \le x_i \le 1$ for $1 \le i \le n+1$, then

$$\begin{aligned}\prod_{i=1}^{n+1}(1-x_i) &= (1-x_{n+1})\prod_{i=1}^{n}(1-x_i)\\ &\ge (1-x_{n+1})\left(1-\sum_{i=1}^{n}x_i\right)\\ &= 1 - \sum_{i=1}^{n}x_i - x_{n+1} + x_{n+1}\sum_{i=1}^{n}x_i\\ &\ge 1 - \sum_{i=1}^{n+1}x_i,\end{aligned}$$

and $(*)$ is true for $n+1$.

By the principle of mathematical induction, $(*)$ is true for all $n \ge 1$.

$\square$

Lemma 2. For $n \geq 1$ and $z \in \mathbb{C}$, we have

$$\left| e^z - \left(1 + \frac{z}{n}\right)^n \right| \leq \left| e^{|z|} - \left(1 + \frac{|z|}{n}\right)^n \right| \leq \frac{|z|^2 e^{|z|}}{2n}.$$

Proof. We have

$$\begin{aligned}
\left| e^z - \left(1 + \frac{z}{n}\right)^n \right| &= \left| \sum_{k=0}^{\infty} \frac{z^k}{k!} - \sum_{k=0}^{n} \binom{n}{k} \frac{z^k}{n^k} \right| && \text{(Maclaurin expansion of } e^z \text{ and binomial theorem)} \\
&= \left| \sum_{k=2}^{\infty} \frac{z^k}{k!} - \sum_{k=2}^{n} \binom{n}{k} \frac{z^k}{n^k} \right| && \text{(first two terms the same in both sums)} \\
&= \left| \sum_{k=n+1}^{\infty} \frac{z^k}{k!} + \sum_{k=2}^{n} \left[1 - \frac{k!}{n^k} \binom{n}{k}\right] \frac{z^k}{k!} \right| \\
&= \left| \sum_{k=n+1}^{\infty} \frac{z^k}{k!} + \sum_{k=2}^{n} \underbrace{\left[1 - \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)\right]}_{\geq 0} \frac{z^k}{k!} \right| \\
&\leq \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} + \sum_{k=2}^{n} \left[1 - \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)\right] \frac{|z|^k}{k!} \\
&= \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} + \sum_{k=2}^{n} \left[1 - \frac{k!}{n^k} \binom{n}{k}\right] \frac{|z|^k}{k!} \\
&= \left| \sum_{k=2}^{\infty} \frac{|z|^k}{k!} - \sum_{k=2}^{n} \binom{n}{k} \frac{|z|^k}{n^k} \right| \\
&= \left| \sum_{k=0}^{\infty} \frac{|z|^k}{k!} - \sum_{k=0}^{n} \binom{n}{k} \frac{|z|^k}{n^k} \right| \\
&= \left| e^{|z|} - \left(1 + \frac{|z|}{n}\right)^n \right|,
\end{aligned}$$

therefore

$$\left| e^z - \left(1 + \frac{z}{n}\right)^n \right| \leq \left| e^{|z|} - \left(1 + \frac{|z|}{n}\right)^n \right|$$

for all $n \geq 1$ and $z \in \mathbb{C}$.

In order to prove the second inequality, from the above, we have

$$\begin{aligned}
\left| e^{|z|} - \left(1+\frac{z}{n}\right)^n \right| &\le \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} + \sum_{k=2}^{n} \left[1 - \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right] \frac{|z|^k}{k!} \\
&\le \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} + \sum_{k=2}^{n} \left(\sum_{i=1}^{k-1} \frac{i}{n}\right) \frac{|z|^k}{k!} \qquad \text{(Lemma 1)} \\
&\le \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} + \sum_{k=2}^{n} \frac{k(k-1)}{2n} \frac{|z|^k}{k!} \\
&= |z|^2 \sum_{k=n+1}^{\infty} \frac{|z|^{k-2}}{k!} + \frac{|z|^2}{2n} \sum_{k=2}^{n} \frac{|z|^{k-2}}{(k-2)!} \\
&= \frac{|z|^2}{n(n+1)} \sum_{\ell=n-1}^{\infty} \frac{n(n+1)\,|z|^\ell}{(\ell+2)!} + \frac{|z|^2}{2n} \sum_{\ell=0}^{n-2} \frac{|z|^\ell}{\ell!} \\
&\le \frac{|z|^2}{2n} \sum_{\ell=n-1}^{\infty} \frac{|z|^\ell}{\ell!} + \frac{|z|^2}{2n} \sum_{\ell=0}^{n-2} \frac{|z|^\ell}{\ell!} \qquad (\text{ since } \ (\ell+2)! \ge n(n+1)\cdot \ell! \ \text{ for } \ \ell \ge n-1\ ) \\
&= \frac{|z|^2}{2n} \sum_{\ell=0}^{\infty} \frac{|z|^\ell}{\ell!} \\
&= \frac{|z|^2}{2n}\, e^{|z|}
\end{aligned}$$

that is,

$$\left| e^{|z|} - \left(1+\frac{|z|}{n}\right)^n \right| \le \frac{|z|^2}{2n}\, e^{|z|}$$

for $n \ge 1$ and $z \in \mathbb{C}$.

□

Theorem. If $z \in \mathbb{C}$, then $\lim_{n\to\infty} \left(1+\frac{z}{n}\right)^n = e^z$.

Proof. Let n tend to infinity in the inequality

$$\left| e^{z} - \left(1+\frac{z}{n}\right)^n \right| \le \frac{|z|^2}{2n}\, e^{|z|}.$$

□