Math 311 Spring 2014
Theory of Functions of a Complex Variable
An Example from Euclidean Geometry

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Example. Show that $z_1,\ z_2,\ z_3$ are the vertices of an equilateral triangle, if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1. \qquad (*)$$

SOLUTION: We will show that the identity $(*)$ is true if and only if $z_1,\ z_2,\ z_3$ are the vertices of an equilateral triangle. If $(*)$ holds, we rearrange the identity as follows,

$$\begin{aligned}
0 &= z_1 z_2 - z_1^2 + z_2 z_3 - z_2^2 + z_3 z_1 - z_3^2 \\
&= z_1(z_2 - z_1) + z_2(z_3 - z_2) + z_3(z_1 - z_3) \\
&= z_1(z_2 - z_1) - z_2(z_2 - z_1) + z_2(z_2 - z_1) + z_2 z_3 - z_2^2 + z_3 z_1 - z_3^2 \\
&= -(z_1 - z_2)^2 + z_2^2 - z_1 z_2 + z_2 z_3 - z_2^2 + z_3 z_1 - z_3^2 \\
&= -(z_1 - z_2)^2 + z_2(z_3 - z_1) + z_3(z_1 - z_3) \\
&= -(z_1 - z_2)^2 + (z_2 - z_3)(z_3 - z_1),
\end{aligned}$$

that is,

$$(z_1 - z_2)^2 = (z_2 - z_3)(z_3 - z_1),$$

and therefore

$$(z_1 - z_2)^3 = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1).$$

Using symmetry arguments or arguments similar to the above, we see that

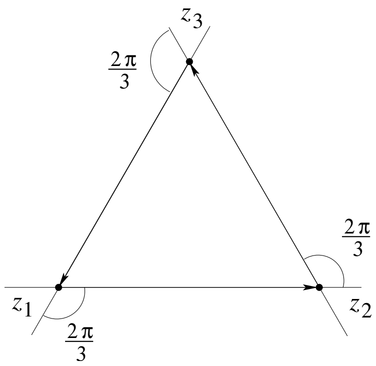
$$(z_3 - z_2)^3 = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1) = (z_1 - z_3)^3,$$

therefore $|z_1 - z_2|^3 = |z_2 - z_3|^3 = |z_3 - z_2|^3$, so that $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$, and $z_1,\ z_2,\ z_3$ are the vertices of an equilateral triangle.

Conversely, suppose that $z_1,\ z_2,\ z_3$ are the vertices of an equilateral triangle, then

$$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|,$$

and as can be seen from the figure



$$z_3 - z_2 = (z_2 - z_1)e^{\frac{i2\pi}{3}} \qquad \text{and} \qquad z_2 - z_1 = (z_1 - z_3)e^{\frac{i2\pi}{3}}.$$

Therefore

$$(z_1 - z_2)^2 = (z_2 - z_3)e^{-\frac{i2\pi}{3}}(z_3 - z_1)e^{\frac{i2\pi}{3}} = (z_2 - z_3)(z_3 - z_1),$$

and from this we see that

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$