Math 311 Spring 2014
Theory of Functions of a Complex Variable

Differentiation and Integration of Power Series

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In this note we will show that a power series $\sum_{n=0}^{\infty} a_n z^n$ can be differentiated or integrated term by term inside its circle of convergence. First we need the following lemma.

Lemma. If $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, then

$$\sum_{n=0}^{\infty} n a_n z^{n-1} \qquad \text{and} \qquad \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$$

also have radius of convergence R.

Proof. Let $z \in \mathbb{C}$ with $|z| < R$ and choose r with $|z| < r < R$, then

$$\left| n a_n z^{n-1} \right| = \left| n a_n \left(\frac{z}{r}\right)^{n-1} \cdot r^{n-1} \right| = \frac{n}{r} \left|\frac{z}{r}\right|^{n-1} \cdot |a_n| r^n,$$

however,

$$\lim_{n\to\infty} n \left|\frac{z}{r}\right|^{n-1} = \lim_{n\to\infty} n e^{(n-1)\ln|z/r|} = 0$$

since $|z| < r$, so there exists an integer n_0 such that

$$\frac{n}{r} \left|\frac{z}{r}\right|^{n-1} < 1$$

for all $n \geq n_0$, and so

$$\left| n a_n z^{n-1} \right| < |a_n| r^n$$

for all $n \geq n_0$.

Now, $\displaystyle\sum_{n=0}^{\infty} |a_n| r^n$ converges since $r < R$, and from the comparison test this implies that $\displaystyle\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges absolutely for all $z \in \mathbb{C}$ with $|z| < R$.

On the other hand, if $z \in \mathbb{C}$ with $|z| > R$, we can choose a positive number r such that $R < r < |z|$, and if $\displaystyle\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges, then $r < |z|$ implies that $\displaystyle\sum_{n=1}^{\infty} n |a_n| r^{n-1}$ converges. However,

$$n |a_n| r^{n-1} \geq \frac{1}{r} |a_n| r^n$$

implies that $\displaystyle\sum_{n=1}^{\infty} |a_n| r^n$ converges, which in turn implies that $\displaystyle\sum_{n=1}^{\infty} a_n r^n$ converges. This is a contradiction, since $r > R$. Therefore, $\displaystyle\sum_{n=1}^{\infty} n a_n z^{n-1}$ diverges for all $z \in \mathbb{C}$ with $|z| > R$.

We have shown that $\sum_{n=1}^{\infty} na_n z^{n-1}$ converges for $|z| < R$, and diverges for $|z| > R$, that is, the radius of convergence of this power series is also R.

Now let $R' > 0$ be the radius of convergence of $\sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$, then from the above, the series $\sum_{n=0}^{\infty} a_n z^n$ also has radius of convergence R', and therefore $R' = R$. □

And now the promised result, with nary a mention of uniform convergence.

Theorem. If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, then the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < R$, is differentiable (and therefore continuous) for $|z| < R$; and

(a) $f'(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$ for $|z| < R$,

(b) $\int_0^z f(s)\,ds = \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$ for any path C joining 0 and z which lies entirely inside the circle of convergence.

Proof.

(a) Let $z \in \mathbb{C}$ with $|z| < R$, and choose $H > 0$ so that $|z| + H < R$ (z and H are **fixed**).
Now let $h \in \mathbb{C}$ be such that $0 < |h| \le H$ and define

$$f_1(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$$

for $|z| < R$, then

$$\begin{aligned}
\frac{f(z+h) - f(z)}{h} - f_1(z) &= \sum_{n=0}^{\infty} a_n \left\{ \frac{(z+h)^n - z^n}{h} \right\} - \sum_{n=1}^{\infty} nz^{n-1} \\
&= \sum_{n=1}^{\infty} a_n \left\{ \frac{(z+h)^n - z^n}{h} \right\} - \sum_{n=1}^{\infty} nz^{n-1} \\
&= \sum_{n=1}^{\infty} a_n \left\{ \frac{(z+h)^n - z^n - nz^{n-1}h}{h} \right\} \\
&= \sum_{n=2}^{\infty} a_n \left\{ \frac{(z+h)^n - z^n - nz^{n-1}h}{h} \right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\frac{f(z+h)-f(z)}{h} - f_1(z)\right| &= \left|\sum_{n=2}^{\infty} a_n \left\{\frac{(z+h)^n - z^n - nz^{n-1}h}{h}\right\}\right| \\
&= \left|\sum_{n=2}^{\infty} a_n \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^{k-1}\right| \\
&\leq |h| \cdot \sum_{n=2}^{\infty} |a_n| \sum_{k=2}^{n} \binom{n}{k} |z|^{n-k} |h|^{k-2} \\
&\leq |h| \cdot \sum_{n=2}^{\infty} |a_n| \sum_{k=2}^{n} \binom{n}{k} |z|^{n-k} H^{k-2} \\
&= \frac{|h|}{H^2} \cdot \sum_{n=2}^{\infty} |a_n| \sum_{k=2}^{n} \binom{n}{k} |z|^{n-k} H^{k} \\
&\leq \frac{|h|}{H^2} \cdot \sum_{n=1}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} |z|^{n-k} H^{k} \\
&= \frac{|h|}{H^2} \cdot \sum_{n=1}^{\infty} |a_n| (|z|+H)^n \\
&\leq \frac{|h|}{H^2} \cdot M
\end{aligned}$$

where $M = \sum_{n=1}^{\infty} |a_n|(|z|+H)^n < \infty$ since $|z|+H < R$.

Therefore, we have

$$\left|\frac{f(z+h)-f(z)}{h} - f_1(z)\right| \leq \frac{|h|}{H^2} \cdot M$$

for $|h| \leq H$. Letting $h \to 0$, then

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f_1(z)$$

for $|z| < R$, that is,

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

for $|z| < R$.

(b) Now let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < R$, and define

$$F(z) = \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$$

for $|z| < R$, then from part (a), f is analytic in the domain $|z| < R$, and so is continuous there, and

$$F'(z) = \sum_{n=0}^{\infty} a_n z^n$$

for all $|z| < R$, that is, F is an antiderivative of f in the domain $|z| < R$.

Therefore, if $|z| < R$ and C is any contour joining 0 and z which lies entirely inside the circle of convergence, then

$$\int_C f(s)\, ds = F(z) - F(0) = \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$$

for $|z| < R$.

□

Note: If the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $R > 0$, then $f'(z)$ exists for all z with $|z| < R$, that is, **any power series is an analytic function in its circle of convergence**.