Math 311 Spring 2014

Theory of Functions of a Complex Variable

The Field of Complex Numbers: $\mathbb{C}$

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- If $\mathbb{C}$ is the set of all complex numbers:

$$\mathbb{C} = \{\, z = (x, y) \,|\, x, y \in \mathbb{R} \,\}$$

with addition and multiplication defined as follows

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$, then $\mathbb{C}$ is a field, that is, $\mathbb{C}$ together with these operations of addition and multiplication satisfies the following axioms:

a_1 : $z_1 + z_2 = z_2 + z_1$ for all $z_1, z_2 \in \mathbb{C}$ (*commutative law*)

a_2 : $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$ (*associative law*)

a_3 : there exists an element 0 in $\mathbb{C}$ such that $z + 0 = z = 0 + z$ for all $z \in \mathbb{C}$ (*additive identity*)

a_4 : for each $z \in \mathbb{C}$, there exists a $w \in \mathbb{C}$ such that $z + w = 0 = w + z$ (*additive inverse*)

a_5 : $z_1 \cdot z_2 = z_2 \cdot z_1$ for all $z_1, z_2 \in \mathbb{C}$ (*commutative law*)

a_6 : $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$ (*associative law*)

a_7 : $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$, $(z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$ (*distributive laws*).

a_8 : there exists an element $1 \in \mathbb{C}$ such that $z \cdot 1 = z = 1 \cdot z$ for all $z \in \mathbb{C}$ (*multiplicative identity*)

a_9 : $1 \neq 0$

a_{10} : for each $z \in \mathbb{C}$ with $z \neq 0$, there exists a $w \in \mathbb{C}$ such that $z \cdot w = 1 = w \cdot z$ (*multiplicative inverse*)

You should verify each of these axioms.

For example, verify that the additive identity in $\mathbb{C}$ is $0 = (0, 0)$, and that the multiplicative identity in $\mathbb{C}$ is $1 = (1, 0)$.

- Note that if $i \in \mathbb{C}$ is the ordered pair $i = (0, 1)$, then

$$i^2 = (0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -(1, 0) = -1,$$

and if $z = (x, y) \in \mathbb{C}$, then

$$z = (x, 0) + (0, y) = (x, 0) + (0, 1)(y, 0) = x + iy$$

where we have identified the ordered pair $(x, 0)$ with the real number x and the ordered pair $(y, 0)$ with the real number y.

Therefore, instead of writing $z = (x, y)$, do as Cardano did, write $z = x + iy$ and manipulate expressions as usual (that is, as if the numbers were real), and replace i^2 by -1 whenever it occurs.

- **Exercise.** Show that we can also define $\mathbb{C}$ as the set of all 2×2 matrices with real entries of the form

$$z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

that is,

$$\mathbb{C} = \left\{ z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

with the usual definition of matrix addition and matrix multiplictation.

Here you have to show first that $\mathbb{C}$ is closed under addition and multiplication. Note that

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix},$$

while

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix}.$$

Clearly, the additive and multiplicative identities are

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

respectively, while

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$