Math 311 Spring 2014
Theory of Functions of a Complex Variable

The Binomial Series

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The following expansion is known as the *binomial series* :

Theorem. Let α be any complex number that is not a nonnegative integer. Define the binomial coefficients

$$\binom{\alpha}{n} = \begin{cases} \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, & \text{for} \quad n \geq 1, \\ 1, & \text{for} \quad n = 0. \end{cases}$$

The binomial series

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \tag{$*$}$$

has radius of convergence $R = 1$, it converges absolutely if $|z| < 1$, and it diverges if $|z| > 1$.

On the circle of convergence $|z| = 1$, we have

(i) If $\mathrm{Re}(\alpha) > 0$ and $|z| = 1$, then $(*)$ converges absolutely.

(ii) If $-1 < \mathrm{Re}(\alpha) \leq 0$ and $z = -1$, then $(*)$ diverges.

(iii) If $-1 < \mathrm{Re}(\alpha) \leq 0$ and $0 < \delta < 2$, then $(*)$ converges uniformly on

$$\{z \in \mathbb{C} : |z| \leq 1, \ |1+z| \geq \delta\}.$$

The convergence is not absolute if $|z| = 1$.

(iv) If $\mathrm{Re}(\alpha) \leq -1$ and $|z| = 1$, then $(*)$ diverges.

Finally, in all cases for which $(*)$ converges, we have

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \tag{$**$}$$

where

$$(1+z)^\alpha = e^{\alpha \mathrm{Log}(1+z)}$$

is the principal value of $(1+z)^\alpha$ if $z \neq -1$ and $0^\alpha = 0$ (recall $\alpha \neq 0$).

Proof. We show only that the series converges to $(1+z)^\alpha$ for all z in the open disk $|z|<1$, the remainder of the proof can be found in the text *An Introduction to Classical Real Analysis*, by Karl R. Stromberg.

For $n \geq 0$, let

$$a_n = \left| \binom{\alpha}{n} z^n \right|,$$

then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left| \frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}} z \right| = \lim_{n\to\infty} \frac{n-\alpha}{n+1}|z| = |z|,$$

and by the ratio test, the series converges absolutely for $|z|<1$.

Now define the function g on the disk $|z|<1$ as follows,

$$g(z) = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \quad |z|<1,$$

differentiating, we have

$$g'(z) = \sum_{n=1}^{\infty} n\binom{\alpha}{n} z^{n-1} = \alpha \sum_{n=1}^{\infty} \binom{\alpha-1}{n-1} z^{n-1},$$

so that

$$\begin{aligned}
(1+z)g'(z) &= \alpha \sum_{n=1}^{\infty} \binom{\alpha-1}{n-1} z^{n-1} + \alpha \sum_{n=1}^{\infty} \binom{\alpha-1}{n-1} z^n \\
&= \alpha \sum_{n=0}^{\infty} \binom{\alpha-1}{n} z^n + \alpha \sum_{n=1}^{\infty} \binom{\alpha-1}{n-1} z^n \\
&= \alpha \left[1 + \sum_{n=1}^{\infty} \left\{ \binom{\alpha-1}{n} + \binom{\alpha-1}{n-1} \right\} z^n \right] \\
&= \alpha \left[1 + \sum_{n=1}^{\infty} \binom{\alpha}{n} z^n \right] \\
&= \alpha g(z),
\end{aligned}$$

that is,

$$g'(z) = \frac{\alpha g(z)}{1+z}$$

for $|z|<1$.

Now define

$$h(z) = \frac{g(z)}{(1+z)^\alpha}$$

for $|z|<1$, where we use the principal value of the logarithmic function for the power, then

$$h'(z) = \frac{g'(z)}{(1+z)^\alpha} - \frac{\alpha g(z)}{(1+z)^{\alpha+1}} = \frac{\alpha g(z)}{(1+z)^{\alpha+1}} - \frac{\alpha g(z)}{(1+z)^{\alpha+1}} = 0$$

for all $|z|<1$, so that $h(z)$ is constant on the disk $|z|<1$.

Since $h(0)=1$, then $h(z)=1$ for all $|z|<1$, that is,

$$(1+z)^\alpha = g(z) = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$$

for $|z|<1$.

$\square$