Math 311 Spring 2014

Theory of Functions of a Complex Variable

The Binomial Theorem

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Binomial Theorem. If a and b are complex numbers and n is a positive integer, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}. \qquad (*)$$

We will prove this using the principle of mathematical induction, but first we give a definition of the binomial coefficients, and a lemma.

Definition. If n and k are nonnegative integers, the **binomial coefficient** $\binom{n}{k}$ is defined to be

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{for} \quad 0 \le k \le n \\ 0 & \text{for} \quad k > n, \end{cases}$$

where by convention $0! = 1$.

Pascal's Lemma. If n and k are integers with $1 \le k \le n$, then

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Proof. Let n and k be integers with $1 \le k \le n$, then

$$\binom{n}{k-1}+\binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!}+\frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k+1)!}\,[k+n+1-k] = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}.$$

□

Proof of Binomial Theorem. We assume first that both a and b are nonzero. In this case, $a^0 = 1$ and $b^0 = 1$.

Base case: If $n = 1$, then

$$(a+b)^1 = a + b = \binom{1}{0} a^0 b^{1-0} + \binom{1}{1} a^1 b^{1-1} = \sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k}$$

and $(*)$ is true for $n = 1$.

Inductive Step: If $n \geq 1$ and $(*)$ is true for n, then

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b)\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n+1-k}$$

$$= \binom{n}{n}a^{n+1} + \underbrace{\sum_{k=0}^{n-1}\binom{n}{k}a^{k+1}b^{n-k}}_{\ell=k+1} + \sum_{k=1}^{n}\binom{n}{k}a^k b^{n+1-k} + \binom{n}{0}b^{n+1}$$

$$= \binom{n+1}{n+1}a^{n+1} + \sum_{\ell=1}^{n}\binom{n}{\ell-1}a^{\ell}b^{n+1-\ell} + \sum_{k=1}^{n}\binom{n}{k}a^k b^{n+1-k} + \binom{n+1}{0}b^{n+1}$$

$$= \binom{n+1}{n+1}a^{n+1} + \sum_{k=1}^{n}\binom{n}{k-1}a^{k}b^{n+1-k} + \sum_{k=1}^{n}\binom{n}{k}a^k b^{n+1-k} + \binom{n+1}{0}b^{n+1}$$

$$= \binom{n+1}{n+1}a^{n+1} + \sum_{k=1}^{n}\underbrace{\left[\binom{n}{k-1} + \binom{n}{k}\right]}_{\text{Pascal's Lemma}}a^k b^{n+1-k} + \binom{n+1}{0}b^{n+1}$$

$$= \binom{n+1}{n+1}a^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}a^k b^{n+1-k} + \binom{n+1}{0}b^{n+1}$$

$$= \sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{n+1-k},$$

and $(*)$ is true for $n+1$ also.

Therefore, by the principle of mathematical induction, $(*)$ is true for all $n \geq 1$.

In the case that one (or both) of a or b is zero, then $a+b=b$ or $a+b=a$, and the sum is either b^n or a^n, and the theorem is true in this case also, provided we interpret the term 0^0 in the sum as 1. ∎

Example. If we take $a=-1$ and $b=1$ in the binomial theorem, then

$$0 = (-1+1)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k} = \sum_{k \text{ even}}\binom{n}{k} - \sum_{k \text{ odd}}\binom{n}{k}.$$

In the first sum the index k runs through all the even integers from 0 to n, and the last term in the sum occurs when $2k \leq n < 2k+1$, that is, when $k = \left\lfloor \frac{n}{2} \right\rfloor$, that is, when k is the greatest integer less than or equal to $\frac{n}{2}$.

In the second sum the index k runs through all the odd integers from 1 to n, and the last term in the sum occurs when $2k-1 \leq n < 2k$, that is, when $k = \left\lfloor \frac{n+1}{2} \right\rfloor$, that is, when k is the greatest integer less than or equal to $\frac{n+1}{2}$. Therefore,

$$0 = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k} - \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor}\binom{n}{2k-1},$$

and $$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k} = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor}\binom{n}{2k-1}.$$

Exercise. Prove the identity

$$2^{\frac{n}{2}}\sin\tfrac{n\pi}{4} = \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^k\binom{n}{2k+1}$$

for $n \geq 1$.

Solution: From the binomial theorem we have

$$(1+i)^n = \sum_{k=0}^{n}\binom{n}{k}i^k$$

and

$$(1-i)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k i^k,$$

so that

$$(1+i)^n - (1-i)^n = \sum_{k=0}^{n}\binom{n}{k}i^k\left[1-(-1)^k\right],$$

and the only terms that survive are the terms when k is odd.

Now if $k = 2m+1$, then

$$i^{2m+1} = i\cdot i^{2m} = i(-1)^m,$$

and since we want $2m+1 \leq n$, then we must have $m \leq \left\lfloor\frac{n-1}{2}\right\rfloor$, and

$$(1+i)^n - (1-i)^n = 2i\sum_{m=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2m+1}(-1)^m,$$

and therefore,

$$\frac{(1+i)^n - (1-i)^n}{2i} = \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2k+1}(-1)^k$$

for $n \geq 0$.

Also,

$$\begin{aligned}\frac{1}{2i}\left[(1+i)^n - (1-i)^n\right] &= \frac{1}{2i}\left[\left(\tfrac{1+i}{\sqrt{2}}\right)^n - \left(\tfrac{1-i}{\sqrt{2}}\right)^n\right]\cdot 2^{\frac{n}{2}}\\ &= \frac{1}{2i}\left[e^{\frac{in\pi}{4}} - e^{-\frac{in\pi}{4}}\right]\cdot 2^{\frac{n}{2}}\\ &= 2^{\frac{n}{2}}\sin\tfrac{n\pi}{4},\end{aligned}$$

and therefore,

$$2^{\frac{n}{2}}\sin\tfrac{n\pi}{4} = \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^k\binom{n}{2k+1}$$

for $n \geq 0$.

Note: We have used Euler's formulas

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{and} \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

to write

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

in the above.