Math 311 - Spring 2014

Assignment # 4

Completion Date: Friday May 16, 2014

Question 1. [p 77, #1 (a)]

Apply the theorem in Sec. 22 to verify that the function

$$f(z) = 3x + y + i\,(3y - x)$$

is entire.

Question 2. [p 77, #1 (c)]

Apply the theorem in Sec. 22 to verify that the function

$$f(z) = e^{-y}\sin x - i\,e^{-y}\cos x$$

is entire.

Question 3. [p 77, #4 (c)]

For the function

$$f(z) = \frac{z^2+1}{(z+2)(z^2+2z+2)},$$

determine the singular points of the function and state why the function is analytic everywhere except at those points.

Ans: $z = -2,\ -1 \pm i$.

Question 4. [p 78, #6]

Use the results in Sec. 23 to verify that the function

$$g(z) = \ln r + i\,\theta \quad (r > 0,\ 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative $g'(z) = \dfrac{1}{z}$. Then show that the composite function $G(z) = g(z^2+1)$ is analytic in the quadrant $x > 0,\ y > 0$, with derivative

$$G'(z) = \frac{2z}{z^2+1}.$$

Suggestion: Observe that $\mathrm{Im}(z^2+1) > 0$ when $x > 0,\ y > 0$.

Question 5. [p 76, Example 4]

Let $f(z)$ be analytic in a domain D. Prove that $f(z)$ must be constant throughout D if $|f(z)|$ is constant throughout D.

Suggestion: Observe that

$$\overline{f(z)} = \frac{c^2}{f(z)} \quad \text{if} \quad |f(z)| = c \ (c \neq 0);$$

then use the main result in Example 3, Sec. 25.

Question 6. [p 81, #1 (c)]

Show that the function

$$u(x, y) = \sinh x \sin y$$

is harmonic in some domain and find a harmonic conjugate $v(x, y)$.

Ans: $v(x, y) = -\cosh x \cos y$.

Question 7. [p 82, #6]

Verify that the function $u(r, \theta) = \ln r$ is harmonic in the domain $r > 0,\ 0 < \theta < 2\pi$ by showing that it satisfies the polar form of Laplace's equation, obtained in Exercise 5. Then use the technique in Example 5, Sec. 26, but involving the Cauchy-Riemann equations in polar form (Sec. 23), to derive the harmonic conjugate $v(r, \theta) = \theta$. (Compare Exercise 6, Sec. 25)

Question 8. [p 88, #4]

We know from Example 1, Sec. 22, that the function

$$f(z) = e^x e^{iy}$$

has a derivative everywhere in the finite plane. Point out how it follows from the reflection principle (Sec. 28) that

$$\overline{f(z)} = f(\overline{z})$$

for each z. Then verify this directly.

Question 9. [p 92, #1]

Show that

(a) $\exp(2 \pm 3\pi i) = -e^2$; (b) $\exp\left(\frac{2 + \pi i}{4}\right) = \sqrt{\frac{e}{2}}(1 + i)$; (c) $\exp(z + \pi i) = -\exp z$.

Question 10. [p 92, #3]

Use the Cauchy-Riemann equations and the theorem in Sec. 21 to show that the function

$$f(z) = \exp \overline{z}$$

is not analytic anywhere.

Question 11. [p 92, #5]

Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp(2z + i) + \exp(iz^2)| \leq e^{2x} + e^{-2xy}.$$

Question 12. [p 92, #6]

Show that $|\exp(z^2)| \leq \exp(|z|^2)$.

Question 13. [p 92, #8]

Find all values of z such that

(a) $e^z = -2$; (b) $e^z = 1 + \sqrt{3}\, i$; (c) $\exp(2z - 1) = 1$.

Ans:

(a) $z = \ln 2 + (2n+1)\pi\, i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

(b) $z = \ln 2 + \left(2n + \dfrac{1}{3}\right)\pi\, i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

(c) $z = \dfrac{1}{2} + n\pi\, i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

Question 14. [p 92, #13]

Let the function $f(z) = u(x, y) + i\, v(x, y)$ be analytic in some domain D. State why the functions

$$U(x, y) = e^{u(x,y)} \cos v(x, y), \qquad V(x, y) = e^{u(x,y)} \sin v(x, y)$$

are harmonic in D and why $V(x, y)$ is, in fact, a harmonic conjugate of $U(x, y)$.