Math 311 Spring 2014
Theory of Functions of a Complex Variable
Principal Arguments

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We showed earlier that the principal values of the arguments and logarithms for products, quotients, and powers may not be just the sum or difference of the principal values of the product or quotient of the complex numbers involved.

In fact, we have the following result.

Theorem. If the complex numbers z, z_1, and z_2 are different from zero, then the principal values of the arguments and logarithms of the product, quotient, and powers are given by

$$\text{Arg}(z_1 \cdot z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2\pi n_1(z_1, z_2),$$

$$\text{Log}(z_1 \cdot z_2) = \text{Log}(z_1) + \text{Log}(z_2) + 2\pi i n_1(z_1, z_2),$$

$$\text{Arg}(z_1/z_2) = \text{Arg}(z_1) - \text{Arg}(z_2) + 2\pi n_2(z_1, z_2),$$

$$\text{Log}(z_1/z_2) = \text{Log}(z_1) - \text{Log}(z_2) + 2\pi i n_1(z_1, z_2),$$

$$\text{Arg}(z^n) = n\text{Arg}(z) + 2\pi k(z, n),$$

$$\text{Log}(z^n) = n\text{Log}(z) + 2\pi i k(z, n),$$

where n is any integer, and n_1 and n_2 take on the values -1, 0, $+1$ as follows:

$$n_1(z_1, z_2) = \begin{cases} -1, & \text{if } \quad \pi < \text{Arg}(z_1) + \text{Arg}(z_2) \le 2\pi \\ 0, & \text{if } \; -\pi < \text{Arg}(z_1) + \text{Arg}(z_2) \le \pi \\ +1, & \text{if } \; -2\pi < \text{Arg}(z_1) + \text{Arg}(z_2) \le -\pi \end{cases}$$

$$n_2(z_1, z_2) = \begin{cases} -1, & \text{if } \quad \pi < \text{Arg}(z_1) - \text{Arg}(z_2) \le 2\pi \\ 0, & \text{if } \; -\pi < \text{Arg}(z_1) - \text{Arg}(z_2) \le \pi \\ +1, & \text{if } \; -2\pi < \text{Arg}(z_1) - \text{Arg}(z_2) \le -\pi \end{cases}$$

and k is the integer given by the greatest integer function

$$k(z, n) = \left\lfloor \frac{1}{2} - \frac{n}{2\pi}\text{Arg}(z) \right\rfloor.$$

Proof. We will prove the first staement and leave the rest as an exercise.

If $z_1 \neq 0$ and $z_2 \neq 0$, then we can write

$$z_1 = |z_1| \cdot e^{i\text{Arg}(z_1)} \qquad \text{and} \qquad z_2 = |z_2| \cdot e^{i\text{Arg}(z_2)}$$

where $-\pi < \text{Arg}(z_1) \le \pi$ and $-\pi < \text{Arg}(z_2) \le \pi$, so that

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot e^{i\text{Arg}(z_1)} \cdot e^{i\text{Arg}(z_2)} = |z_1 z_2| \cdot e^{i(\text{Arg}(z_1)+\text{Arg}(z_2))}$$

In order to find the principal value of $\text{Arg}(z_1 z_2)$ we need to know where $\text{Arg}(z_1) + \text{Arg}(z_2)$ is.

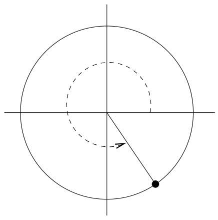
We consider three cases, as shown below. In each case, the angle shown is $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$.

Case 1: If

$$\pi < \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \leq 2\pi,$$

then

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) - 2\pi.$$

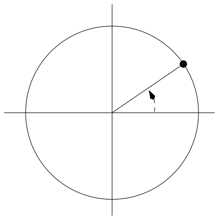


Case 2: If

$$-\pi < \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \leq \pi,$$

then

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2).$$

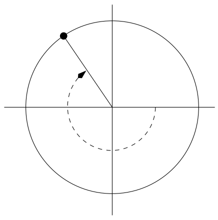


Case 3: If

$$-2\pi < \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \leq -\pi,$$

then

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) + 2\pi.$$



From Case 1, Case 2, and Case 3, we have

$$\mathrm{Arg}(z_1 \cdot z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) + 2\pi n_1(z_1, z_2),$$

where

$$n_1(z_1, z_2) = \begin{cases} -1, & \text{if} \quad \pi < \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \leq 2\pi \\ 0, & \text{if} \quad -\pi < \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \leq \pi \\ +1, & \text{if} \quad -2\pi < \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \leq -\pi. \end{cases}$$

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