MATH 309

Solutions to Midterm Examination I

DATE: Monday June 11, 2018 TIME: 50 Minutes

INSTRUCTOR: I. E. Leonard (Section C1)

Question 1. Find two numbers whose sum is 2 and whose product is 17.

SOLUTION: Suppose that $z_1, z_2 \in \mathbb{C}$, we want $z_1 + z_2 = 2$ and $z_1 \cdot z_2 = 17$. The obvious solution is $z_1 = 1 + 4i$ and $z_2 = 1 - 4i$, since $z_1 + z_2 = 1 + 1 = 2$, and $z_1 \cdot z_2 = 1^2 + 4^2 = 17$.

If we want $z_1 + z_2 = 2$ and $z_1 \cdot z_2 = 17$, then we must have

$$z_1 + \frac{17}{z_1} = 2,$$

so that z_1 satisfies the quadratic equation: ${z_1}^2 - 2z_1 + 17 = 0$. Completing the square, we have

$$(z_1 - 1)^2 = -16,$$

so that $z_1 - 1 = \pm 4i$, and therefore $z_1 = 1 + 4i$ and $z_2 = 1 - 4i$ obviously.

Question 2. If $k > 0$, show that the locus of points z for which

$$\left|\frac{1+z}{1-z}\right| = k$$

is a circle if $k \neq 1$, and a straight line if $k = 1$.

SOLUTION: If $k = 1$, then the set of all points $z \in \mathbb{C}$ such that $|z + 1| = |1 - z| = |z - 1|$ is the set of all points that are equidistant from the points $p_1 = 1$ and $p_2 = -1$; that is, the line L which is the perpendicular bisector of the line segment $[p_1, p_2]$ joining the points p_1 and p_2.

If $k \neq 1$, then we want the set of all points in the complex plane for which $|z + 1| = k|1 - z| = k|z - 1|$, and since the absolute value is a nonnegative real number, we may assume that $k > 0$. Also, we may also assume that $0 < k < 1$, since if $k > 1$ we only need to divide the equation by k to get the

Now if k is a positive real number such that $0 < k < 1$, then the set of all point z in the complex plane such that $|z+1| = k|z-1|$ is the **circle of Apollonius** for 1, -1 and k. We can show this as follows: $|z+1| = k|z-1|$ if and only if

$$\begin{aligned} |z+1|^2 &= (x+1)^2 + y^2 = k^2\left[(x-1)^2 + y^2\right] \\ &= x^2 + 2x + 1 + y^2 = k^2\left[x^2 - 2x + 1 + y^2\right] \\ &= (1-k^2)x^2 + (1+k^2)2x + (1-k^2)y^2 + (1-k^2) = 0, \end{aligned}$$

that is,

$$(1-k^2)x^2 + (1+k^2)2x + (1-k^2)y^2 + (1-k^2) = 0.$$

Completing the square, we have the equivalent statement

$$x^2 + \frac{1+k^2}{1-k^2}2x + \frac{(1+k^2)^2}{(1-k^2)^2} + y^2 = \frac{(1+k^2)^2}{(1-k^2)^2} - 1 = \frac{4k^2}{(1-k^2)^2},$$

So that z lies on the circle of radius $R = \dfrac{2k}{1-k^2}$ with center at $z_0 = \dfrac{1+k^2}{1-k^2}$ on the line joining $p = (1,0)$ and $q = (-1,0)$.

Question 3. Let S be the set of all points $a + ib$ in the complex plane, where a and b are rational real numbers, such that $0 \leq a \leq 1$ and $0 \leq b \leq 1$.

Answer the following questions concerning the set S. **Give a brief reason for your answer to each part.**

(a) Is S bounded?

(b) What are the accumulation points of S, if any?

(c) Is S closed?

(d) What are the interior and boundary points of S.

(e) Is S open?

(f) Is S connected?

(g) Is S a domain?

(h) What is the closure of S?

SOLUTION:

(b) Let T be the (solid) square of side 1:

$$T = \{z \in \mathbb{C} \,|\, z = a + ib,\ 0 \le a \le 1,\ 0 \le b \le 1\},$$

then every point of T is an accumulation point of S.

(c) S is not closed, since it does not contain all of its accumulation points.

(d) S has no interior points, and every point of T is a boundary point of S.

(e) S is not open, it has no interior points, so $\text{int}(S) = \emptyset$.

(f) S is not connected, since any line segment joining two points of S must contain points in S and points not in S.

(g) S is not a domain, it is neither open nor connected.

(h) The closure of S is T the solid square.

Question 4. Find the image of the circle $|z + i| = 1$ under the map $w = \dfrac{1}{z}$.

Solution: The image of the circle $z = x + iy$ under the map $w = \dfrac{1}{z}$, $z \neq 0$, is the line $v = \dfrac{1}{2}$, where $w = u + iv$.

If $w = u + iv$ is in the w-plane, and z is on the circle, then

$$z + i = \frac{1}{w} + i,$$

and

$$|z + i| = \left|\frac{1}{w} + i\right| = \frac{|1 + iw|}{|w|} = \left|\frac{1 - v + iu}{u + iv}\right| = 1,$$

that is, $(1 - v)^2 + u^2 = u^2 + v^2$, or $1 - 2v = 0$. so the image of the circle $|z + i| = 1$ is the line $v = \dfrac{1}{2}$, $-\infty < u < \infty$ if $w = u + iv$.