



Math 309 - Spring-Summer 2018

Solutions to Problem Set # 2

Question 1.

Show that $\lim_{z \rightarrow \infty} \frac{4z^2}{(z-1)^2} = 4$.

SOLUTION: We have

$$\lim_{z \rightarrow \infty} \frac{4z^2}{(z-1)^2} = \lim_{z \rightarrow 0} \frac{4(1/z)^2}{(1/z-1)^2} = \lim_{z \rightarrow 0} \frac{4}{(1-z)^2} = 4.$$

Question 2.

Show that a set is unbounded if and only if every neighborhood of the point at infinity contains at least one point in S .

SOLUTION: If $S \subseteq \mathbb{C}$ is unbounded, then for each $n \geq 1$, there is a point $z_n \in S$ with $|z_n| \geq n$.

Now, given $\epsilon > 0$, choose $n_0 \geq 1$ with $0 < \frac{1}{n_0} < \epsilon$, then

$$|z_{n_0}| \geq n_0 > \frac{1}{\epsilon},$$

and z_{n_0} is in the ϵ -neighborhood $\{z : |z| > 1/\epsilon\}$ of the point of infinity. Thus, every neighborhood of the point at infinity contains at least one point in S .

Conversely, if every neighborhood of the point at infinity contains at least one point in S , then for each $n \geq 1$, we can choose a point $z_n \in S$ with $|z_n| \geq n$ (that is, z_n is in the $1/n$ -neighborhood of the point at infinity). Then S cannot be bounded, since $z_n \in S$ for all $n \geq 1$, and $\lim_{n \rightarrow \infty} |z_n| = +\infty$, and there is **no** $M > 0$ such that $|z| \leq M$ for all $z \in S$.

Question 3.

Find $f'(z)$ when

- (a) $f(z) = 3z^2 - 2z + 4$; (b) $f(z) = (1 - 4z^2)^3$;
(c) $f(z) = \frac{z-1}{2z+1}$ ($z \neq -1/2$); (d) $f(z) = \frac{(1+z^2)^4}{z^2}$ ($z \neq 0$).

SOLUTION:

(a) $f'(z) = 6z - 2$.

(b) $f'(z) = 3(1 - 4z^2)^2(-8z) = -24z(1 - 4z^2)^2$.

$$(c) \quad f'(z) = \frac{1 \cdot (2z+1) - (z-1) \cdot 2}{(2z+1)^2} = \frac{3}{(2z+1)^2}, \text{ for } z \neq -1/2.$$

$$(d) \quad f'(z) = \frac{4(1+z^2)^3 \cdot 2z \cdot z^2 - (1+z^2)^4 \cdot 2z}{z^4} = \frac{2(1+z^2)^3}{z^3} \cdot (3z^2 - 1), \text{ for } z \neq 0.$$

Question 4.

Apply the definition of the derivative to give a direct proof that

$$f'(z) = -\frac{1}{z^2} \quad \text{when} \quad f(z) = \frac{1}{z} \quad (z \neq 0).$$

SOLUTION: If $f(z) = \frac{1}{z}$ for $z \neq 0$, then

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{z+h} - \frac{1}{z}}{h} = \lim_{h \rightarrow 0} \frac{z - (z+h)}{hz(z+h)},$$

that is,

$$f'(z) = \lim_{h \rightarrow 0} \frac{-h}{hz(z+h)} = \lim_{h \rightarrow 0} \frac{-1}{z(z+h)} = -\frac{1}{z^2}$$

for $z \neq 0$.

Question 5.

Show that $f'(z)$ does not exist at any point z when $f(z) = \text{Im } z$.

SOLUTION: Let $f(z) = \text{Im}(z)$, then for $z, h \in \mathbb{C}$, with $h \neq 0$, we have

$$\frac{\text{Im}(z+h) - \text{Im}(z)}{h} = \frac{\text{Im}(z) + \text{Im}(h) - \text{Im}(z)}{h} = \frac{\text{Im}(h)}{h}.$$

Now, if $h \rightarrow 0$ through real values, $\text{Im}(h) = 0$, and

$$\lim_{\substack{h \rightarrow 0 \\ h \text{ real}}} \frac{\text{Im}(z+h) - \text{Im}(z)}{h} = \lim_{\substack{h \rightarrow 0 \\ h \text{ real}}} \frac{0}{h} = 0. \quad (1)$$

However, if $h \rightarrow 0$ through imaginary values, say $h = it$ where $t \in \mathbb{R}$ and $t \rightarrow 0$, then

$$\frac{\text{Im}(h)}{h} = \frac{t}{it} = -i,$$

and

$$\lim_{\substack{h \rightarrow 0 \\ h \text{ imag}}} \frac{\text{Im}(z+h) - \text{Im}(z)}{h} = \lim_{\substack{h \rightarrow 0 \\ h \text{ imag}}} (-i) = -i. \quad (2)$$

Therefore, (1) and (2) imply that

$$\lim_{h \rightarrow 0} \frac{\text{Im}(z+h) - \text{Im}(z)}{h}$$

doesn't exist, that is, $f'(z)$ does not exist for any $z \in \mathbb{C}$.

Question 6.

Show that $f'(z)$ does not exist at any point if

- (a) $f(z) = \bar{z}$; (b) $f(z) = z - \bar{z}$;
 (c) $f(z) = 2x + ixy^2$; (d) $f(z) = e^x e^{-iy}$.

SOLUTION:

- (a) If $f(z) = \bar{z} = x - iy$, then $u(x, y) = x$ and $v(x, y) = -y$, so that

$$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y},$$

and the Cauchy-Riemann equations do not hold at **any** point $z \in \mathbb{C}$. Therefore $f'(z)$ does not exist for any $z \in \mathbb{C}$.

- (b) If $f(z) = z - \bar{z} = 2iy$, then $u(x, y) = 0$ and $v(x, y) = 2y$, so that

$$\frac{\partial u}{\partial x} = 0 \neq 2 = \frac{\partial v}{\partial y},$$

and again the Cauchy-Riemann equations do not hold at **any** point $z \in \mathbb{C}$. Therefore $f'(z)$ does not exist for any $z \in \mathbb{C}$.

- (c) If $f(z) = 2x + ixy^2$, then $u(x, y) = 2x$ and $v(x, y) = xy^2$, so that

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2, & \frac{\partial v}{\partial y} &= 2xy \\ \frac{\partial u}{\partial y} &= 0, & \frac{\partial v}{\partial x} &= y^2. \end{aligned}$$

Now the Cauchy-Riemann equations are

$$\begin{aligned} 2 &= 2xy \\ 0 &= -y^2 \end{aligned}$$

and these equations have **no** solutions (x, y) . Therefore, $f'(z)$ does not exist at any point $z \in \mathbb{C}$.

- (d) If $f(z) = e^x e^{-iy} = e^x \cos y - ie^x \sin y$, then

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y, & \frac{\partial v}{\partial y} &= -e^x \cos y \\ \frac{\partial u}{\partial y} &= -e^x \sin y, & \frac{\partial v}{\partial x} &= -e^x \sin y. \end{aligned}$$

Now the Cauchy-Riemann equations are

$$\begin{aligned} 2e^x \cos y &= 0 \\ 2e^x \sin y &= 0 \end{aligned}$$

and since $e^x \neq 0$ for all $x \in \mathbb{R}$, these equations are

$$\cos y = 0 \quad \text{and} \quad \sin y = 0,$$

but this is impossible since $\cos^2 y + \sin^2 y = 1$. Therefore, there are **no** solutions to the Cauchy-Riemann equations, and $f'(z)$ does not exist for **any** $z \in \mathbb{C}$.

Question 7.

Determine where $f'(z)$ exists and find its value when

$$(a) f(z) = \frac{1}{z}; \quad (b) f(z) = x^2 + i y^2; \quad (c) f(z) = z \operatorname{Im} z;$$

$$\text{Ans: (a) } f'(z) = -\frac{1}{z^2} \ (z \neq 0); \quad (b) f'(x + i x) = 2x; \quad (c) f'(0) = 0.$$

SOLUTION:

(a) If $f(z) = \frac{1}{z}$, then

$$f(z) = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2},$$

so that

$$u(x, y) = \frac{x}{x^2 + y^2} \quad \text{and} \quad v(x, y) = -\frac{y}{x^2 + y^2}$$

for $z \neq 0$. Now,

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}.$$

Since the partial derivatives are all continuous at each $z \in \mathbb{C}$, $z \neq 0$, and the Cauchy-Riemann equations hold at each $z \in \mathbb{C}$, $z \neq 0$, then $f'(z)$ exists for all $z \neq 0$, and

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{2ixy}{(x^2 + y^2)^2},$$

that is,

$$f'(z) = -\frac{x^2 - y^2 - 2ixy}{(x^2 + y^2)^2} = -\frac{(\bar{z})^2}{|z|^4} = -\frac{1}{z^2}$$

for $z \neq 0$.

(b) If $f(z) = x^2 + i y^2$, then $u(x, y) = x^2$ and $v(x, y) = y^2$, and

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x, & \frac{\partial v}{\partial y} &= 2y \\ \frac{\partial u}{\partial y} &= 0, & \frac{\partial v}{\partial x} &= 0, \end{aligned}$$

so the Cauchy-Riemann equations hold only for those points $z = x + i y$ with $x = y$. Since all partial derivatives are continuous everywhere, $f'(z)$ exists only for the points $z = x + i x = x(1 + i)$, $x \in \mathbb{R}$, and

$$f'(x + i x) = \frac{\partial u}{\partial x}(x, x) + i \frac{\partial v}{\partial x}(x, x) = 2x.$$

(c) If $f(z) = z \operatorname{Im}(z) = (x + i y) \cdot y = xy + i y^2$, then $u(x, y) = xy$ and $v(x, y) = y^2$, so that

$$\begin{aligned} \frac{\partial u}{\partial x} &= y, & \frac{\partial v}{\partial y} &= 2y \\ \frac{\partial u}{\partial y} &= x, & \frac{\partial v}{\partial x} &= 0, \end{aligned}$$

and the Cauchy-Riemann equations hold if and only if

$$y = 2y \quad \text{and} \quad x = 0,$$

that is, if and only if $x = y = 0$. Since all partial derivatives are continuous, then $f'(z)$ exists only for $z = 0$, and

$$f'(0) = \frac{\partial u}{\partial x}(0,0) + i \frac{\partial v}{\partial x}(0,0) = 0.$$

Question 8.

Show that the function

$$f(z) = \sqrt{r}e^{i\theta/2} \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

is differentiable in the indicated domain of definition, and then find $f'(z)$.

Ans: $f'(z) = \frac{1}{2f(z)}.$

SOLUTION: If $f(z) = \sqrt{r}e^{i\theta/2}$, $r > 0$, $\alpha < \theta < \alpha + 2\pi$, then

$$f(z) = \sqrt{r}(\cos \theta/2 + i \sin \theta/2),$$

so that $u(r, \theta) = \sqrt{r} \cos \theta/2$ and $v(r, \theta) = \sqrt{r} \sin \theta/2$.

Now,

$$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}} \cos \theta/2 \quad \text{and} \quad \frac{\partial v}{\partial \theta} = \frac{\sqrt{r}}{2} \cos \theta/2,$$

so that

$$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}} \cos \theta/2 = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

for $r > 0$, $\alpha < \theta < \alpha + 2\pi$.

Also,

$$\frac{\partial u}{\partial \theta} = -\frac{\sqrt{r}}{2} \sin \theta/2 \quad \text{and} \quad \frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}} \sin \theta/2,$$

so that

$$\frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{1}{2\sqrt{r}} \sin \theta/2 = -\frac{\partial v}{\partial r}$$

for $r > 0$, $\alpha < \theta < \alpha + 2\pi$.

The partial derivatives are all continuous for each (r, θ) with $r > 0$, $\alpha < \theta < \alpha + 2\pi$, and the Cauchy-Riemann equations hold for each such point, so that $f'(z)$ exists for all such points, and

$$f'(z) = e^{-i\theta} \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right) = \frac{1}{2\sqrt{r}} (\cos \theta/2 + i \sin \theta/2) e^{-i\theta} = \frac{1}{2\sqrt{r}} e^{-i\theta/2} = \frac{1}{2f(z)},$$

for $r > 0$, $\alpha < \theta < \alpha + 2\pi$.

Question 9.

Show that when $f(z) = x^3 + i(1-y)^3$, it is legitimate to write

$$f'(z) = u_x + i v_x = 3x^2$$

only when $z = i$.

SOLUTION: If $f(z) = x^3 + i(1-y)^3$, then $u(x, y) = x^3$ and $v(x, y) = (1-y)^3$, so that

$$\begin{aligned} \frac{\partial u}{\partial x} &= 3x^2, & \frac{\partial v}{\partial y} &= -3(1-y)^2 \\ \frac{\partial u}{\partial y} &= 0, & \frac{\partial v}{\partial x} &= 0. \end{aligned}$$

The Cauchy-Riemann equations become

$$3x^2 + 3(1 - y)^2 = 0$$

and these hold **only** at the point $z = (0, 1) = i$. Since all partial derivatives are continuous everywhere, then $f'(i)$ exists, and

$$f'(i) = \frac{\partial u}{\partial x}(0, 1) + i \frac{\partial v}{\partial x}(0, 1) = 0.$$

Question 10.

(a) Recall that if $z = x + iy$ then

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}.$$

By *formally* applying the chain rule in calculus to a function $F(x, y)$ of two real variables, derive the expression

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right).$$

(b) Define the operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

suggested by part (a), to show that if the first-order partial derivatives of the real and imaginary parts of a function $f(z) = u(x, y) + i v(x, y)$ satisfy the Cauchy-Riemann equations, then

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} [(u_x - v_y) + i(v_x + u_y)] = 0.$$

Thus derive the *complex form* $\frac{\partial f}{\partial \bar{z}} = 0$ of the Cauchy-Riemann equations.

SOLUTION:

(a) Since $z = x + iy$ and $\bar{z} = x - iy$, then

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i},$$

and

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} = \frac{1}{2} \frac{\partial F}{\partial x} + \left(-\frac{1}{2i}\right) \cdot \frac{\partial F}{\partial y} = \frac{1}{2} \left(\frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right).$$

(b) Now, if $f(z) = u(x, y) + i v(x, y)$ and the real-valued functions u and v satisfy the Cauchy-Riemann equations, then from part (a), we have

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \frac{\partial}{\partial x} (u + i v) + \frac{i}{2} \frac{\partial}{\partial y} (u + i v) = \frac{1}{2} \frac{\partial u}{\partial x} + \frac{i}{2} \frac{\partial v}{\partial x} + \frac{i}{2} \frac{\partial u}{\partial y} - \frac{1}{2} \frac{\partial v}{\partial y},$$

that is,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0,$$

since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.