Math 309 - Spring-Summer 2018

Solutions to Problem Set # 1

Question 1.

In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain squares, and point out which is the principal root:

(a) $(-16)^{1/4}$; (b) $\left(-8-8\sqrt{3}\,i\right)^{1/4}$.

Ans: (a) $\pm\sqrt{2}\,(1+i)$, $\pm\sqrt{2}\,(1-i)$; (b) $\pm(\sqrt{3}-i)$, $\pm(1+\sqrt{3}\,i)$.

SOLUTION:

(a) Note that

$$-16 = 16e^{i[\pi+2k\pi]}$$

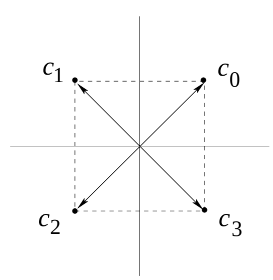
for $k = 0, \pm 1, \pm 2, \ldots$, so the four fourth roots of -16 are

$$c_k = 2e^{i[\pi/4+k\pi/2]}$$

for $k = 0, 1, 2, 3$. Therefore,

$$\begin{aligned}
c_0 &= 2e^{i\pi/4} = 2\left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = \sqrt{2}(1+i) \\
c_1 &= 2e^{i(\pi/4+\pi/2)} = 2e^{i3\pi/4} = 2\left[-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = -\sqrt{2}(1-i) \\
c_2 &= 2e^{i(\pi/4+\pi)} = 2e^{i5\pi/4} = -2\left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = -\sqrt{2}(1+i) \\
c_3 &= 2e^{i(\pi/4+3\pi/2)} = 2e^{i7\pi/4} = -2\left[-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = \sqrt{2}(1-i)
\end{aligned}$$

The four roots are the vertices of a square centered at the origin with side of length $2\sqrt{2}$ as shown in the figure, the principal root is $c_0 = \sqrt{2}(1+i)$.



(b) Note that

$$-8-8\sqrt{3}\,i=-16\left(\frac{1}{2}+\frac{\sqrt{3}\,i}{2}\right)=16e^{i\pi}e^{i\pi/3}=16e^{i4\pi/3}=16e^{i[4\pi/3+2\pi k]}$$

for $k=0,\,\pm1,\,\pm2,\,\dots$, so the four fourth roots of $-8-8\sqrt{3}\,i$ are

$$c_k=2e^{i[\pi/3+\pi k/2]}$$

for $k=0,1,2,3$. Therefore,

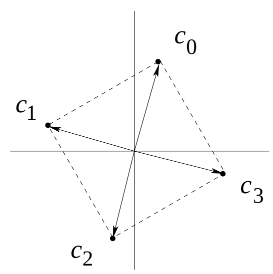
$$c_0=2e^{i\pi/3}=2\left(\frac{1}{2}+\frac{\sqrt{3}\,i}{2}\right)=1+\sqrt{3}\,i$$

$$c_1=2e^{i[\pi/3+\pi/2]}=2e^{i5\pi/6}=2e^{i\pi}e^{-i\pi/6}=-2\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)=-\sqrt{3}+i$$

$$c_2=2e^{i[\pi/3+\pi]}=2e^{i\pi}e^{i\pi/3}=-2\left(\frac{1}{2}+\frac{\sqrt{3}\,i}{2}\right)=-(1+\sqrt{3}\,i)$$

$$c_3=2e^{i[\pi/3+3\pi/2]}=2e^{i11\pi/6}=2e^{i2\pi}e^{-i\pi/6}=2\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)=\sqrt{3}-i$$

The four roots are the vertices of a square centered at the origin with side of length $2\sqrt{2}$ as shown in the figure, the principal root is $c_0=1+\sqrt{3}\,i$.



Note: These roots are probably not in the same order as the roots you found if you used the principal argument of $-8-8\sqrt{3}\,i$ as $-2\pi/3$, and the principal root would be $c_3=\sqrt{3}-i$ in this case.

Question 2.

In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

(a) $(-1)^{1/3}$; (b) $8^{1/6}$.

Ans: (b) $\pm\sqrt{2},\ \pm\dfrac{1+\sqrt{3}\,i}{\sqrt{2}},\ \pm\dfrac{1-\sqrt{3}\,i}{\sqrt{2}}$.

Solution:

(a) Note that

$$-1=e^{i\pi}=e^{i[\pi+2k\pi]}$$

for $k=0,\,\pm1,\,\pm2,\,\dots$, so the three cube roots of -1 are

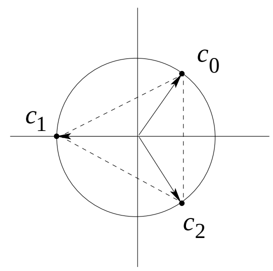
$$c_k=e^{i[\pi/3+2k\pi/3]}$$

for $k=0,1,2$.

Therefore,

$$c_0 = e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}\,i}{2}$$
$$c_1 = e^{i[\pi/3+2\pi/3]} = e^{i\pi} = -1$$
$$c_2 = e^{i[\pi/3+4\pi/3]} = e^{i5\pi/3} = e^{i2\pi}e^{-i\pi/3} = \frac{1}{2} - \frac{\sqrt{3}\,i}{2}$$

The three third roots of -1 are the vertices of an equilateral triangle inscribed in the unit circle, as shown in the figure below, the principal root is $c_0 = \frac{1}{2} + \frac{\sqrt{3}\,i}{2}$.



(b) Note that

$$8 = 8e^{i0} = 8e^{i2k\pi}$$

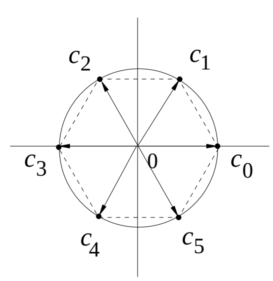
for $k = 0, \pm1, \pm2, \ldots$, so the six sixth roots of 8 are

$$c_k = \sqrt{2}e^{i2k\pi/6} = \sqrt{2}e^{ik\pi/3}$$

for $k = 0, 1, 2, 3, 4, 5$. Therefore,

$$c_0 = \sqrt{2}e^{0} = \sqrt{2}$$
$$c_1 = \sqrt{2}e^{i\pi/3} = \sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}\,i}{2}\right) = \frac{1+\sqrt{3}\,i}{\sqrt{2}}$$
$$c_2 = \sqrt{2}e^{i2\pi/3} = \sqrt{2}e^{i\pi}e^{-i\pi/3} = -\sqrt{2}\left(\frac{1}{2} - \frac{\sqrt{3}\,i}{2}\right) = \frac{-1+\sqrt{3}\,i}{\sqrt{2}}$$
$$c_3 = \sqrt{2}e^{i3\pi/3} = \sqrt{2}e^{i\pi} = -\sqrt{2}$$
$$c_4 = \sqrt{2}e^{i4\pi/3} = \sqrt{2}e^{i\pi}e^{i\pi/3} = -\sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}\,i}{2}\right) = -\frac{1+\sqrt{3}\,i}{\sqrt{2}}$$
$$c_5 = \sqrt{2}e^{i5\pi/3} = \sqrt{2}e^{i2\pi}e^{-i\pi/3} = \sqrt{2}\left(\frac{1}{2} - \frac{\sqrt{3}\,i}{2}\right) = \frac{1-\sqrt{3}\,i}{\sqrt{2}}$$

The six sixth roots of 8 are the vertices of a regular hexagon inscribed in a circle of radius $\sqrt{2}$, and are shown in the figure below.



The principal root is $c_0 = \sqrt{2}$.

Question 3.

Sketch the following sets and determine which are domains:

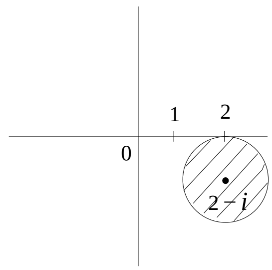
(a) $|z - 2 + i| \leq 1$; (b) $|2z + 3| > 4$; (c) $\text{Im } z > 1$;

(d) $\text{Im } z = 1$; (e) $0 \leq \arg z \leq \pi/4$ $(z \neq 0)$; (f) $|z - 4| \geq |z|$.

Ans: (b), (c) are domains.

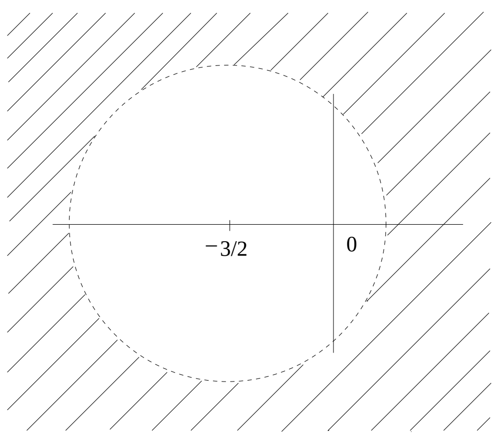
SOLUTION:

(a) The set $A = \{z \in \mathbb{C} : |z - 2 + i| \leq 1\}$ is the closed disk of radius 1 centered at the point $z_0 = 2 - i$, and is *not* a domain.



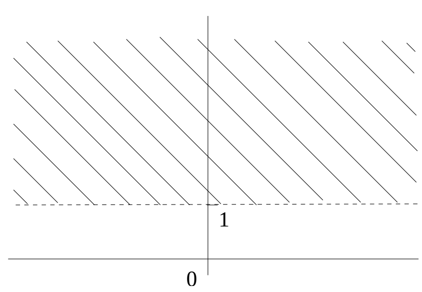
It is connected, but is not open, since for example, the point $z = 2$ is in A, but is not an interior point of A. (For any $\epsilon > 0$, the ϵ-neighborhood of $z = 2$ contains points that are not in A)

(b) The set $B = \{z \in \mathbb{C} : |2z + 3| > 4\}$ is the exterior of the closed disk of radius 2 centered at the point $z_0 = -\dfrac{3}{2}$, and it *is* a domain.



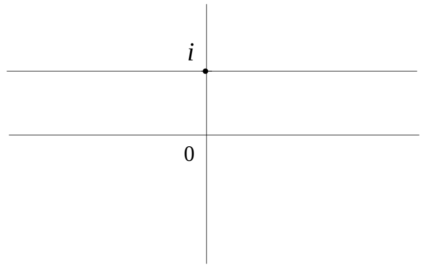
It is open and connected, and is therefore a domain.

(c) The set $C = \{z \in \mathbb{C} : \text{Im } z > 1\}$ is the half-plane $y > 1$, and it *is* a domain.



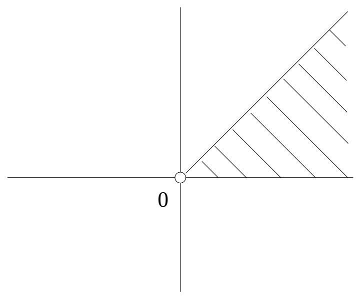
It is open and connected, and is therefore a domain.

(d) The set $D = \{z \in \mathbb{C} : \text{Im } z = 1\}$ is the set of points $z = x + iy$ where $y = 1$, and it is *not* a domain.



It is connected, but it is not open, since for example, the point $z = i$ is not an interior point of D. (For any $\epsilon > 0$, the ϵ-neighborhood of $z = i$ contains points that are not in D)

(e) The set $E = \{z \in \mathbb{C} : 0 \leq \arg z \leq \pi/4\}$ is the set of all nonzero points in the first quadrant between the real axis and the line $y = x$, and it is *not* a domain.



It is connected, but it is not open, since for example, any nonzero point on the real axis is not an interior point of E.

(f) The set $F = \{z \in \mathbb{C} : |z - 4| \geq |z|\}$ is the set of all points z such that the distance from z to 4 is greater than or equal to the distance from z to 0, and this is precisely the set of points $z = x + iy$ such that $x \leq 2$, that is, the half-plane $x \leq 2$.

To see this, note that since the absolute value is a nonnegative real number, then

$$|z - 4| \geq |z| \quad \text{if and only if} \quad |z - 4|^2 \geq |z|^2,$$

that is, if and only if

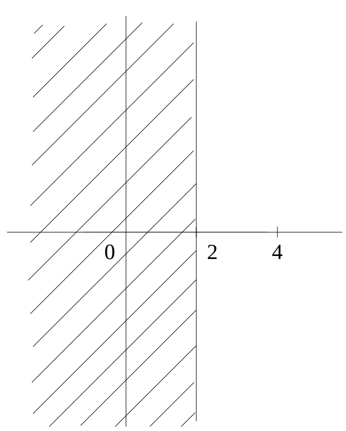
$$(x - 4)^2 + y^2 \geq x^2 + y^2,$$

that is, if and only if

$$-8x + 16 \geq 0,$$

that is, if and only if

$$x \leq 2.$$



Again, the set F is connected but is not open, so that F is *not* a domain.

Question 4.

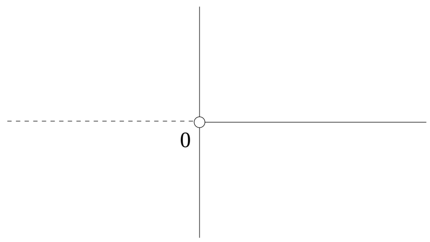
In each case, sketch the closure of the set:

(a) $-\pi < \arg z < \pi \ (z \neq 0)$; (b) $|\text{Re } z| < |z|$;

(c) $\text{Re}\left(\frac{1}{z}\right) \leq \frac{1}{2}$; (d) $\text{Re}\left(z^2\right) > 0$.

SOLUTION:

(a) The set $A = \{z \in \mathbb{C} : -\pi < \arg z < \pi, \ z \neq 0\}$ consists of the entire complex plane **except** for the negative real axis and the point 0.



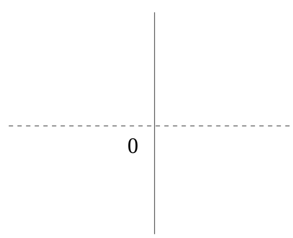
The closure of A is the entire complex plane since the boundary of A is just

$$\text{bdy}(A) = \{z \in \mathbb{C} : z = x, \ x \leq 0\}$$

and $\text{cl}(A) = A \cup \text{bdy}(A) = \mathbb{C}$.

(b) The set $B = \{z \in \mathbb{C} : |\text{Re } z| < |z|\}$ consists of the entire complex plane **except** the real axis $y = 0$, since

$$|x| < \sqrt{x^2 + y^2} \quad \text{if and only if} \quad x^2 < x^2 + y^2 \quad \text{if and only if} \quad y^2 > 0 \quad \text{if and only if} \quad y \neq 0.$$



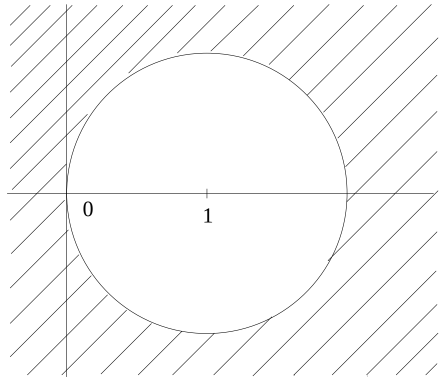
The closure of B is the entire complex plane since the boundary of B is just the real axis

$$\text{bdy}(B) = \{z \in \mathbb{C} : z = x, \ -\infty < x < \infty\}$$

and $\text{cl}(B) = B \cup \text{bdy}(B) = \mathbb{C}$.

(c) The set $C = \left\{z \in \mathbb{C} : \text{Re}\left(\frac{1}{z}\right) \leq \frac{1}{2}\right\}$ consists of the exterior of the open disk centered at $z = 1$ with radius 1, since

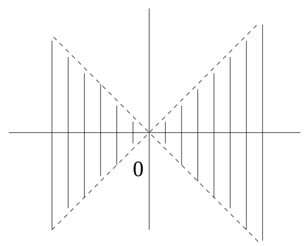
$$\text{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2 + y^2} \leq \frac{1}{2} \quad \text{if and only if} \quad (x-1)^2 + y^2 \geq 1.$$



Since the open disk is an open set, then its complement is closed, and therefore $\mathrm{cl}(C) = C$.

(d) The set $D = \{z \in \mathbb{C} : \mathrm{Re}(z^2) > 0\}$ consists of the points lying strictly between the line $y = x$ and the line $y = -x$, not including the origin, since

$$\mathrm{Re}(x^2 - y^2 + 2ixy) > 0 \quad \text{if and only if} \quad x^2 - y^2 > 0 \quad \text{if and only if} \quad |x| > |y|.$$



The closure is the entire wedge-shaped region since

$$\mathrm{bdy}(D) = \{z \in \mathbb{C} : z = x(1+i),\ -\infty < x < \infty\} \cup \{z \in \mathbb{C} : z = x(1-i),\ -\infty < x < \infty\},$$

and $\mathrm{cl}(D) = \{z \in \mathbb{C} : \mathrm{Re}(z^2) \geq 0\} = \{z \in \mathbb{C} : z = x + iy,\ |x| \geq |y|\}$.

Question 5.

Write the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x, y) + i\,v(x, y)$.

Ans: $(x^3 - 3xy^2 + x + 1) + i\,(3x^2y - y^3 + y)$.

Solution: If $z = x + iy$, then

$$f(z) = (x + iy)^3 + (x + iy) + 1 = (x + iy)(x^2 - y^2 + 2ixy) + x + iy + 1,$$

that is,

$$f(z) = x^3 - xy^2 + 2ix^2y + ix^2y - iy^3 - 2xy^2 + x + iy + 1,$$

that is,

$$f(z) = x^3 - 3xy^2 + x + 1 + i(3x^2y - y^3 + y).$$

Therefore, $f(z) = u(x, y) + iv(x, y)$, where

$$u(x, y) = x^3 - 3xy^2 + x + 1 \qquad \text{and} \qquad v(x, y) = 3x^2y - y^3 + y.$$

Question 6. Suppose that $f(z) = x^2 - y^2 - 2y + i\,(2x - 2xy)$, where $z = x + i\,y$. Use the expressions

$$x = \frac{z + \overline{z}}{2} \qquad \text{and} \qquad y = \frac{z - \overline{z}}{2i}$$

to write $f(z)$ in terms of z and simplify the result.

Ans: $\overline{z}^2 + 2i\,z$.

SOLUTION: We have

$$\begin{aligned} f(z) &= x^2 - y^2 - 2y + i(2x - 2xy) \\ &= x^2 - y^2 - 2ixy + i2x - 2y \\ &= (x - iy)^2 + i(2x + 2iy) \\ &= \overline{z}^2 + 2iz, \end{aligned}$$

so that $f(z) = \overline{z}^2 + 2iz$.

Question 7. Find a domain in the z plane whose image under the transformation $w = z^2$ is the square domain in the w plane bounded by the lines $u = 1$, $u = 2$, $v = 1$, and $v = 2$.

SOLUTION: Under the transformation $w = f(z) = z^2$, that is,

$$\begin{aligned} u &= x^2 - y^2 \\ v &= 2xy, \end{aligned}$$

the vertical line $u = 1$ in the w-plane is the image of the right branch of the hyperbola $x^2 - y^2 = 1$, while the vertical line $u = 2$ in the w-plane is the image of the right branch of the hyperbola $x^2 - y^2 = 2$. Therefore, the vertical strip between $u = 1$ and $u = 2$ is the image under $w = z^2$ of the region between the two hyperbolae $x^2 - y^2 = 1$ and $x^2 - y^2 = 2$.

The horizontal line $v = 1$ in the w-plane is the image of the upper branch of the hyperbola $2xy = 1$, while the horizontal line $v = 2$ in the w-plane is the image of the upper branch of the hyperbola $2xy = 2$. Therefore, the horizontal strip between $v = 1$ and $v = 2$ is the image under $w = z^2$ of the region between the two hyperbolae $2xy = 1$ and $2xy = 2$.

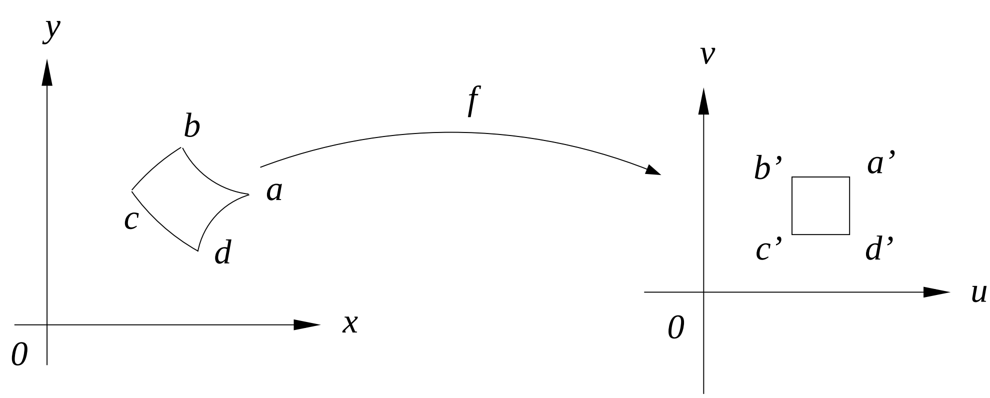
The domain

$$T = \{(u, v) \,:\, 1 < u < 2,\, 1 < v < 2\}$$

in the w-plane is the image under the map $w = z^2$ of the domain

$$S = \{(x, y) \,:\, 1 < x^2 - y^2 < 2\} \cap \{(x, y) \,:\, 1 < 2xy < 2\}$$

in the z-plane. The regions are sketched below.



Question 8. Sketch the region onto which the sector $r \le 1,\ 0 \le \theta \le \pi/4$ is mapped by the transformation

(a) $w = z^2$;   (b) $w = z^3$;   (c) $w = z^4$.

Solution: If $w = \rho e^{i\phi}$, and if $z = re^{i\theta}$, where $r \le 1,\ 0 \le \theta \le \pi/4$, then

(a) For $w = z^2$, we have $w = r^2 e^{i2\theta}$, so that $\rho = r^2$ and $\phi = 2\theta$, and

$$\rho = r^2 \le 1 \qquad \text{and} \qquad 0 \le \phi \le 2\pi/4 = \pi/2.$$

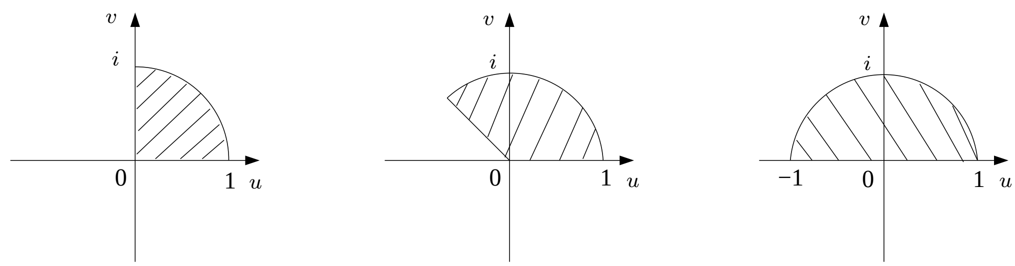
(b) For $w = z^3$, we have $w = r^3 e^{i3\theta}$, so that $\rho = r^3$ and $\phi = 3\theta$, and

$$\rho = r^3 \le 1 \qquad \text{and} \qquad 0 \le \phi \le 3\pi/4.$$

(c) For $w = z^4$, we have $w = r^4 e^{i4\theta}$, so that $\rho = r^4$ and $\phi = 4\theta$, and

$$\rho = r^4 \le 1 \qquad \text{and} \qquad 0 \le \phi \le 4\pi/4 = \pi.$$

The regions are as shown below.



(a)   (b)   (c)

Question 9.

(a) Describe and sketch the set

$$\mathcal{D} = \left\{ z \in \mathbb{C} \mid 2\,\mathrm{Re}(z^2) = |z|^2 \right\}.$$

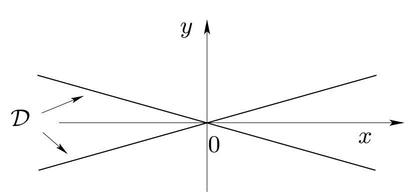
(b) Describe and sketch the set

$$\mathcal{D} = \left\{ z \in \mathbb{C} \mid \mathrm{Im}\left(\frac{1}{z}\right) > 1 \right\}.$$

Solution:

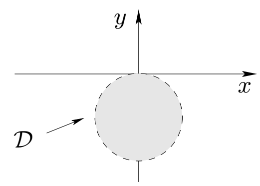
(a) Note that $2\,\mathrm{Re}(z^2) = |z|^2$ if and only if $2(x^2 - y^2) = x^2 + y^2$, if and only if $x^2 = 3y^2$, if and only if, $|y| = \frac{1}{\sqrt{3}}|x|$, if and only if $y = \pm\frac{1}{\sqrt{3}}x$.

Therefore, $z = x + iy$ is in $\mathcal{D}$ if and only if z is on one of the lines $y = \frac{1}{\sqrt{3}}x$ or $y = -\frac{1}{\sqrt{3}}x$, as in the figure below.



(b) Note that $\operatorname{Im}\left(\frac{1}{z}\right) > 1$ if and only if $-\dfrac{y}{x^2+y^2} > 1$, if and only if $-y > x^2+y^2$, if and only if $x^2 + \left(y+\frac{1}{2}\right)^2 < \frac{1}{4}$.

Therefore, $z = x+iy$ is in $\mathcal{D}$ if and only if z is in the **interior** of the disk centered at $\left(0, -\frac{1}{2}\right)$ with radius $\frac{1}{2}$, as in the figure below.



Question 10.

(a) Given a positive integer $n > 2$, find all complex numbers $z \in \mathbb{C}$ satisfying

$$\overline{z} = z^{n-1}.$$

(b) Let ω_n be the primitive n^{th} root of unity given by $e^{\frac{2\pi i}{n}}$, $n \geq 2$, calculate

$$1 + 2\,\omega_n + 3\,\omega_n^2 + \cdots + n\,\omega_n^{n-1}.$$

SOLUTION:

(a) If $z \neq 0$ and $n > 2$, then $\overline{z} = z^{n-1}$ if and only if $z\overline{z} = z^n$, if and only if $|z|^2 = z^n$. However, if $|z|^2 = z^n$, then $|z|^2 = |z^n| = |z|^n$, and since $n > 2$, this implies that $|z| = 1$, and therefore, $z^n = |z|^2 = 1$.

Conversely, if $z^n = 1$, then $|z^n| = |z|^n = 1$, so that $|z| = 1$, and hence $|z|^2 = z^n$.

Thus, the solutions to $\overline{z} = z^{n-1}$ are $z = 0$ **and** the n^{th} roots of unity $z_k = e^{\frac{2\pi i k}{n}}$, $k = 0, 1, 2, \ldots, n-1$.

(b) If $w_n = e^{\frac{2\pi i}{n}}$, then

$$\begin{aligned}(1 + 2\,\omega_n + 3\,\omega_n^2 + \cdots + n\,\omega_n^{n-1})(1-\omega_n) &= 1 + 2\,\omega_n + 3\,\omega_n^2 + \cdots + n\,\omega_n^{n-1} \\ &\quad - \omega_n - 2\,\omega_n^2 - \cdots - (n-1)\,\omega_n^{n-1} - n\,\omega_n^n \\ &= 1 + \omega_n + \omega_n^2 + \cdots + \omega_n^{n-1} - n \\ &= 0 - n = -n.\end{aligned}$$

Therefore,

$$1 + 2\,\omega_n + 3\,\omega_n^2 + \cdots + n\,\omega_n^{n-1} = \frac{n}{\omega_n - 1},$$

since $\omega_n \neq 1$.