Math 309 - Spring-Summer 2018

Problem Set # 1

Question 1.

In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain squares, and point out which is the principal root:

(a) $(-16)^{1/4}$; (b) $\left(-8-8\sqrt{3}\,i\right)^{1/4}$.

Ans: (a) $\pm\sqrt{2}\,(1+i)$, $\pm\sqrt{2}\,(1-i)$; (b) $\pm(\sqrt{3}-i)$, $\pm(1+\sqrt{3}\,i)$.

Question 2.

In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

(a) $(-1)^{1/3}$; (b) $8^{1/6}$.

Ans: (b) $\pm\sqrt{2}$, $\pm\dfrac{1+\sqrt{3}\,i}{\sqrt{2}}$, $\pm\dfrac{1-\sqrt{3}\,i}{\sqrt{2}}$.

Question 3.

Sketch the following sets and determine which are domains:

(a) $|z-2+i|\le 1$; (b) $|2z+3|>4$; (c) $\text{Im } z>1$;
(d) $\text{Im } z=1$; (e) $0\le \arg z\le \pi/4\ (z\ne 0)$; (f) $|z-4|\ge |z|$.

Ans: (b), (c) are domains.

Question 4.

In each case, sketch the closure of the set:

(a) $-\pi<\arg z<\pi\ (z\ne 0)$; (b) $|\text{Re } z|<|z|$;
(c) $\text{Re}\left(\dfrac{1}{z}\right)\le\dfrac{1}{2}$; (d) $\text{Re}\left(z^2\right)>0$.

Question 5.

Write the function $f(z)=z^3+z+1$ in the form $f(z)=u(x,y)+i\,v(x,y)$.

Ans: $(x^3-3xy^2+x+1)+i\,(3x^2y-y^3+y)$.

Question 6.

Suppose that $f(z) = x^2 - y^2 - 2y + i\,(2x - 2xy)$, where $z = x + i\,y$. Use the expressions

$$x = \frac{z + \overline{z}}{2} \qquad \text{and} \qquad y = \frac{z - \overline{z}}{2i}$$

to write $f(z)$ in terms of z and simplify the result.

Ans: $\overline{z}^2 + 2i\,z$.

Question 7.

Find a domain in the z plane whose image under the transformation $w = z^2$ is the square domain in the w plane bounded by the lines $u = 1,\ u = 2,\ v = 1,$ and $v = 2$.

Question 8.

Sketch the region onto which the sector $r \leq 1,\ 0 \leq \theta \leq \pi/4$ is mapped by the transformation

(a) $w = z^2$; (b) $w = z^3$; (c) $w = z^4$.

Question 9.

(a) Describe and sketch the set

$$\mathcal{D} = \left\{\, z \in \mathbb{C} \mid 2\,\mathrm{Re}(z^2) = |z|^2 \,\right\}.$$

(b) Describe and sketch the set

$$\mathcal{D} = \left\{\, z \in \mathbb{C} \mid \mathrm{Im}\left(\frac{1}{z}\right) > 1 \,\right\}.$$

Question 10.

(a) Given a positive integer $n > 2$, find all complex numbers $z \in \mathbb{C}$ satisfying

$$\overline{z} = z^{n-1}.$$

(b) Let ω_n be the primitive n^{th} root of unity given by $e^{\frac{2\pi i}{n}}$, $n \geq 2$, calculate

$$1 + 2\,\omega_n + 3\,\omega_n^2 + \cdots + n\,\omega_n^{n-1}.$$