MATH 309 NOTES

SPRING - SUMMER 2018

by

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I. Introduction

I.a *Fundamental Properties of Complex Numbers*

It was found convenient, for a variety of theoretical and practical questions, to introduce a new "number" j (often also denoted in some books by i) such that $j^2 = -1$. This gives rise to "complex numbers" $z = a + jb$ with a, b real (i.e., ordinary) numbers.

Given a real number $z = a + jb$, we term a the *Real Part of* z (denoted by $a = \mathrm{Re}\,(z)$) and b the *Imaginary Part of* z (denoted by $b = \mathrm{Im}\,(z)$). If $\mathrm{Im}\,(z) = 0$, we say z is a real number. If $\mathrm{Re}\,(z) = 0$, z is termed (purely) imaginary.

The next question is how elementary arithmetical operations (equality, addition, subtraction, division) should be defined for complex numbers. For the first three operations, we are governed by the same rules as for 2-d vectors. Specifically: Let $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$. Then

(1) $z_1 = z_2$ if and only if $a_1 = a_2$ and $b_1 = b_2$,

(2) $z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2)$,

(3) $z_1 - z_2 = (a_1 - a_2) + j(b_1 - b_2)$.

The operations of multiplication and division have no clear counterpart in vector operations. We multiply using distribution, i.e.,

$$\text{(4)}\quad \begin{aligned} z_1 \cdot z_2 &= (a_1 + jb_1)(a_2 + jb_2) = a_1a_2 + ja_1b_2 + jb_1a_2 + j^2b_1b_2 \\ &= (a_1a_2 - b_1b_2) + j(a_1b_2 + b_1a_2) \end{aligned}$$

where $j^2 = -1$ was used. We define division by noting that we can divide by real numbers, so division is turned into a multiplication of complex numbers followed by division by a *real* number. To best see this, it's useful to introduce $\overline{z}$ (also denoted in some books by z^*), which is the complex conjugate of z: If $z = a + jb$, then $\overline{z} = a - jb$. We observe some key properties that connect z and $\overline{z}$:

(a) $\overline{\overline{z}} = z$,

(b) $z\,\overline{z} = (a + jb)(a - jb) = a^2 - jab + jab - j^2b^2 = a^2 + b^2$,

(so $z\,\overline{z}$ is *real. This is a key property*!)

(c) $\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$,

(d) $\overline{z_1 z_2} = \overline{z}_1 \cdot \overline{z}_2$.

So, using $\overline{z}$ we can divide:

$$(5)\quad \frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z}_2}{z_2 \cdot \overline{z}_2} = \frac{(a_1 + jb_1)(a_2 - jb_2)}{a_2^2 + b_2^2} = \frac{(a_1a_2 + b_1b_2)}{a_2^2 + b_2^2} + j\,\frac{(b_1a_2 - b_2a_1)}{a_2^2 + b_2^2}.$$

We consider some examples:

(a) $(1 + j) + (2 - 3j) = (1 + 2) + (1 - 3)j = 3 - 2j$.

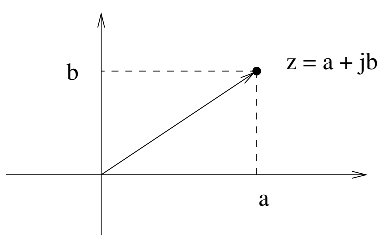
(b) $(17 + j)(1 + 2j) = 17 + 34j + j - 2 = 15 + 35j$.

(c) $\overline{(1 - j)} = 1 + j$.

(d) $\dfrac{1 + 2j}{1 - j} = \dfrac{(1 + 2j)(1 + j)}{(1 - j)(1 + j)} = \dfrac{(1 - 2) + j(2 + 1)}{1^2 + 1^2} = \left(-\dfrac{1}{2}\right) + j\left(\dfrac{3}{2}\right)$.

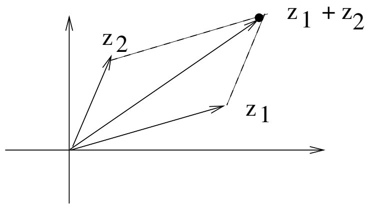
We remark that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$ and if z is real, then $\overline{z} = z$ and the above operations are just the usual ones (for real numbers). Note that division by 0 is still forbidden.

We can now add, subtract, multiply and divide and our next task is to visualize what these operations mean. To do this we introduce the *complex plane*. A plane is clearly needed to plot z (and the operations on z) since z has two parts (a and b) which are not related to each other. So: given a $z = a + jb$ we associate with it the point (a, b) of the (complex) plane.



Note that the x-axis is the "real (number) axis" and the y-axis is the "imaginary (number) axis."

Addition and subtraction, being identical to the same operations for vectors, can then be visualized via the parallelogram law with which we are already familiar.

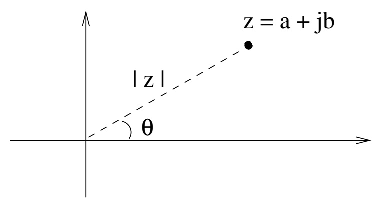


The next question is: how do we visualize multiplication, division and taking the complex conjugate. To do this we use another form of expressing complex numbers: *polar form*.

I.b *Polar Form, Magnitude, Argument, Euler's Formula*

Consider once again $z = a + jb$. Motivated by vectors, we set

$$|z| = \text{magnitude of } z = \sqrt{a^2 + b^2}.$$



The magnitude of z (a *real* number) has the same properties as the magnitude of a vector (notice in particular $|z_1+z_2| \le |z_1|+|z_2|$, and $|z_1-z_2|$ = distance from z_1 to z_2), but it also has a very special connection with multiplication and division as we shall see below. So if $z = a + jb$, then we can write z in polar form by dividing and multiplying by $|z|$, i.e.,

$$z = a + jb = |z| \left(\frac{a}{|z|} + j \, \frac{b}{|z|} \right) = |z|(\cos\theta + j\sin\theta)$$

where θ is shown in the diagram.

We remark right away that θ may be replaced by $\theta \pm 2k\pi$, $k = 0, 1, 2, \ldots$. That is,

$$z = |z|(\cos(\theta \pm 2k\pi) + j\sin(\theta \pm 2k\pi)).$$

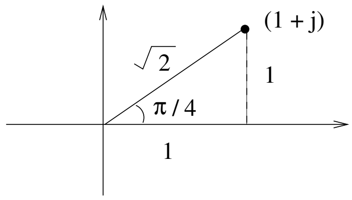
This is due to the periodicity of sin and cos. So there are ∞ many angles for z. We choose the one between $-\pi$ and π and call it the *Principal Argument of* z, denoted by $\mathrm{Arg}\,(z)$. Specifically, $\mathrm{Arg}\,(z)$ is the θ with $-\pi < \theta \le \pi$. We denote the collection of *all possible* θ by $\arg\,(z)$. Thus we have:

$$\arg\,(z) = \mathrm{Arg}\,(z) \pm 2k\pi, \qquad k = 0, 1, \ldots.$$

Note that $\mathrm{Arg}\,(0)$ and $\arg\,(0)$ are *not* defined. Some examples:

Example 1. Let $z = 1 + j$. Write z in polar form, find $|z|$, $\mathrm{Arg}\,(z)$, $\arg\,(z)$.

Answer. $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$,



so

$$
\begin{aligned}
z &= \sqrt{2}\left(\frac{1}{\sqrt{2}} + j\,\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + j\,\sin\frac{\pi}{4}\right) \\
&= \sqrt{2}\left(\cos\left(\frac{\pi}{4} \pm 2k\pi\right) + j\sin\left(\frac{\pi}{4} \pm 2k\pi\right)\right), \qquad k = 0, 1, 2, \ldots.
\end{aligned}
$$

So $|z| = \sqrt{2}$, $\mathrm{Arg}\,(z) = \pi/4$, $\arg\,(z) = \pi/4 \pm 2k\pi$, $k = 0, 1, 2, \ldots$.

We now return to the visualization of $z_1 z_2$. Suppose $z_1 = a_1 + jb_1$, $z_2 = a_2 + jb_2$. To visualize $z_1 z_2$ we write z_1, z_2 in *polar form*:

$$
z_1 = |z_1|(\cos\theta_1 + j\sin\theta_1), \qquad z_2 = |z_2|(\cos\theta_2 + j\sin\theta_2)
$$

and so

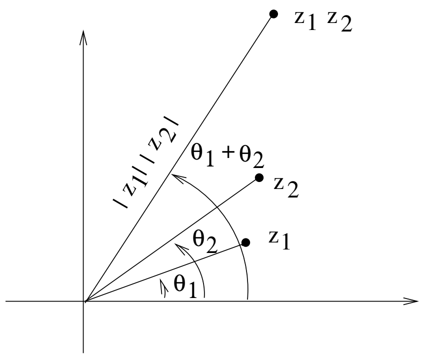
$$
z_1 z_2 = |z_1|\,|z_2|\big([\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2] + j[\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1]\big).
$$

Recalling the trigonometric identities (which we shall not need in the future—see below) gives

$$
z_1 z_2 = |z_1|\,|z_2|[\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)].
$$

So the product $z_1 z_2$ is the number obtained by

1. *multiplying* the magnitudes,
2. *adding* the arguments.



We could do the visualization of division z_1/z_2 in the same way. It looks like we shall need lots of trigonometric identities, but one *key formula* eliminates this need and is very useful for many other results. This is *Euler's Formula.* Let θ be a real number. Then

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}.$$

This formula is shown precisely in Math courses, but from an Engineering point of view it can be justified from the Taylor expansion of both sides:

$$e^{j\theta} = \sum_{n=0}^{\infty} \frac{(j\theta)^n}{n!} = 1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j^3\theta^3}{3!} + \cdots$$

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}}{(2n+1)!} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{(2n)!} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots .$$

Using $j^2 = -1$ gives the result. An Engineering indication that such a result must be true also comes from the differential equation for an LRC circuit. If we solve the equation by setting $i(t) = e^{\alpha t}$ (or $v(t) = e^{\alpha t}$), we find that for some values of L,

R, C the resulting α's are complex. In such cases, direct experimental observations show that $i(t)$ (or $v(t)$) are oscillatory.

Since $\cos\theta$, $\sin\theta$ are periodic, this formula shows that the exponential of a complex number is very different from that of a real number. Note that if a, b are different real numbers, then $e^a \neq e^b$, but here $e^{j\theta} = e^{j(\theta \pm 2k\pi)}$! Since we know $e^{j\theta}$ we can define $e^z = e^{x+jy}$ by letting y play the role θ, i.e.,

$$e^z = e^{x+jy} = e^x[\cos y + j\sin y].$$

This exponential function preserves standard formulas of the "real" exponential: $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$, $e^{z_1-z_2} = e^{z_1}/e^{z_2}$, and yet the function e^z is very different in some ways from e^x as we have noted. We shall consider e^z more later but now we consider some examples to familiarize ourselves with $e^{j\theta}$.

Example 1. Note that $|e^{j\theta}| = 1$ for any θ, $e^{2\pi j} = 1$.

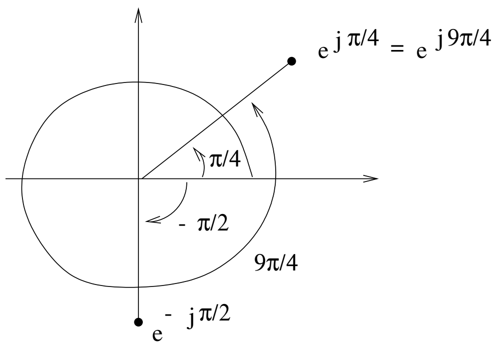
Example 2. Plot $e^{j\pi/4}$, $e^{-j\pi/2}$, $e^{j9\pi/4}$, $7e^{j\pi/4}$.

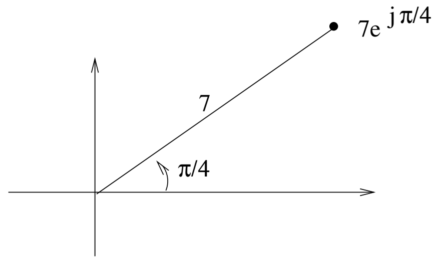
Answer.

$$\begin{aligned}
e^{j\frac{\pi}{4}} &= \cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}, \\
e^{-j\frac{\pi}{2}} &= \cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right) = -j, \\
e^{j\frac{9\pi}{4}} &= \cos\frac{9\pi}{4} + j\sin\frac{9\pi}{4} = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2},
\end{aligned}$$

alternatively,

$$\begin{aligned}
e^{j\frac{9\pi}{4}} &= e^{j(2\pi+\frac{\pi}{4})} = e^{2\pi j} \cdot e^{\frac{\pi}{4}j} = 1 \cdot e^{j\frac{\pi}{4}} \\
7e^{j\frac{\pi}{4}} &= 7\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) = 7\left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right).
\end{aligned}$$





Observe that now we can write:

$$z_1 = a_1 + jb_1 = |z_1|(\cos\theta_1 + j\sin\theta_1) = |z_1|e^{j\theta_1},$$

$$z_2 = a_2 + jb_2 = |z_2|(\cos\theta_2 + j\sin\theta_2) = |z_2|e^{j\theta_2},$$

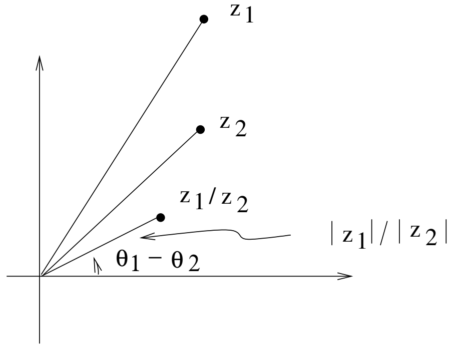
and so

$$\begin{aligned} z_1 \cdot z_2 &= |z_1|\,|z_2|e^{j\theta_1}e^{j\theta_2} = |z_1|\,|z_2|e^{j(\theta_1+\theta_2)} \\ &= |z_1|\,|z_2|\big(\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)\big), \end{aligned}$$

exactly the same as before! In the same way

$$\frac{z_1}{z_2} = \frac{|z_1|e^{j\theta_1}}{|z_2|e^{j\theta_2}} = \frac{|z_1|}{|z_2|}\,e^{j(\theta_1-\theta_2)} = \frac{|z_1|}{|z_2|}(\cos(\theta_1-\theta_2) + j\sin(\theta_1-\theta_2).$$

So we *divide* the magnitudes and *subtract* the angles.



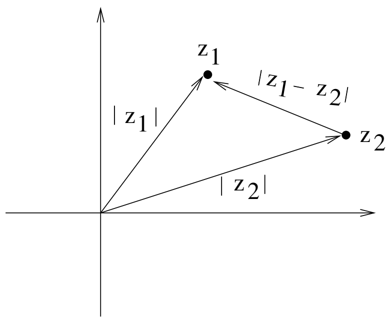
Remark. Note that z_1/z_2, $z_1 \cdot z_2$ can also still be *calculated* as we did at the beginning, particularly if z_1, z_2 are given in the form $a + jb$ rather than $re^{j\theta}$. We can now appreciate what those operations mean, rather than just carrying them out.

We remark, finally, that we have found the following connection between magnitude and the operations of multiplication/division:

(a) $|z_1 z_2| = |z_1|\,|z_2|$,

(b) $|z_1/z_2| = |z_1|/|z_2|$.

Furthermore, since $|z_1 - z_2|$ = distance from z_1, then $|z_1| \le |z_1 - z_2| + |z_2|$ as a consequence of the fact that the shortest distance from the origin to z_1 is along a straight line.



For the same reason,

$$|z_2| \le |z_1 - z_2| + |z_1|$$

and so

$$|z_1| - |z_2| \le |z_1 - z_2|,$$

$$|z_2| - |z_1| \le |z_1 - z_2|.$$

Since *one* of $|z_1| - |z_2|$, $|z_2| - |z_1|$ must be $\big||z_1| - |z_2|\big|$, we conclude

$$\big||z_1| - |z_2|\big| \le |z_1 - z_2|.$$

I.c *Exponential, Logarithm, Trigonometric Functions, Hyperbolic Functions, Roots and Powers*

We introduced the exponential in the last section: Let $z = x + jy$. Then

$$e^z = e^{x+jy} = e^x \cdot e^{jy} = e^x(\cos y + j \sin y).$$

We also noted that

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$$
$$e^{z_1-z_2} = \frac{e^{z_1}}{e^{z_2}}.$$

However, we emphasize that unlike the real case, different z's may have the *same* e^z. For example: if we set $z_k = x + j(y \pm 2k\pi)$ for $k = 0, 1, \ldots,$ then

$$e^{z_k} = e^x \cdot e^{j(y \pm 2k\pi)} = e^{x+jy} = e^{z_0}$$

i.e., e^{z_k} is always the same for any k! So if we know e^z, we can't find z exactly. Note that for a given z, there is *exactly* one e^z.

Motivated by the exponential function, we define $w = \log z$ to be *all* the w's such that $e^w = z$. So let $z = |z|e^{j\theta}$ (note that it's easier to work with the polar form of z than with $z = x + jy$!). We seek *all* w such that

$$e^w = |z|e^{j\theta}.$$

Put $w = u + jv$ (notice: we use this form for w, but the polar form for z), then

$$e^{u+jv} = |z|e^{j\theta}.$$

Take magnitude of both sides:

$$|e^{u+jv}| = |e^u|\,|e^{jv}| = |e^u| = e^u,$$
$$||z|e^{j\theta}| = |z|.$$

And so: $|z| = e^u$, but u and $|z|$ are *real* and thus $u = \mathrm{Log}\,|z|$ (understood as the Log of a *real* number). Then we must have

$$e^{jv} = e^{j\theta}.$$

So we seek *all* v which satisfy this relationship and get

$$v = \theta \pm 2k\pi, \qquad k = 0, 1, 2, \ldots .$$

In conclusion,

$$w = \log z = \mathrm{Log}\,|z| + (\theta \pm 2k\pi)j, \qquad k = 0, 1, 2, \ldots .$$

Before we pass to examples we note: Suppose $\theta = \mathrm{Arg}\,(z)$, i.e., $-\pi < \theta \leq \pi$. Then $\mathrm{Log}\,|z| + \theta j$ is called the *principal value of* $\log z$ and is denoted by $\mathrm{Log}\,(z)$. Observe that there are ∞ many values to $\log z$, but there is only one $\mathrm{Log}\,z$!

Example 1. Evaluate $e^{\frac{\pi}{2}+\frac{\pi}{4}j}$.

Answer.

$$e^{\frac{\pi}{2}+\frac{\pi}{4}j} = e^{\frac{\pi}{2}} \cdot e^{\frac{\pi}{4}j} = e^{\frac{\pi}{2}} \left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) = e^{\frac{\pi}{2}} \left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right).$$

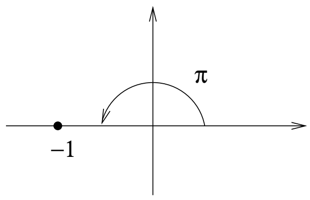
Example 2. Find all z such that $e^z = -1$, i.e., find $\log(-1)$.

Remark. Note that if x is *any* real number, $e^x > 0$, and yet there are z such that

$e^z = -1$!

Answer. Put $z = x + jy$. Then we seek z such that $e^{x+jy} = -1$. We write -1 in *polar form*, i.e., $-1 = |-1|e^{j\pi} = 1 \cdot e^{j\pi}$. Thus

$$e^x \cdot e^{jy} = 1 \cdot e^{j\pi}.$$



Taking magnitudes gives $e^x = 1$ or $x = 0$ (remember: x is *real*). To finish, we need to find *all* y for which

$$e^{jy} = e^{j\pi},$$

and so

$$y = \pi \pm 2k\pi, \qquad k = 0, 1, 2, \ldots .$$

Finally,

$$z = 0 + (\pi \pm 2k\pi)j, \qquad k = 0, 1, 2, \ldots .$$

To define the trigonometric functions, we observe that, for θ *real*

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta.$$

So, adding and subtracting give:

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

We wish to define $\cos z$, $\sin z$, so that *when* z *is real these are the same as* $\cos x$,

$\sin x$. Based on these observations, we define, for $z = x + jy$:

$$\boxed{\cos z = \frac{e^{jz} + e^{-jz}}{2}}, \qquad \boxed{\sin z = \frac{e^{jz} - e^{-jz}}{2j}}.$$

The other trigonometric functions then follow:

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}, \qquad \sec z = \frac{1}{\cos z}, \text{ etc.}$$

Once again the key formulas are preserved. For example, we have the following.

Example 3. Show $\cos^2 z + \sin^2 z = 1$.

Answer.

$$\begin{aligned}\cos^2 z + \sin^2 z &= \frac{(e^{jz} + e^{-jz})^2}{4} + \frac{(e^{jz} - e^{-jz})^2}{-4} \\ &= \frac{(e^{2jz} + 2 + e^{-2jz}) - (e^{2jz} - 2 + e^{-2jz})}{4} = \frac{4}{4} = 1.\end{aligned}$$

Yet there are differences between $\cos z$, $\sin z$ and $\cos x$, $\sin x$. To illustrate this remark and keeping in mind that $-1 \le \cos x, \sin x \le 1$, we shall show later that there are z such that $\sin z = 2$!

Next we recall that

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

and then define:

$$\boxed{\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}}.$$

We can clearly connect $\cosh z$ and $\sinh z$ to the exponential and trigonometric functions, and obtain other useful trigonometric formulas using Euler's formula.

For example,

$$\cosh(jx) = \cos x, \qquad \sinh(jx) = j \sin x,$$

$$\cos z = \cos x \, \cosh y - j \sin x \, \sinh y,$$

etc. Also, since $\cos\theta = (e^{j\theta} + e^{-j\theta})/2$,

$$\cos^n \theta = \left(\frac{e^{j\theta} + e^{-j\theta}}{2} \right)^n .$$

So formulas for $\cos^n \theta$ can be obtained by expanding the right hand side. A similar approach works for $\sin\theta = (e^{j\theta} - e^{-j\theta})/(2j)$.

In an analogous way,

$$(e^{j\theta})^n = e^{jn\theta}.$$

That is,

$$(\cos\theta + j \sin\theta)^n = (\cos(n\theta) + j \sin(n\theta)).$$

And we can get formulas for $\cos(n\theta)$, $\sin(n\theta)$ by expanding the left hand side.

We now pass to the definitions of powers and roots. Suppose first m is a positive integer. We then set $z = |z|e^{j\theta}$,

$$z^m = \underbrace{z \cdots z}_{m \text{ times}} = |z|^m \, e^{jm\theta},$$

and

$$z^{-m} = \frac{1}{z^m} = |z|^{-m} \, e^{-mj\theta}.$$

These are as is expected. Next we look at $z^{1/m}$. It is reasonable to put $z^{1/m}$ to be *all* the complex numbers w such that w^m is the given z. We already know from the real number case that there may be more than one w ($1^{1/2}$ is both $+1$ and -1).

We shall find that *for $z \neq 0$ there are always exactly m mth roots in the complex plane.* To see this, let $z = |z|e^{j\theta}$, $w = |w|e^{j\phi}$ and $w^m = z$. I.e.,

$$|w|^m e^{jm\phi} = |z|e^{j\theta}.$$

We then get $|w|^m = |z|$ or $|w| = |z|^{1/m}$. Since $|w| \geq 0$ and $|z|$, $|w|$ are real numbers, this relationship specifies $|w|$ exactly as $|z|^{1/m}$.

Next, we need $e^{jm\phi} = e^{j\theta}$ and so it seems we should choose $\phi = \theta/m$. This is indeed a correct value, but remember we wish to find *all* w such that $w^m = z$, and keep in mind that $z = |z|e^{j\theta} = |z|e^{j(\theta \pm 2k\pi)}$, $k = 0, 1, \ldots$. So, for w^m to be z it suffices that $m\phi = \theta \pm 2k\pi$ for some k, i.e.,

$$\phi = \frac{\theta \pm 2k\pi}{m}, \qquad k = 0, 1, 2, \ldots.$$

We get ∞ many ϕ, and it looks like we should get ∞ many w as well, but note that

$$w = |z|^{\frac{1}{m}} e^{j(\frac{\theta \pm 2k\pi}{m})}, \qquad k = 0, 1, 2, \ldots.$$

So we get putting $k = 0, 1, 2, \ldots$ in turn:

$$(*) \qquad w = |z|^{\frac{1}{m}} e^{j\frac{\theta}{m}}; \quad |z|^{\frac{1}{m}} e^{j(\frac{\theta+\pi}{m})}; \quad |z|^{\frac{1}{m}} e^{j(\frac{\theta+2\pi}{m})}; \ldots; \quad |z|^{\frac{1}{m}} e^{j(\frac{\theta+(m-1)2\pi}{m})}.$$

The next w would be $|z|^{\frac{1}{m}} e^{j(\frac{\theta+2\pi m}{m})} = |z|^{\frac{1}{m}} e^{\frac{j\theta}{m}}$ and we get again the first root we found. As k increases we simply repeat the roots given in $(*)$, and the same is true for k negative. In summary, the m mth roots of z are

$$|z|^{\frac{1}{m}} e^{j\left(\frac{\theta+2k\pi}{m}\right)} \quad \text{for} \quad k = 0, 1, \ldots, m-1.$$

Before doing examples, some remarks.

Remark 1 There are clearly *two* square roots of $z \neq 0$. Note that if you find *one* of them, call it w_1, then the other w_2 must be $-w_1$. After all, $z = w_1^2 = (-w_1)^2$!

Remark 2 Let m, n be positive integers. Then we set $z^{m/n} = (z^{1/n})^m$ and $z^{-(m/n)} = 1/z^{m/n}$.

Remark 3 For any other number c, not of type m/n (for example, for $\sqrt{2}$, π, e, $1+j$) we define z^c to be $e^{c \log z}$.

Example 4. Find $(1)^{1/3}$.

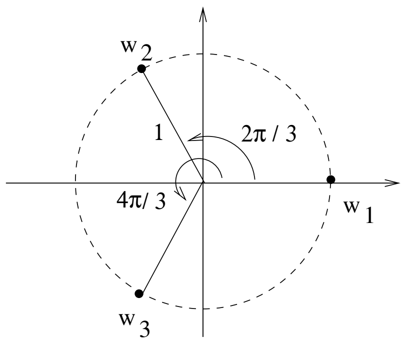
Answer. We know there will be three roots, and we already know one of these is 1. What are the other two? To find them, put $w = |w|e^{j\theta}$, then

$$|w|^3 e^{3j\theta} = 1 = e^{(2\pi k)j}, \qquad k = 0, 1, 2, \ldots .$$

And so $|w| = 1$, $3\theta = 2\pi k$, and as we noted above, we need only use $k = 0, 1, 2$. So

$$w_1 = e^{0j} = 1, \quad w_2 = e^{j\frac{2\pi}{3}}, \quad w_3 = e^{j\frac{4\pi}{3}}.$$

In pictures:



Note that w_1, w_2, w_3 form a symmetric picture on the complex plane: between w_1 and w_2, w_2 and w_3, w_3 and w_1 there is an angle of $2\pi/3$! This symmetry is always found.

Example 5. Find $(1+j)^{\frac{1}{5}}$.

Answer. We expect 5 roots. Again set $w = |w|e^{j\theta}$ and note $1+j = \sqrt{2}e^{j\frac{\pi}{4}}$. Then

$$|w|^5 e^{j5\theta} = \sqrt{2}\, e^{j\frac{\pi}{4}}$$

and so

$$w = 2^{\frac{1}{10}}\, e^{j(\frac{\frac{\pi}{4}+2k\pi}{5})} \quad \text{for} \quad k = 0, 1, 2, 3, 4.$$

The other values of k give repetitions of these roots.

Example 6. Find $(j)^{-\frac{1}{2}}$.

Answer. Note $j^{-1/2} = 1/j^{1/2}$. Since $j^{1/2}$ has 2 roots, there will be 2 values for $j^{-1/2}$.

Now $j = e^{j\frac{\pi}{2}}$ and so $j^{\frac{1}{2}} = e^{j\frac{\pi}{4}}$, $e^{j(\frac{\pi}{4}+\pi)} = -e^{j\frac{\pi}{4}}$. Consequently,

$$j^{-\frac{1}{2}} = e^{-j\frac{\pi}{4}}, \quad -e^{-j\frac{\pi}{4}}.$$

Example 7. Show that the quadratic formula holds, i.e., if

$$az^2 + bz + c = 0$$

with a, b, c complex and $a \neq 0$, then

$$z = -\frac{b}{2a} + \left(\frac{b^2 - 4ac}{4a^2}\right)^{1/2}.$$

(Keep in mind $\left(\frac{b^2-4ac}{4a^2}\right)^{\frac{1}{2}}$ has 2 values.)

Answer. $az^2+bz+c=0$ is the same as $z^2+\frac{b}{a}z+\frac{c}{a}=0$ or

$$\left(z+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}.$$

So z must be such that

$$\left(z+\frac{b}{2a}\right)=\left(\frac{b^2-4ac}{4a^2}\right)^{1/2}.$$

We conclude

$$z=-\frac{b}{2a}+\left(\frac{b^2-4ac}{4a^2}\right)^{1/2}.$$

Example 8. Find $(j+1)^j$.

Answer. Since j is not of type m/n, we have

$$(j+1)^j=e^{j\log(j+1)}.$$

So we need to calculate $\log(j+1)$ first. Now $w=\log(j+1)$ iff $e^w=j+1$. Putting $w=u+jv$, $1+j=\sqrt{2}e^{j\pi/4}$,

$$e^{u+jv}=\sqrt{2}\,e^{j\frac{\pi}{4}}$$

and $e^u=\sqrt{2}$ or $u=\mathrm{Log}\,(\sqrt{2})=\frac{1}{2}\mathrm{Log}\,2$. Next v must be such that

$$e^{jv}=e^{j\frac{\pi}{4}}$$

and so

$$v=\frac{\pi}{4}\pm 2k\pi,\qquad k=0,1,2,\ldots.$$

We have

$$\log(j+1)=\frac{1}{2}\mathrm{Log}\,2+\left(\frac{\pi}{4}\pm 2k\pi\right)j$$

and

$$(j+1)^j = e^{j\left(\frac{\operatorname{Log} 2}{2}+(\frac{\pi}{4}\pm 2k\pi)j\right)} = e^{-\left(\frac{\pi}{4}\pm 2k\pi\right)}\left[\cos\left(\frac{\operatorname{Log} 2}{2}\right)+j\sin\left(\frac{\operatorname{Log} 2}{2}\right)\right].$$

The next example shows a difference between trigonometric functions of complex numbers and those of real numbers.

Example 9. Find all z such that $\sin z = 2$.

Answer. Note that for any real x, $-1 \leq \sin x \leq 1$! Now

$$\sin z = \frac{e^{jz} - e^{-jz}}{2j} = 2$$

and so

$$e^{jz} - e^{-jz} - 4j = 0.$$

Multiply by e^{jz}:

$$(e^{jz})^2 - 4j(e^{jz}) - 1 = 0.$$

By the quadratic formula (earlier example),

$$e^{jz} = 4j + \left((4j)^2 - 4(-1)\right)^{\frac{1}{2}} = 4j + (-16+4)^{\frac{1}{2}} = 4j + (-12)^{\frac{1}{2}}.$$

Now $-12 = 12e^{\pi j}$ and so

$$(-12)^{\frac{1}{2}} = \pm(12)^{\frac{1}{2}}e^{j\frac{\pi}{2}} = \pm(12)^{\frac{1}{2}}j.$$

Here $(12)^{\frac{1}{2}}$ is the unique real number root of 12! So

$$e^{jz} = (4 \pm (12)^{\frac{1}{2}})j = |4 \pm (12)^{\frac{1}{2}}|\, e^{j\frac{\pi}{2}}.$$

Important Remark. Suppose "12" had been replaced by "25" then we would have had:

$$(4 \pm (25)^{\frac{1}{2}})j = (4 \pm 5)j = 9j, \ -j.$$

You would then split the problem since $9j = 9e^{j\frac{\pi}{2}}$, while $-j = e^{-j\frac{\pi}{2}}$! In our case, if $z = x + jy$, then

$$e^{j(x+jy)} = |4 \pm (12)^{\frac{1}{2}}|\, e^{j\frac{\pi}{2}}$$

and so

$$-y = \operatorname{Log}|4 \pm (12)^{\frac{1}{2}}|, \quad x = \frac{\pi}{2} \pm 2k\pi, \qquad k = 0, 1, 2, \dots,$$

i.e.,

$$z = -j\operatorname{Log}|4 \pm (12)^{\frac{1}{2}}| + \left(\frac{\pi}{2} \pm 2k\pi\right).$$

Example 10. Find all z such that $\cos z = 1$.

Answer. We know $\cos x = 1$ iff $x = \pm 2k\pi$ for $k = 0, 1, 2, \dots$. So we expect to find these values of z (i.e., $z = (\pm k\pi)$ and $0j$) and maybe more! Now $\cos z = 1$ means

$$e^{jz} + e^{-jz} = 2$$

or

$$(e^{jz})^2 - 2e^{jz} + 1 = 0.$$

So

$$(e^{jz} - 1)^2 = 0.$$

This means

$$e^{jz} = 1 = e^{0j}$$

and so if $z = x + jy$, then

$$e^{-y} = 1, \quad e^{x} = e^{0j} = e^{\pm 2k\pi} \quad \text{for} \quad k = 0, 1, 2, \dots .$$

I.e., $y = 0$ and $x = \pm 2k\pi$, which yield

$$z = \pm 2k\pi + 0j.$$

So there are no other values except for the “real” ones!

Further Exercises:

Exercise 1. Let $z_1 = 1 - j$, $z_2 = 1 + j$, $z_3 = j$. Find:

(a) $z_1 + z_2$; (b) $z_1 - z_2$; (c) z_1/z_2; (d) $z_1 z_2$; (e) $z_1\overline{z}_3/(z_1 + z_2)$.

Answer.

(a) $z_1 + z_2 = (1 - j) + (1 + j) = (1 + 1) + j(-1 + 1) = 2$.

(b) $z_1 - z_2 = (1 - j) - (1 + j) = -2j$.

(c) $\dfrac{z_1}{z_2} = \dfrac{1-j}{1+j} = \dfrac{(1-j)(1-j)}{(1+j)(1-j)} = \dfrac{1-2j+j^2}{1^2+1^2} = \dfrac{-2j}{2} = -j$.

(d) $z_1 z_2 = (1 - j)(1 + j) = 1 - j + j - j^2 = 1 + 1 = 2$.

(e) $\dfrac{z_1\overline{z}_3}{z_1+z_2} = \dfrac{(1-j)(-j)}{2} = \dfrac{-j+j^2}{2} = \dfrac{-1-j}{2}$.

Exercise 2. Use polar form to find: (a) $(1-j)^{10}$; (b) $\left[\dfrac{1+j}{1-j}\right]^{100}$.

Answer.

(a) $1 - j = \sqrt{2}e^{-\frac{\pi}{4}j}$, so $(1-j)^{10} = (\sqrt{2})^{10}e^{-\frac{10\pi}{4}j} = 2^5 e^{-\frac{10\pi}{4}j}$. Note that $\sqrt{2}$ here means the "real" positive root of 2!

(b) $\left[\dfrac{1+j}{1-j}\right]^{100} = \left[\dfrac{\sqrt{2}e^{\frac{\pi}{4}j}}{\sqrt{2}e^{-\frac{\pi}{4}j}}\right]^{100} = [e^{\frac{\pi}{2}j}]^{100} = e^{50\pi j} = e^{25\cdot 2\pi j} = 1^{25} = 1$.
(Equivalently, $e^{\frac{\pi}{2}j} = j$, so $j^{100} = (j^2)^{50} = (-1)^{50} = 1$)

Exercise 3. Find $(-1)^{1/6}$.

Answer. $(-1) = e^{\pi j} = e^{(\pi+2k\pi)j}$. so $(-1)^{1/6} = e^{\frac{(\pi+2k\pi)}{6}j}$ for $k = 0, 1, \ldots, 5$, i.e.,

$$(-1)^{\frac{1}{6}} = e^{\frac{\pi}{6}j},\ e^{\frac{3\pi}{6}j},\ e^{\frac{5\pi}{6}j},\ e^{\frac{7\pi}{6}j},\ e^{\frac{9\pi}{6}j},\ e^{\frac{11\pi}{6}j}.$$

Note that these roots have magnitude 1 and difference in argument from the preceding one by $2\pi/6$.

Exercise 4. Find $\log(j)$ and $\mathrm{Log}(j)$.

Answer. $j = e^{\frac{\pi}{2}j} = e^{(\frac{\pi}{2} \pm 2k\pi)j}$, $k = 0, 1, 2, \ldots$. So

$$\log(j) = \left(\frac{\pi}{2} \pm 2k\pi\right) j, \quad k = 0, 1, 2, \ldots.$$

Now $\mathrm{Log}(j)$ is the logarithm corresponding to the principal value of the argument of j, i.e., to $\mathrm{Arg}\,(j)$, which is $\pi/2$. So

$$\mathrm{Log}(j) = \frac{\pi}{2} j.$$

Exercise 5. Find all z such that $\cos z = 2$.

Answer. $\cos z = \dfrac{e^{jz} + e^{-jz}}{2}$ and so $\cos z = 2$ iff $\dfrac{e^{jz} + e^{-jz}}{2} = 2$. Multiply by e^{jz} to get: $(e^{jz})^2 + 1 = 4e^{jz}$, i.e., $(e^{jz})^2 - 4e^{jz} + 1 = 0$. Thus

$$e^{jz} = \frac{4 \pm \sqrt{(-4)^2 - 4}}{2} = 2 \pm \sqrt{3}.$$

We recall that by $\sqrt{3}$ we mean the real, positive root of 3. Now

$$2 \pm \sqrt{3} = (2 \pm \sqrt{3})e^{0j} = (2 \pm \sqrt{3})e^{\pm 2k\pi j}, \quad k = 0, 1, 2, \ldots,$$

and if $z = x + jy$, then

$$e^{jx - y} = (2 \pm \sqrt{3})e^{\pm 2k\pi j}.$$

So $x = \pm 2k\pi$ and $e^{-y} = 2 \pm \sqrt{3}$, i.e,

$$-y = \mathrm{Log}(2 \pm \sqrt{3}) \qquad \text{or} \qquad y = -\mathrm{Log}(2 \pm \sqrt{3}).$$

In conclusion, $z = \pm 2k\pi - \mathrm{Log}(2 \pm \sqrt{3})j$, $k = 0, 1, 2, \ldots$.

Remark. Note that Log here means the "real" Log of $2 \pm \sqrt{3}$. This can be done because $\sqrt{3} < 2$. Otherwise, we would have to split the problem. See the earlier notes.

Exercise 6. Find $(1-j)^{1+j}$.

Answer. Since $(1-j)^{1+j} = e^{(\log(1-j))(1+j)}$, we start by finding $\log(1-j)$. Now

$$1 - j = \sqrt{2}e^{[-\frac{\pi}{4} \pm 2k\pi]j}, \quad k = 0, 1, \ldots$$

and so

$$\log(1-j) = \mathrm{Log}(\sqrt{2}) + \left[-\frac{\pi}{4} \pm 2k\pi\right] j.$$

Thus

$$(1-j)^{1+j} = e^{(1+j)\{\mathrm{Log}\sqrt{2} + [-\frac{\pi}{4} \pm 2k\pi]j\}}$$

and

$$\begin{aligned}(1+j)&\left\{\mathrm{Log}(\sqrt{2}) + \left(-\frac{\pi}{4} \pm 2k\pi\right) j\right\} \\ &= \left[\mathrm{Log}(\sqrt{2}) + \left(\frac{\pi}{4} \mp 2k\pi\right)\right] + j\left[-\frac{\pi}{4} \pm 2k\pi + \mathrm{Log}(\sqrt{2})\right]\end{aligned}$$

and finally,

$$(1-j)^{1+j} = e^{[\mathrm{Log}(\sqrt{2}) + \frac{\pi}{4} \mp 2k\pi] + j[-\frac{\pi}{4} \pm 2k\pi + \mathrm{Log}(\sqrt{2})]}.$$

This can be rewritten in various equivalent ways, for example,

$$\begin{aligned}(1-j)^{1+j} = \sqrt{2}e^{\frac{\pi}{4} \mp 2k\pi}&\left[\cos\left(-\frac{\pi}{4} \pm 2k\pi + \mathrm{Log}(\sqrt{2})\right)\right. \\ &\left. + j\left[\sin\left(-\frac{\pi}{4} \pm 2k\pi + \mathrm{Log}(\sqrt{2})\right)\right]\right].\end{aligned}$$

The term $\pm 2k\pi$ in cos and sin functions can also be clearly omitted.

Exercise 7. Solve for z if $z^2 + jz = 1$.

Answer. We can use the quadratic formula:

$$z = \frac{-j \pm \sqrt{j^2+4}}{2} = \frac{-j \pm \sqrt{3}}{2}.$$

We have found

$$z_1 = \frac{\sqrt{3}-j}{2}, \qquad z_2 = \frac{-\sqrt{3}-j}{2}.$$

Note that in this problem it is easy to check the answer, if at all in doubt. In fact,

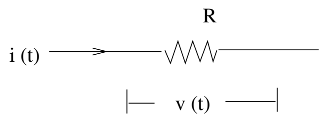
$$z_1^2 + jz_1 = \left(\frac{\sqrt{3}-j}{2}\right)^2 + j\left(\frac{\sqrt{3}-j}{2}\right) = \frac{2-2j\sqrt{3}}{4} + j\left(\frac{\sqrt{3}-j}{2}\right) = 1,$$

as expected.

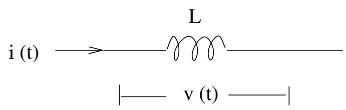
I.d *An Application: Electrical Circuits*

The three fundamental electrical circuit components are: the resistor (of resistance R), the inductor (of inductance L) and the capacitor (of capacitance C). Let t denote time. The voltage drop $v(t)$ across one of these components can be related by observation to the current $i(t)$ flowing through the component according to the formulas:

(a) *for a resistor*: $\quad v(t) = Ri(t)$

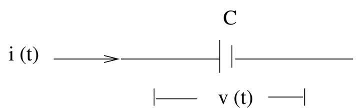


(b) *for an inductor*: $\quad v(t) = L\,\dfrac{di}{dt}$



(c) *for a capacitor*

$$Q(t) = Cv(t) \text{ (where } Q = \text{charge)} \quad \text{or} \quad i(t) = C\,\frac{dv}{dt}.$$



These are the general relationships, and technically speaking could be used—together with Kirchoff's Laws—to derive all that follows by solving differential equations. However, if we deal with sinusoidal current/voltage situations and ignore transients (e.g., charge on capacitor at $t = 0$), then complex number techniques can be used to great advantage since they replace finding a derivative by multiplication and finding an integral by division!

We remark that by using Fourier Series technique, many inputs (square, triangular, rectified sine/cosine, sawtooth, ...) can be expressed as an infinite series of sines/cosines so that sinusoidal inputs are more important than what may be conjectured at first. Specifically, suppose now $v = \mathrm{Im}\,(\widehat{V}e^{jwt})$ with $\widehat{V}$ complex. This corresponds to $v = \mathrm{Im}\,(|\widehat{V}|e^{j(wt+\phi)})$ where $\widehat{V} = |\widehat{V}|e^{j\phi}$, i.e., $v = |\widehat{V}|\sin(wt+\phi)$. Then,

(a) for a *resistor*:

$$Ri(t) = v(t), \quad \text{or} \quad i = \mathrm{Im}\left(\frac{\widehat{V}}{R}\,e^{jwt}\right).$$

So, if we write $i = \mathrm{Im}\,(\widehat{I}e^{jwt})$, then $\widehat{I} = \widehat{V}/R$. Note that this calculation *can be reversed*, i.e., if we know $i = \mathrm{Im}\,(\widehat{I}e^{jwt})$ through the resistor, then $v = \mathrm{Im}\,(\widehat{V}e^{jwt})$ with $\widehat{I}R = \widehat{V}$. We conclude: $\widehat{I}R = \widehat{V}$ for a resistor.

(b) for an *inductor*:

$$L\frac{di}{dt} = v(t) = |\widehat{V}|\,\sin(wt+\phi)$$

and so (since we neglect transients):

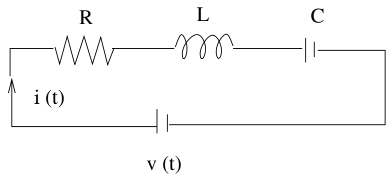
$$\begin{aligned} i &= -\frac{|\widehat{V}|}{Lw}\cos(wt+\phi) = \frac{|\widehat{V}|}{Lw}\sin\left(wt+\phi-\frac{\pi}{2}\right) \\ &= \frac{|\widehat{V}|}{Lw}\,\mathrm{Im}\,\left(e^{j(wt+\phi-\frac{\pi}{2})}\right) = \frac{|\widehat{V}|}{Lw}\,\mathrm{Im}\left(\frac{e^{j(wt+\phi)}}{j}\right) = \mathrm{Im}\left(\frac{\widehat{V}e^{jwt}}{jLw}\right). \end{aligned}$$

If we write $i = \mathrm{Im}\,(\widehat{I}e^{jwt})$, then $\widehat{I} = \widehat{V}/jwL$. Again we can *reverse* this. If we know $i = \mathrm{Im}\,(\widehat{I}e^{jwt})$, then $v = \mathrm{Im}\,(\widehat{V}e^{jwt})$ with $\widehat{V} = \widehat{I}(jwL)$. In summary, $\widehat{V} = \widehat{I}jwL$ for an inductor!

(c) for a *capacitor*: we repeat the same process as we followed for an inductor and conclude $\widehat{I} = \widehat{V}(jwC)$.

These ideas can be combined to deal with more complicated circuits. As a first example we have the following.

Example 1. (**Series Resonance**). Consider the series circuit shown, and suppose $v = \operatorname{Im}(\widehat{V}e^{jwt})$. We seek the current $i(t) = \operatorname{Im}(\widehat{I}e^{jwt})$.



By Kirchoff's Law, $v(t)$ is the sum of the voltage drops across R, L, C. Now across R we have $i = \operatorname{Im}(\widehat{I}e^{jwt})$ and $v_R = \operatorname{Im}(R\widehat{I}e^{jwt})$, across L we have $v_L = \operatorname{Im}(jwL\widehat{I}e^{jwt})$ and across C we have $v_C = \operatorname{Im}((\widehat{I}/jwL)e^{jwt})$ since i is the same. Thus

$$v = v_R + v_C + v_L$$

gives

$$\operatorname{Im}(\widehat{V}e^{jwt}) = \operatorname{Im}(\widehat{I}Re^{jwt}) + \operatorname{Im}(jwt\widehat{I}e^{jwt}) + \operatorname{Im}\left(\frac{\widehat{I}}{jwC}\,e^{jwt}\right).$$

That is,

$$\widehat{V} = \widehat{I}\left(R + jwL + \frac{1}{jwC}\right).$$

We define $Z = \text{impedance} = R + jwL + \dfrac{1}{jwC} = R + j\left(wL - \dfrac{1}{wC}\right)$, and obtain

$$\widehat{I} = \frac{\widehat{V}}{Z} \qquad \text{or}$$

$$i = \operatorname{Im}\left(\frac{\widehat{V}}{Z}\,e^{jwt}\right) = \operatorname{Im}\left(\frac{\widehat{V}}{|Z|}\,e^{j(wt-\phi)}\right)$$

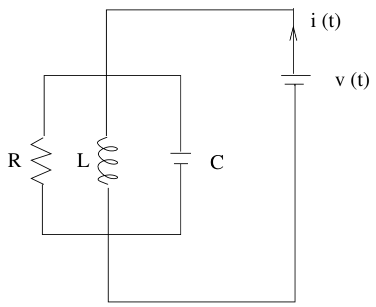
where $Z = |Z|e^{j\phi}$. Now suppose $v = V\sin(wt)$ (with V real note) or, equivalently,

$v = \mathrm{Im}\,(Ve^{jwt})$, then

$$i = \frac{V}{|Z|}\sin(wt-\phi) \quad \text{with} \quad \begin{aligned} |Z| &= \sqrt{R^2 + \left(wL - \frac{1}{wC}\right)^2}, \\ \tan\phi &= \frac{(wL - 1/(wC))}{R}. \end{aligned}$$

Observe that $|Z|$ has a minimum at $wL = 1/(wC)$ (i.e., $w = 1/\sqrt{LC}$) This w is the series circuit *resonance frequency.* For this w, i is maximal and equals $i = (V/R)\sin wt$. It is just as if L and C were not present in the circuit!

Example 2. (**Parallel Resonance**). Consider the parallel circuit shown.



Once again $v = \mathrm{Im}\,(\widehat{V}e^{jwt})$ and we seek $i = \mathrm{Im}\,(\widehat{I}e^{jwt})$. Clearly the same voltage drop occurs across R, L and C and i is the sum of the three currents: through R, through L, through C. Now

$$\widehat{I}_R = \frac{\widehat{V}}{R}, \quad \widehat{I}_L = \frac{\widehat{V}}{jwL}, \quad \widehat{I}_C = \widehat{V}\,jwC$$

and thus

$$\widehat{I} = \widehat{V}\left(\frac{1}{R} + \frac{1}{jwL} + jwC\right).$$

If we write $\widehat{V} = \widehat{I}Z$ or

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{jwL} + jwC = \frac{1}{R} + j\left(wC - \frac{1}{wL}\right),$$

then

$$\begin{cases} \left|\dfrac{1}{Z}\right| = \sqrt{\dfrac{1}{R^2} + \left(wC - \dfrac{1}{wL}\right)^2} \\ \operatorname{Arg}\left(\dfrac{1}{Z}\right) = \tan^{-1}\left(\dfrac{wC - 1/(wL)}{1/R}\right) = \phi, \end{cases} \quad \text{and so} \quad \widehat{I} = \widehat{V} \cdot \frac{1}{Z}.$$

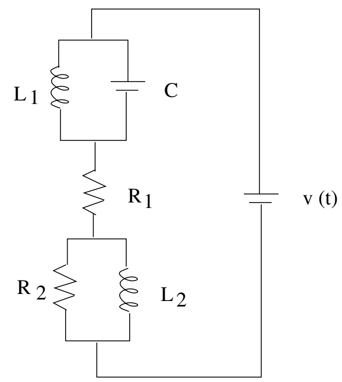
Again if $v(t) = V\sin(wt) = \operatorname{Im}(Ve^{jwt})$, then

$$i(t) = V \cdot \frac{1}{|Z|}\sin(wt+\phi) = V\sqrt{\frac{1}{R^2} + \left(wC - \frac{1}{wL}\right)^2}\,\sin(wt+\phi)$$

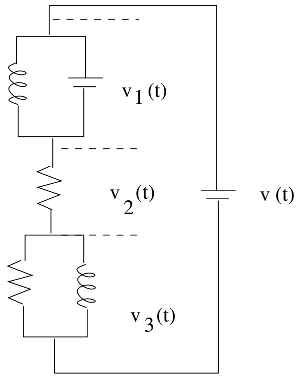
and again if $w = 1/\sqrt{LC}$ (the parallel resonance frequency), then $i = (V/R)\sin wt$. Except, unlike for the series circuit case, this frequency gives a minimum for i, *not* a maximum.

We conclude by remarking that one of the main advantages of using complex numbers for circuit analysis is that this method allows the simple decomposition of complicated circuits into a collection of more elementary circuits. As an example we have

Example 3. Consider the circuit shown. If $v(t) = \operatorname{Im}(\widehat{V}e^{jwt})$ find $i(t)$.



Answer. We split the problem into sub-problems and find the Z for each.



Now $v(t) = v_1(t) + v_2(t) + v_3(t)$. Suppose $i = \operatorname{Im}(\widehat{I}e^{jwt})$. Then

$$\begin{aligned}
v_1(t) &= \operatorname{Im}(\widehat{I}Z_1e^{jwt}) \quad \text{with} \quad \frac{1}{Z_1} = \frac{1}{jwL_1} + jwC, \\
v_2(t) &= \operatorname{Im}(\widehat{I}Z_2e^{jwt}) \quad \text{with} \quad Z_2 = R_1, \\
v_3(t) &= \operatorname{Im}(\widehat{I}Z_3e^{jwt}) \quad \text{with} \quad \frac{1}{Z_3} = \frac{1}{R_2} + \frac{1}{jwL_2}.
\end{aligned}$$

So

$$\operatorname{Im}(\widehat{V}e^{jwt}) = \operatorname{Im}\left(\widehat{I}(Z_1 + Z_2 + Z_3)e^{jwt}\right)$$

and so

$$i(t) = \operatorname{Im}\left(\frac{\widehat{V}}{Z}e^{jwt}\right)$$

with

$$Z = Z_1 + Z_2 + Z_3 = \frac{1}{j(wC - \frac{1}{wL_1})} + R_1 + \frac{R_2 jwL_2}{R_2 + jwL_2}.$$

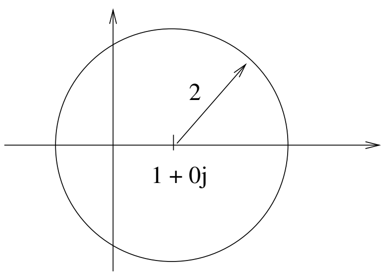
I.e *Graphs, Linear and Bilinear Maps*

We recall that the graph of $y = f(x)$ was of considerable importance in analyzing the properties of f. Graphs are also of significance for complex functions but the situation is more complicated. As we shall see, to really "graph" a complex function we would need four dimensions.

We begin with a simpler problem, that is: find all those z (if any) on the complex plane which satisfy some given relationship. This can also be tough to do, and there are no general rules as to how one should go about it. We present several examples to show how this problem is to be approached.

Example 1. Find all those z such that $|z - 1| = 2$.

Answer. The "honest" way—and the more general way—is as follows: put $z = x + jy$ then $z - 1 = (x - 1) + jy$ and $|z - 1|^2 = (x - 1)^2 + y^2 = 4$. So we get a circle, centered at $x = 1$, $y = 0$ of radius 2. The best way to do this specific problem is to note that $|z - 1| =$ distance from z to 1. So we find all those points z that are 2 units distant from 1. Again a circle of radius 2 centered at 1.



Example 2. Find all z such that

$$\frac{|z - j|}{|z + j|} = 4.$$

Answer. Except for complex variable experts, one should put $z = x + jy$. Then

$$\frac{|z-j|}{|z+j|} = \frac{|x+j(y-1)|}{|x+j(y+1)|} = 4.$$

We square both sides and get

$$x^2 + (y-1)^2 = 4(x^2 + (y+1)^2)$$

or

$$x^2 + y^2 - 2y + 1 = 4x^2 + 4y^2 + 8y + 4.$$

I.e.,

$$3x^2 + 3y^2 + 10y + 3 = 0$$

or

$$x^2 + \left(y + \frac{5}{3}\right)^2 = -1 + \frac{25}{9} = \frac{16}{9},$$

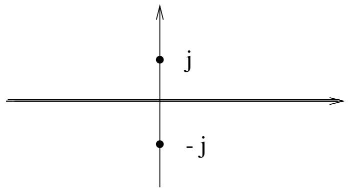
again a circle, centered at $0 - \frac{5}{3}j$ of radius $4/3$.

You should not think that the above problem always has circles for solutions. To see this note:

Example 3. Find all z such that

$$\frac{|z-j|}{|z+j|} = 1.$$

Answer. Now we have $x^2 + (y-1)^2 = x^2 + (y+1)^2$. I.e., $4y = 0$ or $y = 0$. We have no restrictions on x. So $z = x + 0j$, i.e., the x-axis. This makes sense since we are asking for those z which are *equidistant* from j and $-j$!



Example 4. Find all z such that

$$|z - j| + |z + j| = 4.$$

Answer. We want all those z such that the *sum* of the distances from z to j and from z to $-j$ gives 4. We expect therefore to *get an ellipse*. Now we have

$$\sqrt{x^2 + (y-1)^2} + \sqrt{x^2 + (y+1)^2} = 4$$

or

$$x^2 + (y-1)^2 + x^2 + (y+1)^2 + 2\sqrt{x^2 + (y-1)^2}\,\sqrt{x^2 + (y+1)^2} = 16.$$

That is,

$$x^2 + y^2 + 1 + \sqrt{x^2 + (y-1)^2}\,\sqrt{x^2 + (y+1)^2} = 8$$

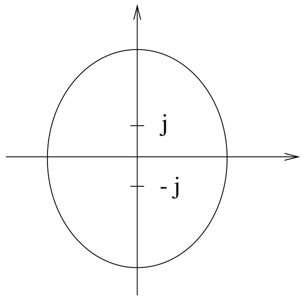
or

$$\sqrt{x^2 + (y-1)^2}\,\sqrt{x^2 + (y+1)^2} = 7 - x^2 - y^2.$$

Squaring again gives

$$16x^2 + 12y^2 = 48$$

which is an ellipse.



One needs again to be a little careful.

Example 5. Find all z such that

$$|z-j|+|z+j|=1.$$

Answer. Since the distance from $-j$ to j is 2, there are no z's whose distance from z to j is *plus* the distance from z to $-j$ can be 1 (otherwise j itself would have to be closer than 2 units from $-j$!). Let us see what happens in the calculations now. We get, again

$$x^2+(y-1)^2+x^2+(y+1)^2+2\sqrt{x^2+(y-1)^2}\,\sqrt{x^2+(y+1)^2}=1,$$

which gives

$$x^2+y^2+1+\sqrt{x^2+(y-1)^2}\,\sqrt{x^2+(y+1)^2}=\frac{1}{2}$$

or

$$\sqrt{x^2+(y-1)^2}\,\sqrt{x^2+(y+1)^2}=-\frac{1}{2}-x^2-y^2.$$

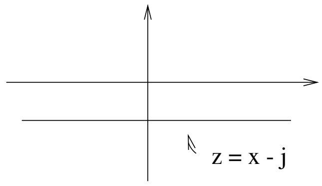
Now this is impossible, since the left hand side is nonnegative, while the right hand side is negative for any (x,y)! So no (x,y) work.

Now for some different examples.

Example 6. Find all z such that

$$\operatorname{Im}(z - j) = -2.$$

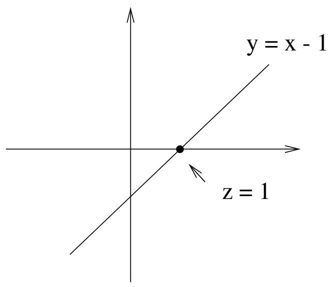
Answer. Put $z = x + jy$. Then $z - j = x + j(y-1)$ and $\operatorname{Im}(z - j) = y - 1$. We thus require $y - 1 = -2$ or $y = -1$. Note that x is not restricted. So $z = x - j$, that is, a straight line.



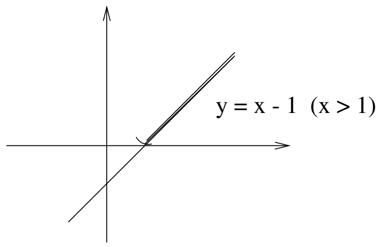
Example 7. Find all those z such that

$$\operatorname{Arg}(z - 1) = \frac{\pi}{4}.$$

Answer. Put $z = x + jy$. Then $z - 1 = (x-1) + jy$ and $\pi/4 = \operatorname{Arg}(z-1) = \phi$ where $\tan\phi = y/(x-1)$. We have $y/(x-1) = \tan\phi = \tan(\pi/4) = 1$ and so $y = x - 1$. It appears that we got the entire straight line, but this is *not* right. The problem comes from the fact that $\tan(-\pi/4)$ is also 1, so that the line $y = x - 1$ includes both z where $\operatorname{Arg}(z-1) = \pi/4$ *and* those z where $\operatorname{Arg}(z-1) = -\pi/4$.



We must remove the latter (we also must be careful at $x = 1$, since we can't divide by 0)! To see how to do this, note that $\text{Arg}\,(z-1) = \pi/4$ also implies that $z-1$ must be on the first quadrant. So $z-1 = x-1+jy$ must have: $x-1 > 0$, $y > 0$. Of course we also have $y = x-1$ from before. So, in summary, we have found $y = x-1$ with $x > 1$ is the 1/2 line we seek.

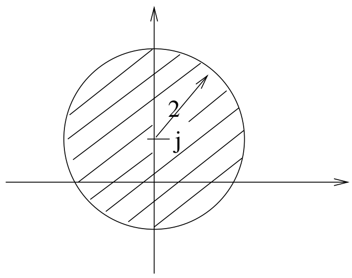


Note that at $x = 1$, $y = 0$ we get $z - 1 = 0$ and $\text{Arg}\,(0)$ is not defined.

Not all graphs are curves.

Example 8. Find all z such that $|z - j| \le 2$.

Answer. Clearly we seek all z whose distance from j is ≤ 2. We thus find a *disc* of radius 2 centered at j.



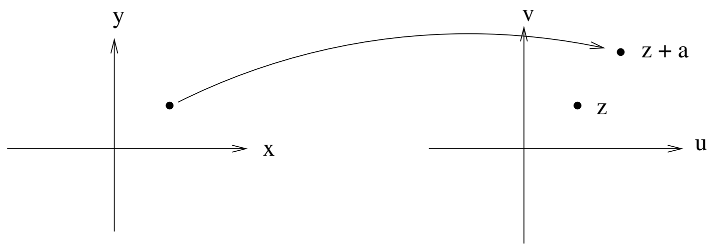
We now pass to the problem of "graphing" functions of z. That is, we suppose $w = f(z)$ and we try to get a "picture" of w. Note that if $z = x + jy$, then in general w will also be complex so we set $w = u + jv$. For example, if $w = z^2$, then

$(u + jv) = (x + jy)^2 = (x^2 - y^2) + 2xyj$. So $u = x^2 + y^2$, $v = 2xy$. To really get a picture of this, we would need 4 axes (as mentioned above): one each for x, y, u, v. This is impossible, nevertheless we can get an idea of the "graph" of f by seeing how f maps certain curves in the (x, y) plane to (different) curves in the (u, v) plane.

We illustrate this process with the simplest functions: the linear maps. Consider the function $w = z+a$. That is: Let $a = \alpha+\beta j$. Then $u+jv = (x+jy)+(\alpha+\beta j)$ and so

$$\begin{cases} u = x + \alpha \\ v = y + \beta. \end{cases}$$

To see what this map does, observe that w takes any z to $z+a$. That is, w *translates every point* of the plane by a.

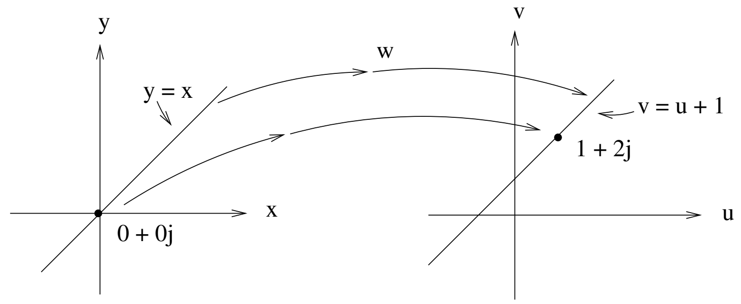


We conclude that any straight line in the (x, y) plane gets mapped by w to a straight line (usually *not* the same one) in the (u, v) plane, and the same is true for circles. As an example we have

Example 9. Find the image of the line $y = x$ and of the circle $(x-1)^2+(y-2)^2 = 4$ under the map $w = z + (1 + 2j)$.

Answer. Put $z = x + jy$, $w = z + (1 + 2j)$. So here $a = 1 + 2j$ and $u = x + 1$, $v = y + 2$. To find what happens to the line $y = x$ we replace y and x in this equation by u and v. I.e., $u - 1 = x = y = v - 2$, and so $u - 1 = v - 2$ or $v = u + 1$.

In pictures:



Note that the point $0 + j0$ in the (x, y) plane gets mapped to $1 + 2j$ in the (u, v) plane.

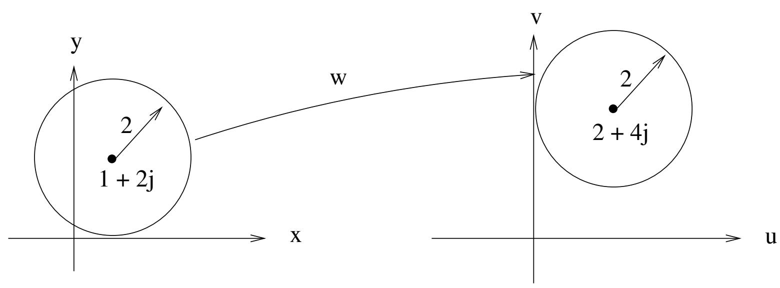
We next consider what happens to the circle $(x-1)^2 + (y-2)^2 = 4$. Again we replace (x, y) by (u, v) and get

$$((u-1)-1)^2 + ((v-2)-2)^2 = 4.$$

That is,

$$(u-2)^2 + (v-4)^2 = 4.$$

As might have been expected, the circle remains a circle with the same radius, but the center has been "translated" to $(2, 4)$.



The next simplest map is a multiplication by a constant: $w = az$ with $a = \alpha + j\beta$. Now $u + jv = (\alpha + j\beta)(x + jy)$ or $u = \alpha x - \beta y$, $v = \alpha y + \beta x$. To see what this map does, write $z = |z|e^{j\theta}$, $a = |a|e^{j\phi}$. Then $w = |a|\,|z|e^{j(\theta+\phi)}$. So under w, the point z gets mapped to a point with magnitude $|a|\,|z|$ (i.e., *dilation* or *magnification* by $|a|$) and Argument $\theta + \phi$ (i.e., *rotation* by ϕ). So $w = az$ is a *dilation* or *magnification* by $|a|$ and a *rotation by* $\mathrm{Arg}\,(a)$ of every z. What does this do to lines and circles? A line $y = mx + b$ is mapped to the (u, v) plane as

$$\frac{\alpha v - \beta u}{\alpha^2 + \beta^2} = m\left(\frac{\alpha u + \beta v}{\alpha^2 + \beta^2}\right) + b$$

or

$$\alpha v - \beta u = m(\alpha u + \beta v) + b(\alpha^2 + \beta^2).$$

Here we have used the fact that $x = (\alpha u + \beta v)/(\alpha^2 + \beta^2)$, $y = (\alpha v - \beta u)/(\alpha^2 + \beta^2)$. So lines stay lines, but rotated and moved (by the magnification). Note that lines through the origin remain lines through the origin! Next suppose we have a circle $(x - x_0)^2 + (y - y_0)^2 = r^2$. We could replace (x, y) by (u, v) just like before, but it is easier to deal with circles by noticing $(x - x_0)^2 + (y - y_0)^2 = r^2$ is the same as $|z - z_0| = r$ where $z_0 = x_0 + jy_0$. Then

$$|a|r = |a|\,|z - z_0| = |az - az_0| = |w - az_0|.$$

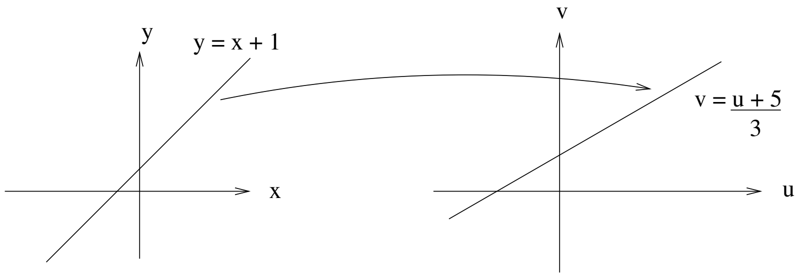
So the circle remains a circle with center az_0 and radius $|a|r$.

Example 10. Find the image of the line $y = x + 1$ and the circle $x^2 + (y - 1)^2 = 2$ under the map $w = (2 - j)z$.

Answer. $w = (2 - j)z = (2 - j)(x + jy) = (2x + y) + j(2y - x)$. So $u = 2x + y$, $v = 2y - x$, or equivalently,

$$x = \frac{2u - v}{5}, \qquad y = \frac{u + 2v}{5}.$$

The line $y = x + 1$ becomes $\dfrac{u+2v}{5} = \dfrac{2u-v}{5} + 1$ or $v = \dfrac{u+5}{3}$.



Now for the circle $x^2+(y-1)^2 = 2$, we rewrite this as $|z-j| = \sqrt{2}$ or $|a||z-j| = \sqrt{2}\,|a|$ where $a = 2-j$ and $|a| = \sqrt{5}$. We conclude $|az-aj| = \sqrt{10}$ or $|w-(1+2j)| = \sqrt{10}$, and the original circle is mapped to a circle centered at $1+2j$ of radius $\sqrt{10}$.

Let us check this by using direct calculations. We have: $x^2 + (y-1)^2 = 2$ becomes

$$\left(\frac{2u-v}{5}\right)^2 + \left(\frac{u+2v}{5} - 1\right)^2 = 2.$$

That is,

$$\left(\frac{4u^2-4uv+v^2}{25}\right) + \left(\frac{u^2+4uv+4v^2-10u-20v+25}{25}\right) = 2$$

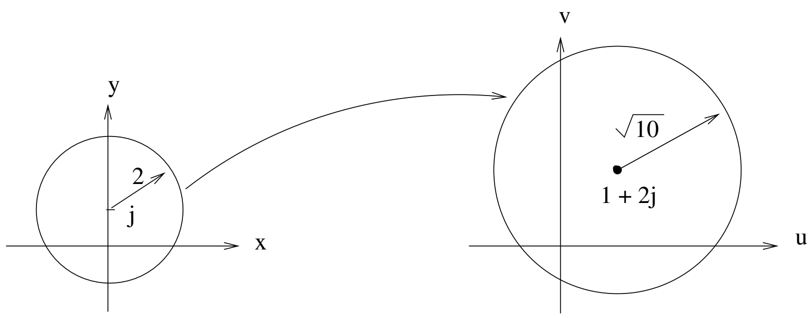
or

$$u^2+v^2-2u-4v = 5,$$

i.e.,

$$(u-1)^2+(v-2)^2 = 10.$$

Just like before!



Observe that a map $w = az + b$ can then be seen as a composition of the earlier maps

$$z \xrightarrow[\text{rotation}]{\text{magnification}} az \xrightarrow{\text{translation}} az + b.$$

So w is a composition of a magnification, followed by a rotation followed by a translation. Note that the *order* is important. A translation (by b) *followed* by rotation and magnification would be the function

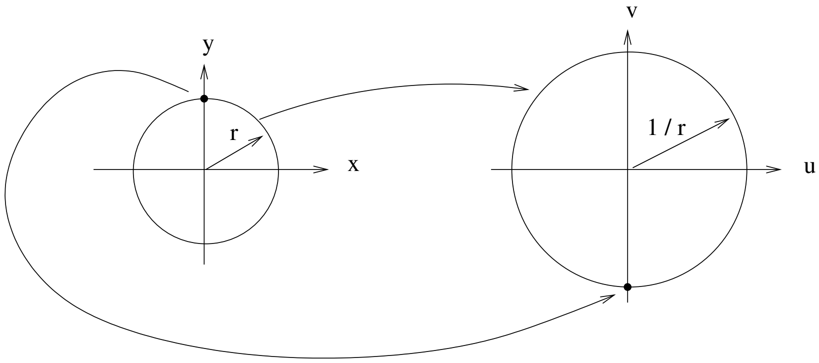
$$w_1 = a(z + b).$$

And w_1 is different from w!

The next map we consider is the *inversion* $w = 1/z$. If we let, one more time, $w = u + jv$, $z = x + jy$, we get $u + jv = (x - jy)/(x^2 + y^2)$ or

$$u = \frac{x}{x^2 + y^2}, \qquad v = \frac{-y}{x^2 + y^2}.$$

We shall use this later. Observe first that $w = 1/z = (1/|z|)e^{-j\theta}$ where $z = |z|e^{j\theta}$. So if $w = |w|e^{j\phi}$, then $|w| = 1/|z|$ and $\phi = -\theta$. The map $w = 1/z$ thus maps a circle of radius r centered at the origin (i.e., $|z| = r$) to a circle of radius $1/r$ also centered at the origin, but the individual points of the circle get mapped to points with negative angles!



To see what happens in general, note that we can get any circle or straight line into the form

$$a(x^2 + y^2) + bx + cy + d = 0$$

with a, b, c, d real numbers. This expression represents:

$$\text{if } a \neq 0 \ \longrightarrow \text{circle} \begin{cases} d = 0 & \text{through the origin} \\ d \neq 0 & \text{not through the origin} \end{cases}$$

$$\text{if } a = 0 \ \longrightarrow \text{straight line} \begin{cases} d = 0 & \text{through the origin} \\ d \neq 0 & \text{not through the origin.} \end{cases}$$

Now $u = x/(x^2 + y^2)$, $v = -y/(x^2 + y^2)$ imply $x = u/(u^2 + v^2)$, $y = -v/(u^2 + v^2)$. So the expression $a(x^2 + y^2) + bx + cy + d$ gets mapped to

$$a \left(\frac{1}{u^2 + v^2} \right) + b \left(\frac{u}{u^2 + v^2} \right) - c \left(\frac{v}{u^2 + v^2} \right) + d = 0$$

or

$$a + bu - cv + d(u^2 + v^2) = 0.$$

So circles not through the origin in (x, y) plane become circles not through the origin in (u, v) plane since a, d are not zero; circles through the origin in (x, y) plane become lines not through the origin in the (u, v) plane since $a \neq 0$ but $d = 0$;

lines through the origin in (x, y) plane become lines through the origin in the (u, v) plane since $a = d = 0$; and finally lines not through the origin become circles through the origin since $a = 0$, $d \neq 0$.

Example 11. Find the image of $x^2 + y^2 + 2x + 2y = 2$ under $w = 1/z$.

Answer. $x^2 + y^2 + 2x + 2y = 2$ is the same as

$$(x + 1)^2 + (y + 1)^2 = 4,$$

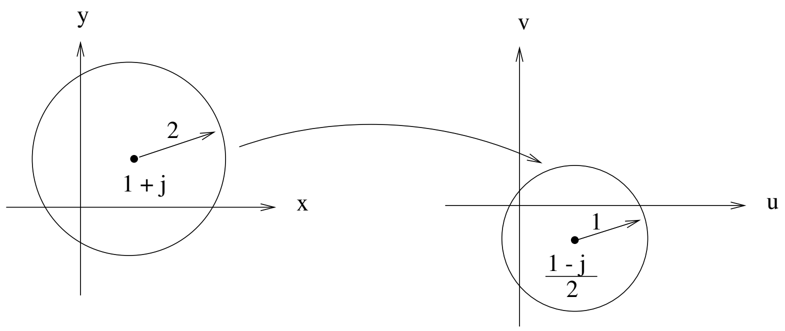
that is, a circle not through the origin (since $x = 0$, $y = 0$ do not satisfy the equation). We expect the image to also be a circle *not* through the origin. We find $x^2 + y^2 + 2x + 2y = 2$ becomes

$$\frac{1}{u^2 + v^2} + \frac{2u}{u^2 + v^2} - \frac{2v}{u^2 + v^2} = 2.$$

Equivalently,

$$1 + 2u - 2v = 2(u^2 + v^2) \qquad \text{or} \qquad \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = 1.$$

In pictures:



The final example of this kind that we present is the *bilinear map*:

$$w = \frac{az + b}{cz + d}$$

with a, b, c, d *complex* and c, d not both zero. We can decompose the bilinear map into a sequence of earlier maps as follows: First note

$$w = \frac{\frac{a}{c}(cz+d) + b - \frac{ad}{c}}{cz+d} = \frac{a}{c} + \frac{\frac{cb-ad}{c}}{cz+d}.$$

So we have

$$\begin{array}{llllll} z \xrightarrow[\text{rotation}]{\text{magnification}} & cz & \xrightarrow{\text{translation}} & cz+d & \xrightarrow{\text{inversion}} & \dfrac{1}{cz+d} \\ \xrightarrow[\text{rotation}]{\text{magnification}} & \dfrac{\frac{cb-ad}{c}}{cz+d} & \xrightarrow{\text{translation}} & \dfrac{a}{c} + \dfrac{\frac{cb-ad}{c}}{cz+d} & & . \end{array}$$

So this is quite a complicated map to visualize. There are some simple properties that we mention.

(1) Given any 3 different numbers z_1, z_2, z_3 on the z plane and 3 different numbers w_1, w_2, w_3 on the w plane we can find a bilinear map w such that

$$w(z_1) = w_1, \quad w(z_2) = w_2, \quad w(z_3) = w_3.$$

There are two ways to do this: one way is just to solve brutally for the a, b, c, d that will do the job. The sneaky way is this:

$$\frac{(w-w_1)}{(w-w_2)}\frac{(w_3-w_2)}{(w_3-w_1)} = \frac{(z-z_1)}{(z-z_2)}\frac{(z_3-z_2)}{(z_3-z_1)}$$

and then solve for w. The idea behind the sneaky way is this: when $z = z_1$, the right hand side is zero and so the left hand side is zero too, and this means $w = w_1$. When $z = z_2$, the left hand side is infinite, so is the right hand side and thus $w = w_2$. Finally when $z = z_3$ the right side is one, then the left hand side is one too and $w = w_3$.

(2) If $w = (az+b)/(cz+d)$, then the inverse of w (i.e., the map which gives the original z for a given value of w) is also a bilinear transformation, usually denoted by w^{-1}. (Don't get mixed up between this and $1/w$!) To find w^{-1},

just invert z and w, i.e.,

$$z = \frac{aw + b}{cw + d},$$

or

$$z(cw + d) = aw + b.$$

And so

$$w = \frac{b - zd}{zc - a}.$$

We have $w^{-1} = (b - zd)/(zc - a)$. Equivalently, solve for z (in terms of w) in $w = (az + b)/(cz + d)$, then $z = (b - wd)/(wc - a)$, the same thing.

(3) Fixed points of w are those values of z (if any) such that $w = z$, i.e., $z = (az+b)/(cz+d)$. Note that there may not be any fixed points (e.g., $w = z+j$) or all points may be fixed points (e.g., $w = z$).

We illustrate these remarks with examples.

Example 12. Find a bilinear map which maps 1, 0, j to 0, j, $-j$, respectively.

Answer. Here

$$\begin{cases} z_1 = 1, & z_2 = 0, & z_3 = j \\ w_1 = 0, & w_2 = j, & w_3 = -j, \end{cases}$$

and so

$$\frac{(w - 0)(-j - j)}{(w - j)(-j - 0)} = \frac{(z - 1)(j - 0)}{(z - 0)(j - 1)}.$$

That is,

$$\frac{-2jw}{(w - j)(-j)} = \frac{(z - 1)\, j}{z(j - 1)}.$$

Simplifying gives

$$w = \frac{w-j}{2} \, \frac{z-1}{z} \, \frac{j}{j-1}.$$

We find

$$w(j-1)2z = w(z-1)j + (z-1).$$

Finally,

$$w = \frac{z-1}{z(j-2)+j}.$$

We check:

$$\begin{aligned}
\text{when} \quad z &= 1, \qquad w = 0 \checkmark \\
z &= 0, \qquad w = \frac{-1}{j} = j \checkmark \\
z &= j, \qquad w = \frac{j-1}{j(j-2)+j} = \frac{j-1}{-1-j}.
\end{aligned}$$

The last w should equal $-j$. Does it? Note

$$\frac{j-1}{-1-j} = \frac{1-j}{1+j} = \frac{(1-j)(1-j)}{(1+j)(1-j)} = \frac{1-2j-1}{1+1} = -j \checkmark.$$

Example 13. Find the inverse map to $w = \dfrac{z-j}{z+2j}$.

Answer. This is the map that, given w, return the original z. The easiest way to find this map is to interchange z, w in the formula for w and then solve for w. Here we have

$$z = \frac{w-j}{w+2j}.$$

Thus

$$zw + 2jz = w - j$$

or

$$w(z-1) = -2jz - j.$$

Finally

$$w = \frac{-2jz - j}{z-1}.$$

Let us see if this works. Suppose we choose $z = 0$ in the original w. Then the original map sends $z = 0$ to $(-j)/(2j)$, i.e., to $-1/2$. The inverse map sends $-1/2$ to $(-2j(-1/2) - j)/(-1/2 - 1) = 0$, which is the original z.

Example 14. Find the fixed points (if any) of

$$w = \frac{jz+1}{(j+1)z+j}.$$

Answer. We seek those z such that

$$z = \frac{jz+1}{(j+1)z+j}$$

or

$$(j+1)z^2 + jz = jz + 1.$$

We have

$$z^2 = \frac{1}{j+1} = \frac{1}{\sqrt{2}\, e^{j\frac{\pi}{4}}}.$$

This gives

$$z = \pm\frac{1}{2^{\frac{1}{4}}}\, e^{-j(\frac{\pi}{8})}.$$

Example 15. Find the image of the circle $|z-j|=2$ under the map

$$w = \frac{jz}{z+1}.$$

Answer. We could try to find equations for x, y in terms of (u,v) and substitute these into $|z-j|=2$. In essence this is what we do, but in a slightly different way. Note that

$$w = \frac{j[(z+1)-1]}{z+1} = j - \frac{j}{z+1}.$$

So

$$\frac{w-j}{j} = \frac{-1}{z+1}.$$

We conclude

$$z+1 = \frac{j}{j-w} \qquad \text{or} \qquad z = \frac{j}{j-w} - 1 = \frac{w}{j-w}.$$

(Note that we have found the *inverse transformation.*) Then $|z-j|=2$ becomes $|w/(j-w)-j|=2$. We simplify this to

$$\left|\frac{(1+j)w+1}{j-w}\right| = 2 \qquad \text{or} \qquad \frac{|w+\frac{1}{j+1}|}{|j-w|} = \frac{2}{|1+j|} = \sqrt{2}.$$

That is,

$$\frac{|w+\frac{1-j}{2}|}{|w-j|} = \sqrt{2}.$$

Put $w = u+jv$ and square both sides to get

$$\left(u+\frac{1}{2}\right)^2 + \left(v-\frac{1}{2}\right)^2 = 2(u^2+(v-1)^2)$$

or

$$u^2+u+\frac{1}{4}+v^2-v+\frac{1}{4} = 2(u^2+v^2-2v+1).$$

We conclude

$$u^2 + v^2 - 3v - u + \frac{3}{2} = 0$$

$$\left(u - \frac{1}{2}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{1}{4} + \frac{9}{4} - \frac{3}{2} = 1.$$

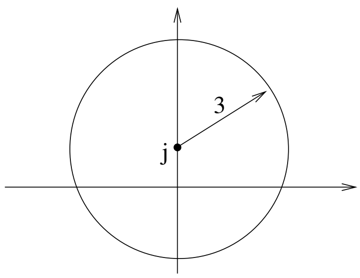
So we get a circle, centered at $z = \frac{1}{2} + \frac{3}{2}j$, radius $= 1$.

Remark. There is another way to do this example, but it is *very special to this type of function* w, namely a bilinear transformation, and to *this type of curve*, namely a circle. Specifically, we use the fact that a bilinear transformation being a composition of translations, inversions, etc. maps circles and lines to either circles or lines. Now $|z - j| = 2$ is a circle so its image under w is either a circle or a line. Note that the points $2 + j$, $-j$, $3j$ are on the circle. The image of $2 + j$ is $w_1 = [j(2+j)]/(3+j)$, that of j is $w_2 = j(j)/(j+1)$, that of $3j$ is $w_3 = j(3j)/(3j+1)$. We plot w_1, w_2, w_3 and determine if these points lie on a circle or a straight line: there are no other possibilities. This still requires some work in this example, but in some cases this is the fastest way. Do *not* use this approach except for bilinear transformations mapping either circles or lines.

Further Exercises:

Exercise 1. Find all z such that $|z - j| = 3$ and sketch.

Answer. $z = x + jy$ will satisfy this equation iff the distance from z to j is 3. We thus have a circle, centered at j, of radius 3.



This is the fastest way to do this. Alternatively and more generally, put $z = x + jy$ into $|z - j| = 3$ and get

$$|x + j(y - 1)| = 3 \text{ or, by squaring, } x^2 + (y - 1)^2 = 9.$$

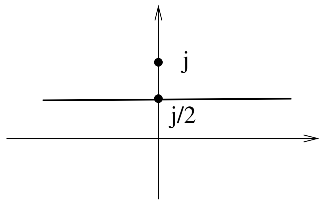
This is the same as before.

Exercise 2. Find all z such that $|z - j| = |z|$ and sketch.

Answer. We seek all points equidistant from j and 0. This can be seen by noting $|z| = |z - 0|$. So the answer is the straight line $z = j/2$. This is the fast way. The slow way (again) is as follows: put $z = x + jy$, then

$$|x+j(y-1)| = |x+jy| \quad \text{or} \quad |x+j(y-1)|^2 = |x+jy|^2 \quad \text{or} \quad x^2+(y-1)^2 = x^2+y^2.$$

Thus $x^2 + y^2 - 2y + 1 = x^2 + y^2$ or $y = 1/2$.



Exercise 3. Find all z such that $|z-1|+|z+1|=2$ and sketch.

Answer. We seek all points z whose sum of the distance to $+1$ and -1 is 2. We expect an ellipse, but since $+1$ and -1 are 2 units apart, the only z which work are $z=x$ with $-1 \le x \le 1$. I.e., the ellipse is degenerate. To see this the slow way, we get

$$\sqrt{(x-1)^2+y^2}+\sqrt{(x+1)^2+y^2}=2,$$

or

$$\begin{aligned}
& (x-1)^2+y^2+(x+1)^2+y^2+2\sqrt{(x-1)^2+y^2}\sqrt{(x+1)^2+y^2}=4 \\
\Longrightarrow \quad & x^2-2x+1+y^2+x^2+2x+1+y^2+2\sqrt{(x-1)^2+y^2}\sqrt{(x+1)^2+y^2}=4 \\
\Longrightarrow \quad & x^2+y^2+\sqrt{(x-1)^2+y^2}\sqrt{(x+1)^2+y^2}=1.
\end{aligned}$$

That is,

$$[(x-1)^2+y^2][(x+1)^2+y^2]=(1-(x^2+y^2))^2$$

which becomes

$$(x-1)^2(x+1)^2+y^2[(x-1)^2+(x+1)^2]+y^4=1-2(x^2+y^2)+(x^2+y^2)^2.$$

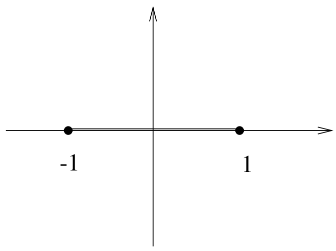
We simplify

$$x^4-2x^2+1+y^2(x^2+1)2+y^4=1-2(x^2+y^2)+x^4+2x^2y^2+y^4$$

or $4y^2 = 0$. This gives us $y = 0$. What about x? It seems that x is free, but note that we have

$$x^2 + \sqrt{(x-1)^2}\sqrt{(x+1)^2} = 1 \quad \text{or} \quad \sqrt{(x-1)^2}\sqrt{(x+1)^2} = 1 - x^2.$$

But $\sqrt{(x-1)^2}\sqrt{(x+1)^2} \geq 0$, so $1 - x^2 \geq 0$, i.e., $x^2 \leq 1$, and we get the same result.



Exercise 4. Find all z such that $|z - j| + |z + j| = 4$.

Answer. There does not seem to be any very simple way to do this problem, so just put $z = x + jy$ and get

$$\sqrt{x^2 + (y-1)^2} + \sqrt{x^2 + (y+1)^2} = 4$$

or

$$x^2 + (y-1)^2 + x^2 + (y+1)^2 + 2\sqrt{x^2 + (y-1)^2}\sqrt{x^2 + (y+1)^2} = 16.$$

This gives

$$x^2 + y^2 + \sqrt{x^2 + (y-1)^2}\sqrt{x^2 + (y+1)^2} = 7$$

or

$$(x^2 + (y-1)^2)(x^2 + (y+1)^2) = (7 - (x^2 + y^2))^2$$

which implies

$$x^4 + x^2[(y-1)^2 + (y+1)^2] + (y-1)^2(y+1)^2 = 49 - 14(x^2+y^2) + x^4 + 2x^2y^2 + y^4.$$

This gives

$$x^4 + x^2(y^2+1)2 + y^4 - 2y^2 + 1 = 49 - 14(x^2+y^2) + x^4 + 2x^2y^2 + y^4$$

or

$$2x^2 - 2y^2 + 14x^2 + 14y^2 = 48.$$

Finally,

$$4x^2 + 3y^2 = 12.$$

Exercise 5. Find the image of the circle $|z - 2j| = 1$ under the map $w = 1/z$.

Answer. Again if $w = u + jv$, $z = x + jy$, then $x = u/(u^2+v^2)$, $y = -v/(u^2+v^2)$. So

$$\sqrt{x^2 + (y-2)^2} = 1 \quad \text{or} \quad x^2 + y^2 - 4y + 4 = 1.$$

We have

$$x^2 + y^2 - 4y = -3 \quad \text{or} \quad \frac{1}{u^2+v^2} + \frac{4v}{u^2+v^2} = -3$$

which gives

$$1 + 4v + 3u^2 + 3v^2 = 0 \quad \text{or} \quad u^2 + v^2 + \frac{4}{3}v + \frac{1}{3} = 0$$

which becomes

$$u^2 + \left(v + \frac{2}{3}\right)^2 = \frac{4}{9} - \frac{3}{9} = \frac{1}{9}.$$

This is a circle centered at $u = 0$, $v = -2/3$ of radius $1/3$.

Exercise 6. Find the image of the line $x = 1$ under the map $w = \dfrac{z-j}{z+j}$.

Answer. We have $(z+j)w = z - j$, or $z(w-1) = -j - jw$. This gives $z = -j(1+w)/(w-1)$. So if $z = x + jy$ and $w = u + jv$, we have

$$x + jy = -j\left(\frac{(u+1)+jv}{(u-1)+jv}\right).$$

This is the same as

$$\begin{aligned} x + jy = \frac{v - j(1+u)}{(u-1)+jv} &= \frac{[v - j(1+u)][(u-1) - jv]}{[(u-1)+jv][(u-1)-jv]} \\ &= \frac{[v(u-1) - v(1+u)] - j[(1+u)(u-1) + v^2]}{(u-1)^2 + v^2}. \end{aligned}$$

So, since $x = 1$, we get

$$1 = \frac{2v(-1)}{(u-1)^2 + v^2}$$

(so can't have $u = 1$, $v = 0$ or get $0/0$) which gives

$$(u-1)^2 + v^2 = -2v.$$

Finally, $(u-1)^2 + (v+1)^2 = 1$, which is a circle of radius 1, centered at $(1, -1)$, *except* for $u = 1$, $v = 0$. Note that $u = 1$, $v = 0$ does not work, since then $1 = (z-j)/(z+j)$, which is impossible.

Exercise 7. Find the fixed points of the map $w = \dfrac{jz - j}{z + 2j}$.

Answer. We seek all z such that $z = (jz - j)/(z + 2j)$. Equivalently,

$$z^2 + 2jz = jz - j \quad \text{or} \quad z^2 + jz + j = 0.$$

From this we obtain $z = (-j + \sqrt{-1-4j})/2$ (both values of $\sqrt{\ }$ to be used). Now $\sqrt{-1-4j} = j\sqrt{1+4j}$, and $1 + 4j = \sqrt{17}e^{j\theta_0}$ where $\tan\theta_0 = 4$. So $\sqrt{1+4j} =$

$-(17)^{\frac{1}{4}}e^{\frac{j\theta_0}{2}}$ and

$$z = \frac{-j \pm (17)^{\frac{1}{4}}e^{\frac{j\theta_0}{2}}}{2} = \frac{-j \pm (17)^{\frac{1}{4}}[\cos(\frac{\theta_0}{2}) + j\sin(\frac{\theta_0}{2})]}{2}.$$

Note $\theta_0 \approx 1.326$, $\cos(\frac{\theta_0}{2}) \approx .7882$, $\sin(\frac{\theta_0}{2}) \approx .6154$.

II. Limits and Derivatives

II.a *Limits*

We do this briefly. Let $w = f(z)$. If we set $w = u + jv$, $z = x + jy$, then $w = f(z)$ gives u and v as functions of (x, y). We term (the *complex* number) L to be the limit as $z \to z_0$ iff as $|z - z_0| \to 0$ (but $z \neq z_0$!) we have $|f(z) - L| \to 0$. Now put $L = \ell_1 + j\ell_2$. Then $|f(z) - L| \to 0$ iff $\sqrt{(u(x,y) - \ell_1)^2 + (v(x,y) - \ell_2)^2} \to 0$, i.e., *both* $|u(x,y) - \ell_1| \to 0$ and $|v(x,y) - \ell_2| \to 0$. In the same way, set $z_0 = x_0 + jy_0$. The $|z - z_0| \to 0$ iff both $x \to x_0$ and $y \to y_0$. So $\lim\limits_{z \to z_0} f(z) = L$ is *exactly* the same as

$$\begin{cases} \lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = \ell_1 \\ \\ \lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = \ell_2. \end{cases}$$

As was done before, we say f is *continuous* at $z = z_0$ iff $\lim\limits_{z \to z_0} f(z) = f(z_0)$. Equivalently, $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u(x_0, y_0)$ *and* $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v(x_0, y_0)$. So this is *exactly* the same as the situation encountered in advanced calculus courses! The problem you recall, is that (x, y) may approach (x_0, y_0) in ∞ many ways (in complex lingo: $z \to z_0$ in ∞ many ways) so that using the definition directly to show there is a limit, is basically futile. We recall the following rules which are useful.

1. Polynomials in (x, y) are continuous.
2. Rational functions (i.e., poly/poly) are continuous (*except where the bottom*$=$ 0).
3. Let $r = f(s)$ be a continuous function of one variable s (for this it more than suffices that f be differentiable) and $s = h(x, y)$ be a continuous function of (x, y). Then $r = f(s(x, y))$ is a continuous function of (x, y).
4. Sums, Differences, Products, Quotients (*except where the bottom*$=0$) of continuous functions are continuous.

Before doing examples, we recall that the definition of limit is directly useful in showing limits do *not* exist. If we choose two different paths on the complex plane passing through z_0 and *one* of u, v has different limits along the two paths as $z \to z_0$, then $\lim\limits_{z\to z_0} f(z)$ does not exist.

Example 1. Show $w = z^2$ is continuous at all z.

Answer. Put $w = u + jv$, $z = x + jy$. Then $w = z^2$ is the same as

$$u = x^2 + y^2, \qquad v = 2xy.$$

Now let $z_0 = x_0 + jy_0$ be chosen at random. Then since both u and v are polynomials in x and y, we have by Rule 1

$$\lim_{(x,y)\to(x_0,y_0)} u = x_0^2 + y_0^2, \qquad \lim_{(x,y)\to(x_0,y_0)} v = 2x_0y_0.$$

So

$$\lim_{z\to z_0} w = (x_0^2 + y_0^2) + 2x_0y_0j = w(x_0, y_0)$$

and w is continuous.

Example 2. Show $w = e^z$ is continuous at all z.

Answer. Same procedure as before: $u = e^x \cos y$, $v = e^x \sin y$. Note that u and v are continuous (since e^x is differentiable and so are $\cos y$ and $\sin y$) by Rules 3 and 4. Then

$$\begin{aligned}\lim_{z\to z_0} w &= \lim_{(x,y)\to(x_0,y_0)} u + j \lim_{(x,y)\to(x_0,y_0)} v\\ &= e^{x_0}\cos y_0 + je^{x_0}\sin y_0 = e^{z_0} = w(z_0).\end{aligned}$$

Example 3. Show if $w = \dfrac{\mathrm{Re}\,(z) - \mathrm{Im}\,(z)}{\mathrm{Re}^{\,2}(z) + \mathrm{Im}^{\,2}(z)} + j\mathrm{Im}\,(z)$, then $\lim_{z \to 0} w$ does not exist.

Answer. Observe that $z \to 0$ means $z \to 0 + j0$, i.e., $x \to 0$ and $y \to 0$. We now have $u = (x-y)/(x^2+y^2)$, $v = y$. Note that $\lim_{(x,y)\to(0,0)} y = 0$. So v *is* continuous. We check u. Note that if $(x,y) \to (0,0)$ along $x = y$ with $x > 0$, we get $u \to 0$ while if $(x,y) \to (0,0)$ along $y = 0$ with $x > 0$, u blows up. So $\lim_{(x,y)\to(0,0)} u(x,y)$ does *not* exist and thus neither does $\lim_{z \to z_0} f(z)$.

Remark. In practice, the complex functions encountered will usually either be continuous (and differentiable even) at the point in question or else blow up there. So the existence/nonexistence of limits will usually be obvious.

We now pass to the definition of the most important limit.

II.b *Derivatives*

In the last section, z sort of disappeared. We did everything in terms of x, y, u, v. Now z returns. We have: If $\lim_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$ exists, we call it *the derivative of f with respect to z at $z = z_0$*, and denote it by $\dfrac{df}{dz}(z_0)$ or $f'(z_0)$. There is no counterpart of this concept for vectors.

When it comes to *calculating* $f'(z_0)$, there are 2 immediate difficulties with this definition.

(1) $\dfrac{f(z)-f(z_0)}{z-z_0}$ is a fraction with the bottom $= 0$ at $z = z_0$, so we can't use our Rules directly to help us find $f'(z_0)$.

(2) We recall that z can approach z_0 in ∞ many ways!

We don't even attempt to use the definition directly to find $f'(z_0)$. You could actually use it for very simple cases, but there is a much better practical way. Before we see what this way is, we introduce some notation. If f is differentiable at z_0 (i.e., $f'(z_0)$ exists) and at all points near z_0, then f is *analytic* (also called *holomorphic*) *at* z_0. If f is analytic at all points of the plane, then f is *entire* (so an entire function is one which has a derivative everywhere). In practice, most functions are either analytic everywhere *except* at some points (where they blow up or have jumps), or entire.

We now see how to calculate $f'(z)$.

II.c *Cauchy-Riemann Equations and Consequences*

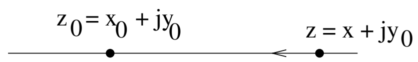
Here we shall use partial derivatives to help us compute $f'(z)$. Suppose first that we know (by some miracle) that $f'(z_0)$ actually exists. So

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

and z can approach z_0 in any of the ∞ many possible ways. This means we can *choose* a specific way and since $f'(z_0)$ is always the same, will actually get $f'(z_0)$ as $z \to z_0$ along this special chosen way.

We now choose a way. We let z approach $z_0 = x_0 + jy_0$ by *keeping y fixed (at y_0)*. So $z = x + jy_0$, and

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + j\left(\frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}\right).$$



Let now $z \to z_0$ along this path. Then $x \to x_0$ and $y \to y_0$ (actually y *is* y_0). So

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{(x, y_0) \to (x_0, y_0)} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + j \lim_{(x, y_0) \to (x_0, y_0)} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}. \end{aligned}$$

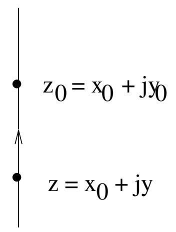
But *we know the right hand side*!!! It is $\partial u/\partial x + j(\partial v/\partial x)$! So if $f'(z_0)$ exists, then

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + j\frac{\partial v}{\partial x}(x_0, y_0).$$

This tells us how to calculate $f'(z_0)$, if *we know it exists.* But when does $f'(z_0)$ exist? After all z can approach z_0 in ∞ many ways, and we have so far considered only one way. Maybe all is fine if z approaches z_0 parallel to the x-axis (as we actually did), but things can go wrong if z approaches z_0 at an angle, say.

Let us, for the moment, have z approach z_0 in another way: parallel to the y-axis. So now $z = x_0 + jy$ and, once again, *if* $f'(z_0)$ exists, then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
$$= \lim_{(x_0, y) \to (x_0, y_0)} \frac{u(x_0, y) - u(x_0, y_0)}{jy} + \lim_{(x_0, y) \to (x_0, y_0)} j \frac{v(x_0, y) - u(x_0, y_0)}{jy}.$$



But *once again we know the right hand side*! It is $\dfrac{\partial u}{\partial y} \dfrac{1}{j} + \dfrac{\partial v}{\partial y}$. So we get

$$f'(z_0) = \frac{\partial u}{\partial y} \frac{1}{j} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - j \frac{\partial u}{\partial y}.$$

We had found earlier that

$$f'(z_0) = \frac{\partial u}{\partial x} + j \frac{\partial v}{\partial x}.$$

We must therefore have

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

at points where $f'(z)$ exists. *These are the Cauchy-Riemann Equations.*

These equations enable us to *find* $f'(z)$ when it exists, but more than that: It can be shown that for functions encountered in practice, if the *Cauchy-Riemann equations* hold in a region R of the plane, then f *is differentiable there.* The basic idea is similar to the calculation of a directional derivative for a "real" function of two variables g in terms of $\partial g/\partial x$ and $\partial g/\partial y$.

In summary,

(1) For functions encountered in practice, $f(z)$ exists wherever the Cauchy-Riemann equations hold;

(2) If the Cauchy-Riemann equations hold, then

$$f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}$$

(or if you do not like this, $f'(z) = \dfrac{\partial v}{\partial y} - j\dfrac{\partial u}{\partial x}$!)

(3) If f is differentiable at z_0 *and at all points near* z_0 we say f is *analytic*. If f is differentiable everywhere f is *entire*.

Now some examples.

Example 1. Show that e^z is differentiable everywhere (*entire*) and find its derivative.

Answer. Can you imagine doing this directly from the definition? Instead we put $u + jv = e^x(\cos y + j\sin y)$. So

$$\frac{\partial u}{\partial x} = e^x \cos y, \qquad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial v}{\partial x} = e^x \sin y, \qquad \frac{\partial v}{\partial y} = e^x \cos y.$$

So $\partial u/\partial x = \partial v/\partial y$ *and* $\partial u/\partial y = -\partial v/\partial x$ for every (x, y), i.e., for every z. Thus e^z is entire and, furthermore, its derivative is

$$\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = e^x(\cos y + j\sin y) = e^z.$$

I.e.,

$$\frac{d}{dz}(e^z) = e^z.$$

Remark. Note that $d(e^z)/dz$ is also

$$\frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y} = e^x(\cos y + j\sin y).$$

Example 2. Find $\dfrac{d}{dz}(z^2)$.

Answer. $u = x^2 - y^2$, $v = 2xy$. So

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x, & \frac{\partial u}{\partial y} &= -2y \\ \frac{\partial v}{\partial x} &= 2y, & \frac{\partial v}{\partial y} &= 2x. \end{aligned}$$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

for all z! So z^2 is also entire, and

$$\frac{d}{dz}(z^2) = 2x + j2y = 2z.$$

Important Remark. From the Cauchy-Riemann Equations, it is easy to show that

(1) all the standard derivative rules (product, quotient, sum, etc.) hold;

(2) all the standard formulas hold:

$$\frac{dz^n}{dz} = nz^{n-1}, \quad \frac{d}{dz}(\sin z) = \cos z, \quad \frac{d}{dz}(\cos z) = -\sin z, \quad \frac{d}{dz}(\cosh z) = \sinh z,$$

etc. This makes calculations often *much* faster and easier. There is a problem with $\log z$ however, because there are many values to $\log z$. If we restrict ourselves to $\operatorname{Log} z$ (i.e., the *principal value*) then indeed, $d(\operatorname{Log} z)/dz = 1/z$! The same comment applies to $z^{\frac{1}{n}}$. Note first that $z^{\frac{1}{n}}$ can be defined as $z^{\frac{1}{n}} = e^{(\log z)\frac{1}{n}}$. By writing out what $\log z$ is, you can check that $z^{\frac{1}{n}}$ given this way is *exactly* the same as $z^{\frac{1}{n}}$ which we had earlier. Again, if we select, amongst all n values of $z^{\frac{1}{n}}$, the one given by $e^{(\operatorname{Log} z)\frac{1}{n}} = |z|^{\frac{1}{n}} e^{\frac{j\theta}{n}}$ with $-\pi < \theta < \pi$, we indeed get the usual derivative formula

$$\frac{d}{dz}(z^{\frac{1}{n}}) = \frac{1}{n} z^{\frac{1}{n}-1}$$

where $z^{\frac{1}{n}}$ is defined in terms of $\operatorname{Log} z$!

Example 3. Find $f'(z)$ if $f(z) = e^{(z^2)} \sin z$.

Answer. We could do this via the Cauchy-Riemann equations, but this is much too long. Instead we use the rules from Remark 1 (namely product rule and chain rule) to get

$$f'(z) = e^{(z^2)}(2z)\sin z + e^{(z^2)}\cos z.$$

Of course, this requires knowing the derivatives of the exponential and the sine. After that, it's just like in the "real" case. Since $f'(z)$ exists for *all* z, f is *entire*.

Example 4. Find $f'(z)$ if $f(z) = (z + 2 + j)^{10}$.

Answer. Again by the chain rule: put $r = z + 2 + j$. Then $f = r^{10}$ and

$$\frac{df}{dz} = \frac{df}{dr}\frac{dr}{dz} = 10r^9 \frac{dr}{dz}.$$

Note that $2+j$ is a constant, so that $dr/dz = 1$ and

$$\frac{df}{dz} = 10(z+2+j)^9.$$

Again, just like the "real" case, and again, f is *entire.*

Example 5. Find all points $z = x + jy$ where the function

$$w = \frac{x}{x^2+y^2} - \frac{jy}{x^2+y^2}$$

is differentiable (w.r.t. z).

Answer. Unless you happen to recognize what w is (in terms of z), it is best to use the Cauchy-Riemann equations. Here

$$u = \frac{x}{x^2+y^2}, \qquad v = \frac{-y}{x^2+y^2},$$

so

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}, \qquad \frac{\partial u}{\partial y} = \frac{-x}{(x^2+y^2)^2} \cdot 2y,$$

$$\frac{\partial v}{\partial x} = \frac{2yx}{(x^2+y^2)^2}, \qquad \frac{\partial v}{\partial y} = \frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}.$$

Now

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \qquad \frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}$$

so the equations are satisfied, *except at* $x = y = 0$ (where the bottom is zero, and w is not defined). So dw/dz exists for all z except $z = 0$ *and*

$$\frac{dw}{dz} = \frac{y^2-x^2}{(x^2+y^2)^2} + j\frac{2xy}{(x^2+y^2)^2}.$$

You may also have noticed that $w = 1/z$. If you did notice, then right away you would find $w' = -1/z^2$ by the quotient rule. Note that

$$-\frac{1}{z^2} = -\frac{(\overline{z})^2}{(z\,\overline{z})^2} = -\frac{(x-jy)^2}{(x^2+y^2)^2} = \frac{(y^2-x^2)+2jxy}{(x^2+y^2)^2}$$

exactly the same as before, however note that we first found dw/dz *without knowing* what w was in terms of z. Note that f is analytic at all points z except for $z = 0$.

Example 6. Same as Example 5 with $w = x - jy^2$.

Answer. Here

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = -2y.$$

So the Cauchy-Riemann Equations hold for $y = -1/2$ and any x. Thus dw/dz exists only on the line $y = -1/2$. Note that w is *nowhere* analytic since it's not differentiable at all points near any given point of the line.

Example 7. Let $w = |z|^2$. Find all points where dw/dz exists.

Answer. Now $w = u + jv = x^2 + y^2$, so $u = x^2 + y^2$, $v = 0$ and thus need to find those points where

$$\frac{\partial u}{\partial x} = 2x = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 2y = 0.$$

There is only one such point: $x = y = 0$, i.e., $z = 0$. We conclude dw/dz exists at $z = 0$ but *nowhere else*. Note that w is *not* analytic at $z = 0$ (since only differentiable at $z = 0$ and not *near* $z = 0$).

Further Exercises:

Exercise 1. Find $\displaystyle\lim_{x\to(1+j)} \frac{z^2-2j}{z-(1+j)}$ or show it does not exist.

Answer. Note that as $z \to 1+j$ we have $z^2-2j \to (1+j)^2-2j = 2j-2j = 0$. So we are in the situation $0/0$ and can't use the rules. We try to divide:

$$
\begin{array}{r|l}
 & z + (1+j) \\
\hline
z-(1+j) & z^2-2j \\
 & z^2 - z(1+j) \\
\cline{2-2}
 & z(1+j)-2j \\
 & z(1+j)-(1+j)^2 \\
\cline{2-2}
 & (1+j)^2-2j
\end{array}
$$

But $(1+j)^2 = 1+2j-j^2 = 2j$! Thus $z^2-2j = [z-(1+j)][z+(1+j)]$ and so

$$\lim_{z\to(1+j)} \frac{z^2-2j}{z-(1+j)} = \lim_{z\to(1+j)} [z+(1+j)] = 2(1+j).$$

Exercise 2. Find $\displaystyle\lim_{z\to 0}\left(\frac{z^2-2}{z-j}\right)$ or show it does not exist.

Answer. Here we use the rules first:

$$\left.\begin{aligned} \lim_{z\to 0}(z-j) &= -j \\ \lim_{z\to 0}(z^2-2) &= -2 \end{aligned}\right\} \text{(since polynomials are continuous).}$$

Thus we obtain

$$\lim_{z\to 0}\left(\frac{z^2-2}{z-j}\right) = \frac{2}{j}\,.$$

Exercise 3. Find $\displaystyle\lim_{z\to 0}\left(\frac{\overline{z}}{z}\right)$ or show it does not exist.

Answer. Note that here we are in the 0/0 case. We have

$$\lim_{z\to 0}\left(\frac{\overline{z}}{z}\right)=\lim_{z\to 0}\left(\frac{x-jy}{x+jy}\right)=\lim_{z\to 0}\left[\frac{(x-jy)^2}{x^2+y^2}\right]=\lim_{(x,y)\to(0,0)}\left[\frac{x^2-y^2}{x^2-y^2}+\frac{-2jyx}{x^2+y^2}\right].$$

Observe

$$\lim_{(x,y)\to(0,0)}\left[\frac{x^2-y^2}{x^2+y^2}\right]=\begin{cases}0 & \text{along } x=y\\ 1 & \text{along } y=0.\end{cases}$$

So it has no limit. We thus conclude that $\lim\limits_{z\to 0}(\overline{z}/z)$ does not exist. Note that we did not need to consider $\lim\limits_{(x,y)\to(0,0)}(-2xy)/(x^2+y^2)$.

Exercise 4. Find all z at which $\sin z$ is continuous.

Answer. Note that $\sin z=(e^{jz}-e^{-jz})/(2j)$. But e^{jz} is continuous for all z, since $e^{jz}=e^{j(x+jy)}=e^{-y}(\cos x+j\sin x)$ and $e^{-y}\cos x$, $e^{-y}\sin x$ are both continuous. The same is true of e^{-jz}, and of $(e^{jz}-e^{-jz})/(2j)$. So $\sin z$ is continuous for all z.

Exercise 5. For what z is $f(z)=(z^2-1)/(z^2+1)$ continuous and why?

Answer. $f(z)=(z^2-1)/(z^2+1)$. Now the bottom and top are polynomials and so $f(z)$ is continuous at all z where the bottom $\neq 0$, i.e., at all z such that $z^2+1\neq 0$. What happens at $z^2+1=0$, i.e., at $z=\pm j$? We look at $\lim\limits_{z\to j}\dfrac{z^2-1}{z^2+1}$. Observe that

$$\begin{aligned}z^2-1 &\quad\to\quad j^2-1=-2,\\ z^2+1 &\quad\to\quad j^2+1=0.\end{aligned}$$

Thus as $z\to j$, $(z^2-1)/(z^2+1)$ blows up. Consequently there is *no limit* at $z=j$, and the function is not continuous at $z=j$. The same thing happens at $z=-j$. In conclusion, $f(z)=(z^2-1)/(z^2+1)$ is continuous at all z not equal to $\pm j$.

Exercise 6. Find all points where $f(z)=\overline{z}$ is differentiable.

Answer. Let $f(z) = \overline{z} = x - jy$. Then $u = x$, $v = -y$ and thus $\partial u/\partial x = 1$, $\partial v/\partial y = -1$. The Caucy-Riemann equations are never satisfied and $f(z)$ is never differentiable.

Exercise 7. Find $d(z \tan z)/dz$.

Answer. We must find $d(\tan z)/dz$. Now $\tan z = \sin z / \cos z$ and $d(\sin z)/dz = \cos z$, $d(\cos z)/dz = -\sin z$. These formulas can be obtained by writing $\sin z$ in terms of $(e^{jz} - e^{-jz})/(2j)$, and repeating with $\cos z$. Now

$$\frac{d(\tan z)}{dz} = \frac{(\cos z)(\cos z) - \sin z(-\sin z)}{\cos^2 z} = \frac{1}{\cos^2 z} = \sec^2 z$$

by the quotient rule. Thus, by the product rule,

$$\frac{d(z \tan z)}{dz} = \tan z + z \sec^2 z.$$

Exercise 8. Find all points where $w = e^{-\mathrm{Im}\,(z)}[\cos(\mathrm{Re}\, z) + j \sin(\mathrm{Re}\, z)]$ is differentiable with respect to z. What is dw/dz at these points?

Answer. We have $w = e^{-y}(\cos x + j \sin x)$. So $u = e^{-y} \cos x$, $v = e^{-y} \sin x$ and

$$\frac{\partial u}{\partial x} = -e^{-y} \sin x, \qquad \frac{\partial v}{\partial y} = -e^{-y} \sin x,$$

$$\frac{\partial u}{\partial y} = -e^{-y} \cos x, \qquad \frac{\partial v}{\partial x} = e^{-y} \cos x.$$

The Cauchy-Riemann equations are always satisfied, and w has a derivative for all z. At any $z = x + jy$,

$$\frac{dw}{dz} = -e^{-y} \sin x + j(e^{-y} \cos x).$$

II.d *Harmonic Functions, Harmonic Conjugates, Singular Points*

Let $f(z) = u + jv$ be analytic in some region of the plane. In this region we then have the Cauchy-Riemann Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If we differentiate these w.r.t. x and y respectively, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

and thus

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0.$$

So u satisfies *Laplace's Equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the region. This is an important equation which arises in a variety of steady-state problems (temperature distribution in bodies, electrostatic potentials, etc.). Its solutions are said to be *harmonic* functions. So if $f(z)$ is analytic, then u is *harmonic.* If we differentiate the original Cauchy-Riemann equations the other way around, then we find that v is also *harmonic.* So if f is analytic, both its real and imaginary parts are harmonic. There is one more thing. Suppose u, v are 2 *harmonic functions which also satisfy the Cauchy-Riemann equations.* Then u, v are *harmonic conjugates* and $f = u + jv$ is *analytic.* Note that the Cauchy-Riemann equations are *not symmetric* (due to the $-$ sign in the second equation). *This fixes which is u and which is v* in $f = u + jv$. *In general these cannot be interchanged.*

Example 1. Show that $r = x^2 - y^2$ is harmonic, find its harmonic conjugate and use the two functions to construct an analytic function with these as real and imaginary parts.

Answer. To see that r is harmonic we compute $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = 2 - 2 = 0$. So r is harmonic. We next find a harmonic conjugate. Suppose we first think of r as "u", then we need v such that

$$2x = \frac{\partial v}{\partial y} \qquad \text{and} \qquad -2y = -\frac{\partial v}{\partial x}.$$

This is the same problem as finding the "potential," or solving "exact differential equations." The first equation gives $v = 2xy + H(x)$ for some function H. The second gives $2y = 2y + H'(x)$. So $H'(x) = 0$, i.e., $H =$ a constant C. We have found $v = 2xy + C$ is a harmonic conjugate *for any* C and thus $f = (x^2 - y^2) + j(2xy + C)$ is the associated analytic function. You may have recognized f as $z^2 + Cj$ for a (real) constant C.

Suppose now we think of r as "v". Then we need u such that

$$\frac{\partial u}{\partial x} = -2y, \qquad \frac{\partial u}{\partial y} = -2x.$$

Now we find $u = -2xy + H(y)$ and then $H(y) = C$. So, in this case $u = -2xy + C$ with C real. The function now becomes $g = (-2xy + C) + j(x^2 - y^2)$. Note that f and g are *different.* But we see the connection after a minute: $f = -jg$, and the arbitrary constants are negatives of each other. Does this make any sense? Yes, if you notice that if $f = -jg$, then *the real part of f is the imaginary part of g and the imaginary part of f is the minus the real part of g.*

In summary, given a *harmonic function* r, we can construct by means of the Cauchy-Riemann equations its *harmonic conjugate.* What this depends on is whether we think of r as the real or as the imaginary part of an analytic function. Usually, if we write u (for r) then we mean that u is to be the *real* part, while if we write v (for r) we mean that v is to be the *imaginary* part of f.

A point z_0 such that f' does *not exist at* z_0*, but does so at all points near* z_0, is called a *singular point* of f. In practice, almost all singular points arise because f is not defined (blows up) at the point in question, for example,

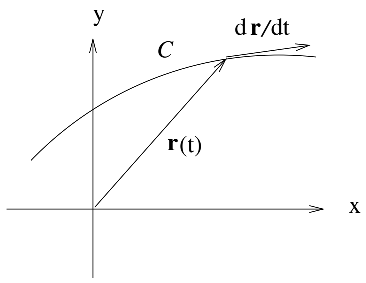
Example 2. Let $f(z) = z/(z^2+1)$. If $z^2+1 \neq 0$, then $f'(z)$ exists: we just apply the quotient rule and the fact that z, z^2+1 are differentiable. If $z^2+1=0$, i.e., $z = \pm j$, then f does not exist, so f' does not exist, so $\pm j$ are singular points.

II.e *Angular Properties: Conformal Maps, Orthogonal Families*

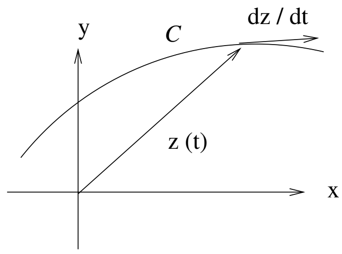
We start by recalling the following. Let

$$\mathcal{C} = \begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad a \le t \le b$$

be a curve in the xy plane. The position vector is $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$. (Note that here $\mathbf{j}$ is a *vector*, not $j = \sqrt{-1}$!) Then the velocity vector is $d\mathbf{r}/dt = x'\,\mathbf{i} + y'\,\mathbf{j}$, and its direction is tangential to $\mathcal{C}$ (in the *direction of increasing* t).



We now pass to the complex plane, and use the analogy between z and 2-d vectors. Again let $\mathcal{C}$ be the given curve, then $z(t) = x(t) + jy(t)$ (now j *is* $\sqrt{-1}$) is a curve in the complex plane. It is of course the same curve as before, just written in different form. We *differentiate* z *with respect to the real variable* t, and get $z'(t) = x'(t) + jy'(t)$. Note that this is *not* a derivative with respect to z. It is just $\lim_{t \to t_0} [z(t) - z(t_0)]/(t - t_0)$, the same as for ordinary differentiation. We observe that $z'(t)$ when plotted, is tangential to $\mathcal{C}$, indeed we have the *same* picture as before.



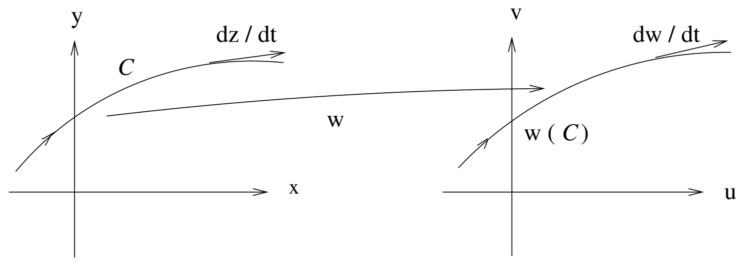
Now suppose $w = f(z)$ is an analytic function. This function maps points from the z-plane to the w-plane. We are interested in what happens to points on the given curve $\mathcal{C}: \ z = x(t) + jy(t)$. They get taken to $w(t) = f(z(t))$, which will vary depending on the specific choice of f (and $\mathcal{C}$).

There is one thing however which we notice. By the Chain Rule,

$$\frac{dw}{dt} = \frac{df}{dz} \cdot \frac{dz}{dt}, \quad \text{so} \quad \left|\frac{dw}{dt}\right| = \left|\frac{df}{dz}\right| \cdot \left|\frac{dz}{dt}\right|$$

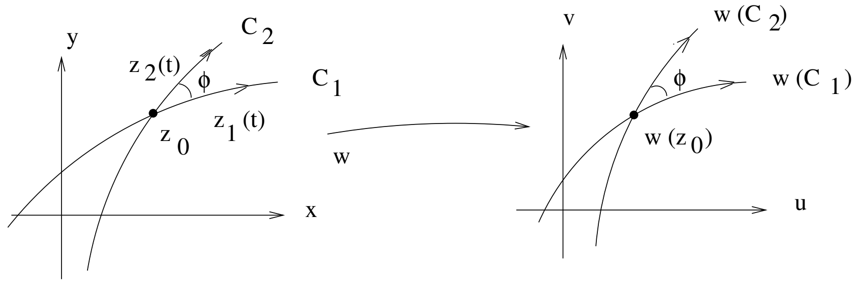
and

$$\arg\left[\frac{dw}{dt}\right] = \arg\left[\frac{df}{dz}\right] + \arg\left[\frac{dz}{dt}\right].$$



We observe that dw/dt is a tangent vector to the image of $\mathcal{C}$ under w, and focus on the second equation. We note that $\arg[dw/dt]$ (i.e., the angle the tangent makes w.r.t. the u-axis) is the original angle of the tangent (i.e., $\arg[dz/dt]$) plus the argument of df/dz, *unless* $df/dz = 0$ *or* $dz/dt = 0$ *in which case* $dw/dt = 0$ *and thus* dw/dt *does not have a defined argument.*

The key observation is that df/dz does not have any dependence on the specific $\mathcal{C}$, i.e., choose a point z_0 and suppose $\underline{2}$ curves C_1 and C_2 pass through z_0, say at $t = 0$. The two curves get mapped by w to $w(C_1)$ and $w(C_2)$ respectively, while z_0 is taken to $w(z_0)$, as in the picture.



Suppose none of $\dfrac{df}{dz}(z_0)$, $\dfrac{dz_1}{dt}(0)$, $\dfrac{dz_2}{dt}(0)$ are *zero*. Then the tangent vectors to C_1, C_2 at the meeting point z_0 get rotated by the *same* angle (namely $\arg[\frac{df}{dz}(z_0)]$), so the angle *between* C_1, C_2 at z_0 and their images $w(C_1)$, $w(C_2)$ at $w(z_0)$ stays the same.

In *summary*: let $w = f(z)$ and suppose $\frac{df}{dz}(z_0) \neq 0$, then w maps curves intersecting at z_0 into the w plane in such a way that the angle between them is preserved at $w(z_0)$. Such a map is called *conformal*. In particular, if two curves meet at right angles in the z-plane, so do their images in the w-plane. *We emphasize that each curve may be distorted by the map w. It is only the angle between them that stays the same.*

The previous discussion had to do with the connection between "angles" in the z plane and in the w plane. There is another "angular" property of analytic functions which should be mentioned, which deals with angles just in the z-plane. Again, let $w = u + jv$ and consider the *families of curves* $u(x, y) = c$, $v(x, y) = d$ (for arbitrary constants c, d) *in the (x, y) plane*. Note that $u(x, y) = c$ will give y, say, as a function of x, denoted by $y_1(x)$. We find, differentiating implicitly w.r.t. x, and treating y_1 as a function of x:

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{dy_1}{dx} = 0.$$

That is,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy_1}{dx} = 0.$$

In the same way, for *any* curve (i.e., for any constant d) of the second family $v(x,y) = d$ we have y as a function of x, denoted by $y_2(x)$ and

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy_2}{dx} = 0.$$

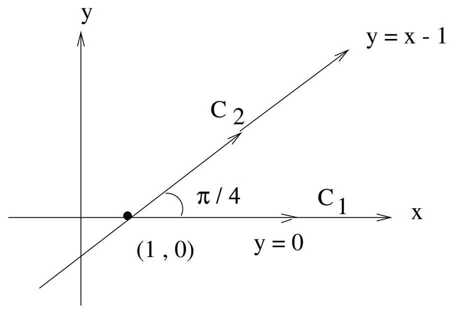
Now suppose a member of the first family meets a member of the second at $z_0 = x_0 + jy_0$. Then at this point we find from the Cauchy-Riemann equations:

$$\frac{dy_1}{dx} \cdot \frac{dy_2}{dx} = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \cdot \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = -1.$$

So the two curves meet at right angles!

In summary, the family of curves $u(x,y) = c$, $v(x,y) = d$ in the (x,y) plane meet at right angles (if at all) at a point where $w = u + jv$ is analytic unless of course one of the derivatives $\partial u/\partial y$, $\partial v/\partial y$ are zero. Since we can however interchange x with y in the previous discussion, all that we need again is that $f'(z) \neq 0$. What happens at this last possibility requires a special treatment, as does the situation where w is not analytic at the point in question, which is beyond the scope of this course. We pass to examples.

Example 1. We consider the situation for a relatively simple case: $w = z^2$. Observe first that $dw/dz = 2z$, so w is *conformal* except at $z = 0$, and angles between curves are preserved. To illustrate this, consider two straight lines on the z-plane $y = 0$ and $y = x - 1$. Clearly these meet at $x = 1$, $y = 0$ with an angle of $\theta = \pi/4$. (To see this, note that a tangent vector to $y = x - 1$ is $\mathbf{i} + \mathbf{j}$.)



Consider their images. Note that $u + jv = (x^2 - y^2) - 2jxy$. So $y = 0$ gets taken to

$$\begin{cases} u = x^2 \\ v = 0 \end{cases} \qquad -\infty < x < \infty.$$

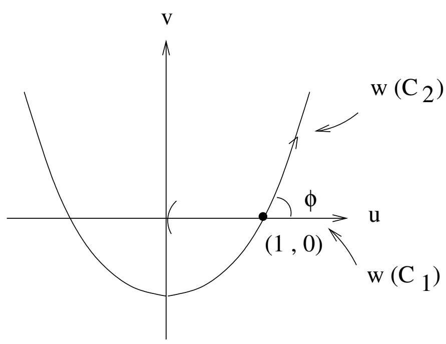
This is the *positive u semiaxis* (crossed over twice note, as x goes from $-\infty$ to ∞). On the other hand, $y = x - 1$ gets taken to:

$$\begin{cases} u = x^2 - (x-1)^2 = 2x - 1 \\ v = 2x(x-1) = 2x^2 - 2x \end{cases}.$$

Solving for x in terms of u and substituting into the equation for v gives

$$v = \frac{(u+1)(u-1)}{2} = \frac{u^2 - 1}{2}$$

which is a parabola. Finally, $z = 1$ becomes $w = 1^2 = 1$, i.e., $u = 1$, $v = 0$. So in the w plane the picture is:

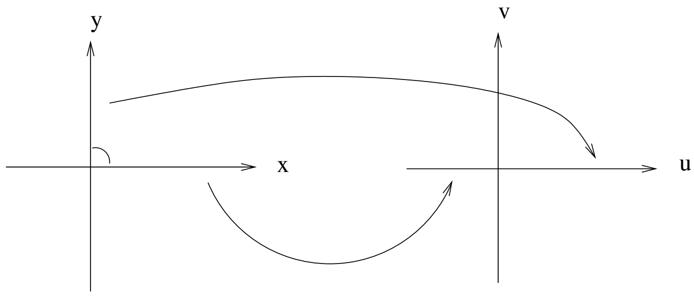


We check what ϕ is. By the theory, it should be $\pi/4$. Now $w(C_2)$ in parametric form is: $u = u$, $v = (u^2 - 1)/2$, so the position vector $\mathbf{r} = u\,\mathbf{i} + (u^2 - 1)/2\,\mathbf{j}$ and the tangent vector is $d\mathbf{r}/du = \mathbf{i} + u\mathbf{j}$ (in the direction of increasing u). So at $w = 1$ (i.e., $u = 1$, $v = 0$) we get $d\mathbf{r}/du = \mathbf{i} + \mathbf{j}$ and we find $\phi = \pi/4$.

Next, let us see what happens to $x = 0$ and $y = 0$ (i.e., the two axes). Clearly these meet at $\pi/2$ in the (x, y) plane. Their images under w are

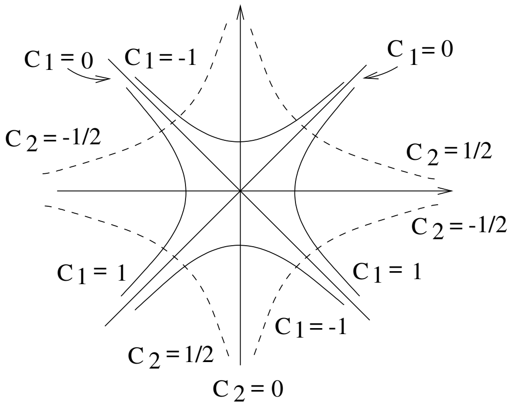
$$\begin{cases} u = -y^2 \\ v = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} u = x^2 \\ v = 0 \end{cases}.$$

So $x = 0$ gets mapped to the negative u semiaxis, while $y = 0$ gets mapped to the positive u semiaxis. The angle between these is clearly *not* $\pi/2$!



The problem is that the x and y axes meet at $(0, 0)$ and there $dw/dz = 2z$ *is zero*, so the map is *not* conformal there.

Finally, consider the family of curves $u = x^2 - y^2 = c_1$, $v = 2xy = c_2$ *in the* (x, y) *plane*, for constants c_1, c_2. We expect these two to meet at right angles, at points where derivatives $\neq 0$ are involved. We sketch a few curves.



We can check (by taking tangents) that indeed these meet at right angles, *except* for the cases $c_1 = c_2 = 0$. Here the curves meet at $(0, 0)$ but not at right angles! Again the difficulty is that $dw/dz = 2z = 0$ there.

In conclusion, these geometric properties of analytic function are useful in solving numerically (and otherwise) a variety of practical problems in Electrical Engineering. In fairness, I feel such an approach was more important in the past, since today we have a variety of different solution procedures on the computer: finite elements, adaptive grids, etc., etc., etc. Furthermore, while most significant practical problems involve 3 space dimensions, analytic function approaches really belong to a 2-d world, i.e., u, v are just functions of (x, y).

Further Exercises:

Exercise 1. Find df/dz if it exists: $f(z) = \sinh(z) + |z - j|^2$.

Answer. Let $f(z) = \sinh z + |z-j|^2$. Now $d(\sinh z)/dz = \cosh z$ everywhere. This can be seen by writing $\sinh z = (e^z - e^{-z})/2$. What about $d(|z-j|^2)/dz$? Now $|z-j|^2 = x^2+(y-1)^2$. So $u = x^2+(y-1)^2$, $v = 0$. The Cauchy-Riemann equations give: $2x = 0$, $2(y-1) = 0$. So we need $x = 0$, $y = 1$ and then the derivative of $|z-j|^2$ is $\partial u/\partial x + j\partial v/\partial x = 0$. So $f(z) = \sinh z + |z-j|^2$ is differentiable only at $z = 0 + 1j = j$.

Exercise 2. Find df/dz if it exists: $f(z) = z^2\overline{z} + e^{(z^2)}$.

Answer. Let $f(z) = z^2\overline{z} + e^{z^2}$. Now e^{z^2} is entire, i.e., differentiable everywhere. In fact, $d(e^{z^2})/dz = e^{z^2}2z$ by the chain rule. Now

$$z^2|\overline{z}| = [(x^2-y^2) + 2xyj][x^2+y^2].$$

So

$$u = (x^2-y^2)(x^2+y^2) = x^4 - y^4, \qquad v = (x^2+y^2)(2xy).$$

Thus

$$\begin{aligned} \frac{\partial u}{\partial x} &= 4x^3, & \frac{\partial v}{\partial x} &= (2x)(2xy) + (x^2+y^2)(2y), \\ \frac{\partial u}{\partial y} &= -4y^3, & \frac{\partial v}{\partial y} &= (2y)(2xy) + (x^2+y^2)(2x). \end{aligned}$$

So

$$\left.\begin{aligned} 4x^3 &= 6xy^2 + 2x^3 \\ -4y^3 &= -(6x^2y + 2y^3). \end{aligned}\right\} \qquad (*)$$

Well, $x = y = 0$ works. Do any other points work? We rewrite the equations of $(*)$ as

$$2x^3 = 6xy^2$$
$$2y^3 = 6x^2y.$$

These show that if *one* of x, y is zero so is the other. We may assume that *both* x, y are $\neq 0$. Then we have $2x^2 = 6y^2$ and $2y^2 = 6x^2$. Consequently we must have $x = y = 0$ and there are no other solutions.

Exercise 3. It is known that $f(z)$ is entire and that $\overline{f(z)} = f(z)$. What is $f(z)$?

Answer. We know that $f(z)$ is entire, so df/dz exists for all z. Now put $f = u+jv$. Then $\overline{f} = f$ and so $u + jv = u - jv$, i.e., $v \equiv 0$. But $\partial u/\partial x = \partial v/\partial y = 0$ and $\partial u/\partial y = 0$ also. So $u =$ constant. In summary, f is a *real* constant.

Exercise 4. Let $u = 2y(1 - x)$. Show that u is harmonic, find its harmonic conjugate v and a function $f(z) = u + jv$ which is entire.

Answer. $u = 2y(1 - x)$. Then we note $u_{xx} = 0$, $u_{yy} = 0$ and thus $u_{xx} + u_{yy} = 0$. We conclude that u is harmonic. Now we find v from the Cauchy-Riemann equatins:

$$\frac{\partial u}{\partial x} = -2y = \frac{\partial v}{\partial y} \quad \longrightarrow \quad v = -y^2 + C(x) \quad \longrightarrow \quad \frac{\partial v}{\partial x} = \frac{dC}{dx}.$$

Then

$$\frac{\partial u}{\partial y} = -2x + 2 = -\frac{\partial v}{\partial x}.$$

We conclude $dC/dx = 2x - 2$, and thus $C = x^2 - 2x + D$ with $D =$ constant. In summary, $v = -y^2 + x^2 - 2x + D$ and $w = 2y(1 - x) + (-y^2 + x^2 - 2x + D)j$.

Exercise 5. Repeat exercise 4, except now $v = 2y(1-x)$ and you are to find u, and $f(z) = u + jv$.

Answer. Now $v = 2y(1-x)$. We already know v is harmonic from exercise 4. We wish to find u. Observe that $\partial u/\partial x = \partial v/\partial y = 2(1-x)$. Consequently,

$$u = 2x - x^2 + C(y).$$

Next

$$\frac{dC}{dy} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 2y.$$

We conclude $dC/dy = 2y$ or $C = y^2 + D$ for a constant D. Finally, $u = 2x - x^2 + y^2 + D$ and $f = (2x - x^2 + y^2 + D) + j2y(1-x)$.

Remark: Note that this f is j times the f found in exercise 4.

Exercise 6. Repeat exercise 4, with $u = \cos x \sinh y$.

Answer. Now $u = \cos x \sinh y$, so $u_{xx} = -\cos x \sinh y$ and $u_{yy} = \cos x \sinh y$. We conclude $u_{xx} + u_{yy} = 0$ and u is harmonic. Now

$$\frac{\partial u}{\partial x} = -\sin x \sinh y = \frac{\partial v}{\partial y}$$

and thus

$$v = -\sin x \cosh y + C(x).$$

But

$$\frac{\partial u}{\partial y} = \cos x \cosh y = -\frac{\partial v}{\partial x} = \cos x \cosh y - \frac{dC}{dx}.$$

Thus $dC/dx = 0$ and $C = \text{constant}$. Finally,

$$f = \cos x \sinh y + (-\sin x \cosh y + C)j.$$

Exercise 7. It is known that both $f = u + jv$ and $g = v + ju$ are entire. What are u and v?

Answer. Now $f = u + jv$ is entire and so

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

But $g = v + ju$ is also entire, i.e.,

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}, \qquad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}.$$

We conclude

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0.$$

But then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \qquad \longrightarrow \qquad u = \text{constant},$$
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \qquad \longrightarrow \qquad v = \text{constant}.$$

So u and v are both constants.

Exercise 8. Construct by complex variable methods a family of curves orthogonal to the family $x + y = c$.

Answer. The first family is $x+y = c$. We recall that if $f = u+jv$ is analytic, with $df/dz \neq 0$, then the families $u = c$, $v = d$ for constants c, d meet at right angles. Now choose $u = x + y$. Now $u_{xx} + u_{yy} = 0$ and we choose v by

$$\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = 1 = -\frac{\partial v}{\partial x}.$$

We have $v = y - x + \text{const}$. So the second family is $y - x = d$. Note that

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = 1 - j \neq 0.$$

So we conclude the members of the family $y - x = d$ meet those of the family $x + y = c$ at right angles.

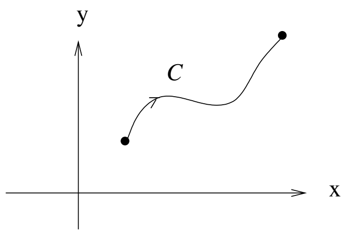
III. Integrals

III.a *Introduction*

We begin by recalling "real" vector path integrals. Let $\mathcal{C}$ be a curve in the (x, y) plane given by

$$\mathcal{C} = \begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad a \le t \le b,$$

and let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field.

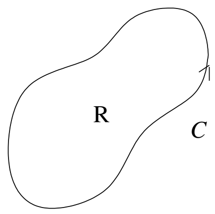


We define

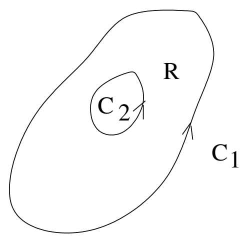
$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{t=a}^{b} \left(M \frac{dx}{dt} + N \frac{dy}{dt} \right) dt.$$

Let us now further suppose that $\mathcal{C}$ is a *simple closed path*, i.e., the start point of $\mathcal{C}$ = end point of $\mathcal{C}$, and $\mathcal{C}$ does not otherwise cross itself. We denote by $\oint \mathbf{F} \cdot d\mathbf{r}$, the integral of $\mathbf{F}$ around $\mathcal{C}$ *in the counterclockwise direction*, i.e., with the "inside" of $\mathcal{C}$ on the left side as we traverse $\mathcal{C}$. We could evaluate such integrals by parametrizing $\mathcal{C}$ and proceeding just as for the first part. It is important for us to note another way, given by *Green's Theorem:* Let $\mathcal{C}$ be a simple closed path and suppose R is the region enclosed by $\mathcal{C}$. Then

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{R} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx\, dy.$$



It is assumed that $\partial N/\partial x$ and $\partial M/\partial y$ both exist and are continuous inside R and on $\mathcal{C}$. This will be the case for what follows, *except* for important situations when N, M blow up at some point inside R. Green's Theorem then fails, and we replace it by the following version: Let R be a region of the plane, with boundary C_1, and a hole with boundary C_2 as shown. (Note that earlier we started with a path $\mathcal{C}$ and R denoted the "inside" of $\mathcal{C}$. Now we *start with* R, and C_1, C_2 denote its two boundaries.)



Then

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx\, dy + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

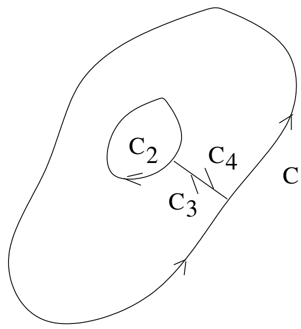
So, if $\partial N/\partial x \equiv \partial M/\partial y$ in R, then

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

This is important for us, because the result holds if $\partial N/\partial x$, $\partial M/\partial y$ both exist and are continuous in R, in particular: M, N could blow up inside "the hole." *What happens inside the "hole" does not matter*, but the price we pay is the calculation of $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

We conclude this review by pointing out that the second result follows from the first, if we make a cut from C_1 to C_2. We have:

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\, dy = \oint_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot d\mathbf{r} + \oint_{C_2} \mathbf{F}\cdot d\mathbf{r} + \int_{C_4} \mathbf{F}\cdot d\mathbf{r}.$$



But C_3 is C_4 traversed backwards while C_2 *is traversed clockwise* and thus

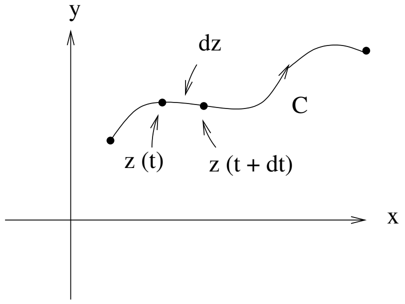
$$\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = -\int_{C_4} \mathbf{F}\cdot d\mathbf{r}$$

$$\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = -\oint_{C_2} \mathbf{F}\cdot d\mathbf{r},$$

and the result.

III.b *Path Integrals in the Complex Plane*

We are now ready to return to the complex plane. Given a path $\mathcal{C}$, we define the integral of a complex function $f(z)$, i.e., $\int_{\mathcal{C}} f(z)\,dz$, just like we did for vectors, keeping in mind that $j^2 = -1$.



$$
\begin{aligned}
\int_{\mathcal{C}} f(z)\,dz &= \int_{\mathcal{C}} (u + jv)(dx + jdy) \\
&= \int_{\mathcal{C}} [(udx - vdy) + j(vdx + udy)] \\
&= \int_{t=a}^{b} \left[u\,\frac{dx}{dt} - v\,\frac{dy}{dt}\right] dt + j\int_{t=a}^{b} \left[v\,\frac{dx}{dt} + u\,\frac{dy}{dt}\right] dt
\end{aligned}
$$

where $\mathcal{C}$ is given by

$$
\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad a \le t \le b.
$$

In more condensed notation:

$$
\int_{\mathcal{C}} f(z)\,dz = \int_{t=a}^{b} [f(z(t))]\left(\frac{dz}{dt}\right) dt.
$$

Note that we can interpret intuitively $\int_{\mathcal{C}} f(z)\,dz$ as follows: Let t vary from t to $t + dt$, then z changes from $z(t)$ to $z(t + dt)$. Thus

$$
f(z)\,dz \cong f(z)[z(t + dt) - z(t)] \cong f(z)\left[\frac{dz}{dt}\right] dt.
$$

Summing as t goes from a to b and taking the limit as $dt \to 0$ gives the result. Observe that there is no immediate physical interpretation to $\int_{\mathcal{C}} f(z)\,dz$ (unlike $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \text{work}$).

Before doing an example, we note

(1) Let $dw/dz = f$ in a region with no holes containing the path $\mathcal{C}$. Then $f\,dz/dt = (dw/dz)(dz/dt) = dw$ and thus $\int_{\mathcal{C}} f\,dz = w(\text{end point}) - w(\text{start point})$.

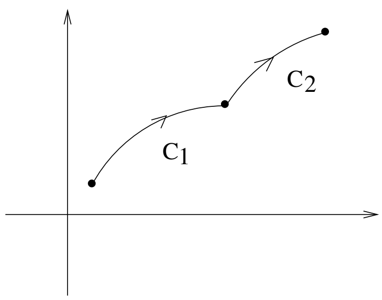
(2) $\int_{\mathcal{C}} f(z)\,dz$ will depend in general not only on the start/end points, but also on the specific path taken.

(3) If we denote by $-\mathcal{C}$, the path $\mathcal{C}$ travelled backwards, then

$$\int_{\mathcal{C}} f(z)\,dz = -\int_{-\mathcal{C}} f(z)\,dz.$$

(4) If the path $\mathcal{C} = C_1 + C_2$ as shown, then

$$\int_{\mathcal{C}} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$



(5) Let

$$\mathcal{C} = \begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad a \le t \le b.$$

Then

$$\int_{t=a}^{t=b} |z'|\,dt = \int_{t=a}^{t=b} \sqrt{(x')^2 + (y')^2}\,dt = L,$$

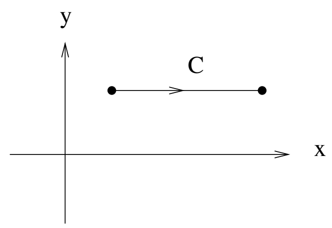
the length of $\mathcal{C}$.

(6) If $\mathcal{C}$ is a path parallel to the x-axis, i.e.,

$$\mathcal{C} = \begin{cases} x = x(t) \\ y = y_0 \end{cases} \qquad a \leq t \leq b$$

with y_0 constant, then $dy = 0$ and

$$\int_{\mathcal{C}} f(z)\, dz = \int_a^b \left(u \frac{dx}{dt} + jv \frac{dx}{dt} \right) dt.$$



Now some examples.

Example 1. Calculate $\int_{\mathcal{C}} z^2\, dz$ where $\mathcal{C}$ is the part of the parabola $y = x^2$ from $z = 0$ to $z = 1 + j$.

Answer. First note that $d(z^3/3)/dz = z^2$. Thus

$$\int_{\mathcal{C}} z^2\, dz = \left. \frac{z^3}{3} \right|_0^{1+j} = \frac{(1+j)^3}{3}.$$

To practice, we also parametrize:

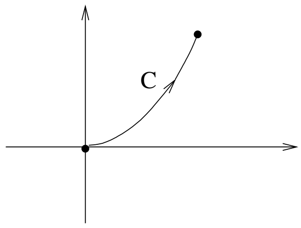
$$\mathcal{C} = \begin{cases} x = t \\ y = t^2 \end{cases} \qquad 0 \leq t \leq 1$$

(of course we could also parametrize $\mathcal{C}$ as

$$\begin{cases} x = x \\ y = x^2 \end{cases} \qquad 0 \leq x \leq 1,$$

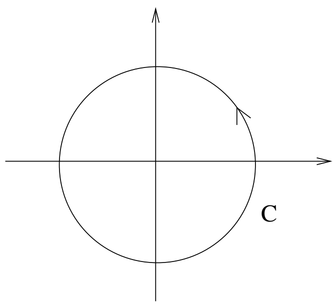
and in other ways as well). Then $z = t + jt^2$ on $\mathcal{C}$ and

$$\int_{\mathcal{C}} z^2\, dz = \int_{t=0}^{1} (t + jt^2)^2 (1 + 2jt)\, dt = \int_{t=0}^{1} (t^2 + 2jt^3 - t^4)(1 + 2jt)\, dt$$
$$= \int_{0}^{1} \left[(t^2 - 5t^4) + j(4t^3 - 2t^5) \right] dt = \left(\frac{1}{3} - 1 \right) + j \left(1 - \frac{2}{6} \right).$$



Example 2. Calculate $\oint_{\mathcal{C}} 1/z\, dz$ if $\mathcal{C}$ is the circle centered at $z = 0$, of radius 1, traversed counterclockwise.

Answer. We first parametrize $\mathcal{C}$, and the angle θ seems useful as a parameter.



So

$$\mathcal{C} = \begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} \qquad 0 \le \theta \le 2\pi.$$

Then, on $\mathcal{C}$, $z = \cos\theta + j\sin\theta = e^{j\theta}$! And

$$\oint_{\mathcal{C}} \frac{1}{z}\, dz = \int_{\theta=0}^{2\pi} \frac{1}{e^{j\theta}} \left(\frac{dz}{d\theta} \right) d\theta = \int_{\theta=0}^{2\pi} \frac{je^{j\theta}}{e^{j\theta}}\, d\theta = j \int_{0}^{2\pi} d\theta = 2\pi j.$$

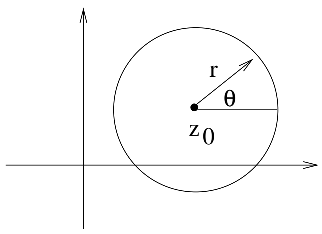
Important Remark. Note that we can't use Green's Theorem directly here—see below—(and indeed $\oint 1/z\,dz \neq 0$) since $1/z$ blows up at $z = 0$. Yet intuitively there should be a connection between $\oint_{\mathcal{C}} f(z)\,dz$ over simple closed paths $\mathcal{C}$—just like our circle—and Green's Theorem. Indeed there is, see below, but for now we do such integrals by parametrizing.

Important Example 3. Let n be an integer, *positive or negative but not equal* -1, and let z_0 be a fixed point of the plane. Calculate

$$\oint_{\mathcal{C}} (z - z_0)^n\,dz$$

where $\mathcal{C}$ is a circle of radius r centered at z_0.

Answer. Again, parametrize $\mathcal{C}$ first. So let $z_0 = x_0 + jy_0$.



Then $\mathcal{C}$ is

$$\begin{cases} x = x_0 + r\cos\theta \\ y = y_0 + r\sin\theta \end{cases} \qquad 0 \leq \theta \leq 2\pi,$$

i.e., $z = z_0 + r(\cos\theta + j\sin\theta) = z_0 + re^{j\theta}$. Then

$$\begin{aligned} \oint_{\mathcal{C}} (z - z_0)^n\,dz &= \int_{\theta=0}^{2\pi} (r^n e^{nj\theta}) rje^{j\theta}\,d\theta \\ &= \int_{\theta=0}^{2\pi} jr^{n+1} e^{(n+1)j\theta}\,d\theta = jr^{n+1} \left.\frac{e^{(n+1)j\theta}}{(n+1)j}\right|_{\theta=0}^{2\pi}. \end{aligned}$$

Here we have used the fact that $n \neq -1$, so $n+1 \neq 0$. So

$$\oint_{\mathcal{C}} (z - z_0)^n \, dz = \frac{r^{n+1}}{n+1} \left[e^{(n+1)j2\pi} - e^0 \right].$$

But

$$e^{j(n+1)2\pi} = \cos(2\pi(n+1)) + j\sin(2\pi(n+1)) = 1 + 0j$$

and thus $\oint_{\mathcal{C}} (z - z_0)^n \, dz = 0$, *regardless of* n *or of* r *or* z_0 (except $n \neq -1$).

Remark. Note that if, actually, $n = -1$, then

$$\oint_{\mathcal{C}} (z - z_0)^{-1} \, dz = \int_{\theta=0}^{2\pi} j \, d\theta = 2\pi j \neq 0\,!$$

So $\oint_{\mathcal{C}} (z - z_0)^{-1} \, dz$ is not zero, and equals $2\pi j$ regardless of z_0 or r.

We now focus on the connection between the integrals of $f(z)$ over simple closed paths $\mathcal{C}$ and the properties of f inside $\mathcal{C}$.

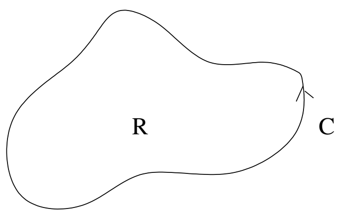
III.c *Cauchy's Theorem, Cauchy's Integral Formula and Consequences*

We have the following result.

Theorem. *Let $\mathcal{C}$ be a simple closed path and $f(z)$ an analytic function inside and on $\mathcal{C}$. Then*

$$\oint_{\mathcal{C}} f(z)\,dz = 0.$$

Proof: We use Green's Theorem. Let $f = u + jv$ and remember $\partial u/\partial x = \partial v/\partial y$, $\partial u/\partial y = -\partial v/\partial x$ inside and on $\mathcal{C}$.



So

$$\begin{aligned}\oint_{\mathcal{C}} f(z)\,dz =& \oint_{\mathcal{C}} (u + jv)(dx + j\,dy) \\ =& \oint_{\mathcal{C}} (u\,dx - v\,dy) + j\oint_{\mathcal{C}} v\,dx + u\,dy.\end{aligned}$$

Now

$$\oint_{\mathcal{C}} (u\,dx - v\,dy) \overset{\text{Green}}{=} \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy \overset{\text{C. R. equations}}{=} 0$$

and

$$\oint_{\mathcal{C}} (v\,dx + u\,dy) \overset{\text{Green}}{=} \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \overset{\text{C. R. equations}}{=} 0.$$

So

$$\oint_{\mathcal{C}} f(z)\,dz = 0.$$

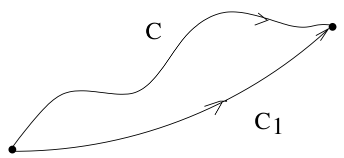
Consequences.

(1) If f is analytic, $\mathcal{C}$ *not* closed, then $\int_{\mathcal{C}} f(z)\,dz$ depends *only* on start/stop points. Indeed, if $\mathcal{C}$ is given, and C_1 is another path with the same start/stop points, $C_1 + (-\mathcal{C})$ make a simple closed path, so then

$$\int_{C_1} f(z)\,dz - \int_{\mathcal{C}} f(z)\,dz = 0,$$

i.e.,

$$\int_{\mathcal{C}} f(z)\,dz = \int_{C_1} f(z)\,dz.$$

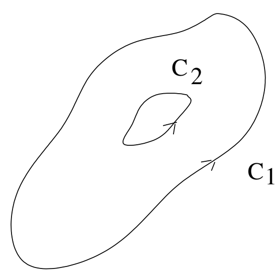


Of course we require f to be analytic on C_1, C_2 and the region between them.

(2) Let $\mathcal{C}$ be *any* simple closed path. Then

$$\oint_{\mathcal{C}} e^z\,dz = 0, \quad \oint_{\mathcal{C}} z^3\,dz = 0, \quad \oint_{\mathcal{C}} \sin z\,dz = 0, \text{ etc.}$$

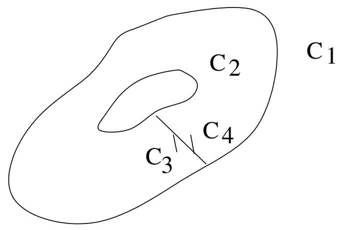
(3) Let C_1, C_2 be as shown.



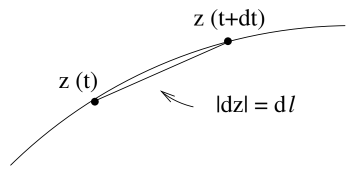
Suppose f is *analytic in the region between C_1 and C_2*. Then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz.$$

To see this, look at cuts, just like before.



We now pass to a closely related result: Cauchy's integral formula. Before we do, we observe a couple of preliminary results. Recall that from the defintion of the integral, $\int_{\mathcal{C}} f(z)\,dz$ is obtained by discretizing $\mathcal{C}$ into pieces as t goes from t to $t+dt$. We then sum $\sum f(z(t))[z(t+dt)-z(t)]$ over all the subsegments into which $\mathcal{C}$ has been decomposed and take limit as $dt \to 0$.



Now we recall that $|a|\,|b| = |ab|$ and $|a+b| \le |a| + |b|$ for any complex a, b. So

$$\left|\sum f(z(t))\,dz\right| \le \sum |f(z(t))|\,|dz|$$

where we have put dz for $z(t+dt)-z(t)$. If we know $|f(z(t))| \le M$ for some M, we conclude that

$$\left|\int_{\mathcal{C}} f(z(t))\,dz\right| \le \int_{\mathcal{C}} M\,|dz|.$$

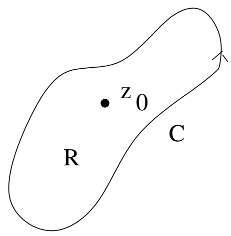
Now

$$|dz| = \sqrt{(dx)^2+(dy)^2} = \sqrt{(x')^2+(y')^2}\,dt = d\ell,$$

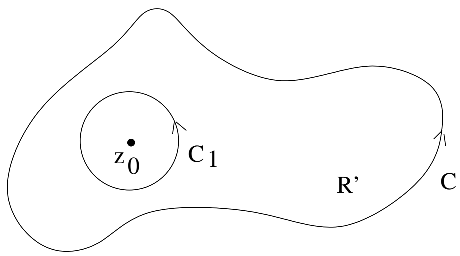
the bit of *length* of $\mathcal{C}$ between $z(t)$ and $z(t+dt)$. So

$$\left|\int_{\mathcal{C}} f(z(t))\, dz\right| \leq M \int_{\mathcal{C}} |dz| = M\ell$$

where ℓ = length of $\mathcal{C}$. Note that this is true for $\mathcal{C}$ closed or not. Now we pass to our next result. Let $\mathcal{C}$ be any simple closed path and z_0 any point inside $\mathcal{C}$. We consider $\oint f(z)/(z-z_0)\, dz$ for *any* f analytic inside and on $\mathcal{C}$.



At first sight, one may think that this integral should be zero, but then one notices that Cauchy's Theorem does *not* hold inside $\mathcal{C}$ since $f(z)/(z-z_0)$ has zero for the bottom at $z = z_0$. Thus $f(z)/(z-z_0)$ does not exist at $z = z_0$, never mind being differentiable. So, we take z_0 "out" by introducing a new path C_1 which is a small circle of radius ε centered at z_0 (as shown).



Let R' be the region between $\mathcal{C}$ and C_1. Then $f(z)/(z-z_0)$ is analytic in R' (remember: the quotient rule holds, and z_0 is *not* in R'). So

$$\oint_{\mathcal{C}} \frac{f(z)}{z-z_0}\, dz = \oint_{C_1} \frac{f(z)}{z-z_0}\, dz,$$

just as noted before. We focus on the right hand side. We add and subtract $f(z_0)$:

$$\oint_{C_1} \frac{f(z)}{z - z_0}\, dz = \oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz + \oint_{C_1} \frac{f(z_0)}{z - z_0}\, dz.$$

Now the second integral is

$$\oint_{C_1} \frac{f(z_0)}{z - z_0}\, dz = f(z_0) \oint_{C_1} \frac{dz}{z - z_0} = f(z_0) \cdot 2\pi j$$

since C_1 is a circle centered at z_0. (This is exactly one of the earlier examples!)

On the other hand, if C_1 has small enough radius, for z on C_1 we have

$$\frac{f(z) - f(z_0)}{z - z_0} \cong \frac{df}{dz}(z_0).$$

Then

$$\begin{aligned}
\left| \oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| &\leq \oint_{C_1} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| |dz| \\
&\leq M \oint_{C_1} |dz| = M \cdot \text{length of } C_1 \\
&= M \cdot 2\pi\varepsilon \longrightarrow 0 \text{ as } \varepsilon \to 0
\end{aligned}$$

where M is an estimate on $|\frac{df}{dz}(z_0)|$. So, we get by letting $\varepsilon \to 0$,

$$\oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz = (2\pi j) f(z_0)$$

or

$$\boxed{f(z_0) = \frac{1}{2\pi j} \oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz}.$$

This is an *amazing* formula. It shows that if f is analytic and we choose *any* $\mathcal{C}$ and any z_0 inside $\mathcal{C}$, then

$$f(z_0) = \frac{1}{2\pi j} \oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz.$$

This means that if I know just $f(z)$ on $\mathcal{C}$, for *any* $\mathcal{C}$ containing a point z_0, then I know $f(z_0)$!

So if an analytic function f is given on a simple closed path $\mathcal{C}$, then we automatically specify f inside $\mathcal{C}$. *Any* changes from these values of f *must* destroy the property of f being differentiable! Totally different from the "real" world! We consider some examples.

Example 1. Evaluate $\frac{1}{2\pi j}\oint_{\mathcal{C}} \frac{f(z)}{z-z_0}\,dz$ if (a) $f(z) = e^z$; (b) $f(z) = \sin z$; (c) $f(z) = z^{10}$; and z_0 is a point inside the simple closed path $\mathcal{C}$.

Answer. (a) e^{z_0}; (b) $\sin(z_0)$; (c) z_0^{10}.

Example 2. Same as Example 1, except z_0 is *outside* $\mathcal{C}$.

Answer. (a) 0; (b) 0; (c) 0.

Note that in Examples 1 and 2 we did not specify $\mathcal{C}$, nor the specific z_0 either inside or outside $\mathcal{C}$!

There are more consequences of this formula. We have

$$f(z_0) = \frac{1}{2\pi j}\oint \mathcal{C}\,\frac{f(z)}{z-z_0}\,dz.$$

It is useful to change notation: let s replace z and z replace z_0. We get

$$f(z) = \frac{1}{2\pi j}\oint_{\mathcal{C}} \frac{f(s)}{s-z}\,ds$$

for any z inside the simple closed contour $\mathcal{C}$. We then have

$$\frac{df(z)}{dz} = \frac{1}{2\pi j}\oint_{\mathcal{C}} \frac{f(s)}{(s-z)^2}\,ds$$

by differentiating both sides with respect to z. If we do it again, we get

$$\frac{d^2 f(z)}{dz^2} = \frac{2}{2\pi j} \oint_{\mathcal{C}} \frac{f(s)}{(s-z)^3}\, ds.$$

Then

$$\frac{d^3 f}{dz^3}(z) = \frac{3 \cdot 2}{2\pi j} \oint_{\mathcal{C}} \frac{f(s)}{(s-z)^4}\, ds$$

and so

$$\frac{d^n f}{dz^n}(z) = \frac{n!}{2\pi j} \oint_{\mathcal{C}} \frac{f(s)}{(s-z)^{n+1}}\, ds.$$

These formulas, connecting integrals and derivatives, will be used in what follows.

Further Exercises:

Exercise 1. Let C_1 be the circle $x^2 + y^2 = 1$, and C_2 be the straight line $y = x$ in the z-plane. These curves meet at $z_0 = 1/\sqrt{2} + 1/\sqrt{2}\,j$. Find the angle between their image curves $w(C_1)$, $w(C_2)$ at the point of intersection $w(z_0)$ if $w = 1/z$.

Answer. Now C_1 is the circle $x^2 + y^2 = 1$ and C_2 is the line $y = x$. So we can parametrize C_1 as

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} \quad 0 \leq \theta \leq 2\pi.$$

Let $\mathbf{r}_1(\theta) = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$ be the position vector for C_1. Then $d(\mathbf{r}_1(\theta))/d\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$. In the same way

$$C_2 = \begin{cases} x = t \\ y = t \end{cases} \quad -\infty < t < \infty, \quad \mathbf{r}_2(t) = t\mathbf{i} + t\mathbf{j}, \quad \frac{d\mathbf{r}_2}{dt} = \mathbf{i} + \mathbf{j}.$$

So C_1, C_2 meet at $1/\sqrt{2} + j1/\sqrt{2}$ (thus $\cos\theta = 1/\sqrt{2}$, $\sin\theta = 1/\sqrt{2}$) at an angle ϕ given by

$$\left(-\frac{1}{\sqrt{2}}\mathbf{i} + \mathbf{j}\frac{1}{\sqrt{2}}\right) \cdot (\mathbf{i} + \mathbf{j}) = \cos\phi \left|-\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right| |\mathbf{i} + \mathbf{j}|.$$

So $\cos\phi = 0$ and $\phi = \pi/2$. Now $w = 1/z$, $dw/dz = -1/z^2$ and $dw/dz \neq 0$ at $z = 1/\sqrt{2} + j\,1/\sqrt{2}$. Thus $w(C_1)$, $w(C_2)$ meet at the same angle as C_1, C_2, namely $\pi/2$.

Evaluate the following integrals: $\int_{\mathcal{C}} f(z)\,dz$.

Exercise 2. $f(z) = (\overline{z})^2$, $\mathcal{C}$ is the piece of parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

Answer.

$$\int_{\mathcal{C}} (\overline{z})^2\,dz = \int_{\mathcal{C}} (x - jy)^2(dx + jdy).$$

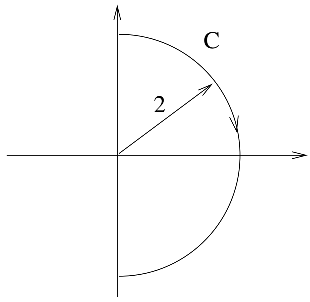
$\mathcal{C}$ can be parametrized as $\begin{cases} x = t \\ y = t^2 \end{cases}$ $0 \le t \le 1$. Thus

$$\int_{\mathcal{C}} (\overline{z})^2 \, dz = \int_{t=0}^{1} (t - jt^2)^2 (1 + 2tj) \, dt = \int_{t=0}^{1} (t^2 - 2jt^3 - t^4)(1 + 2tj) \, dt$$
$$= \left(\frac{1}{3} - \frac{2j}{4} - \frac{1}{5} \right) + 2j \left(\frac{1}{4} - \frac{2j}{5} - \frac{1}{6} \right).$$

Exercise 3. $f(z) = 1/z$, $\mathcal{C}$ is the semicircle $x^2 + y^2 = 4$ from $(0, 2)$ to $(0, -2)$ with $x \ge 0$.

Answer.

$$\int_{\mathcal{C}} \frac{1}{z} \, dz, \qquad \mathcal{C} = \begin{cases} x = 2\sin\theta \\ y = 2\cos\theta \end{cases} \quad 0 \le \theta \le \pi.$$



Since $z = 2(\sin\theta + j\cos\theta) = 2j(\cos\theta - j\sin\theta) = 2je^{-j\theta}$,

$$\int_{\mathcal{C}} \frac{1}{z} \, dz = \int_{\theta=0}^{\pi} (2je^{-j\theta})^{-1} (2j)(-j) e^{-j\theta} \, d\theta = \int_{0}^{\pi} (-j) \, d\theta = -\pi j.$$

Exercise 4. $f(z) = e^z$, $\mathcal{C}$ is any smooth path joining $1 - j$ to j.

Answer. Put $w = e^z$. Then $dw/dz = e^z$. So

$$\int_{\mathcal{C}} f(z) dz = w(\text{end}) - w(\text{start}) = e^j - e^{1-j} = [\cos(1) + j\sin(1)] - e[\cos(1) - j\sin(1)].$$

Exercise 5. $f(z) = \sin z$, $\mathcal{C}$ is any smooth path joining 0 to $2j$.

Answer. Same as 4: $w = -\cos z$. So

$$\int_{\mathcal{C}} f(z)\,dz = -\cos(2j) - (-\cos 0) = 1 - \cos(2j).$$

Now

$$\cos(2j) = \frac{e^{(2j)j} + e^{(-2j)j}}{2} = \frac{e^{-2} + e^{2}}{2}, \quad \text{and} \quad \int_{\mathcal{C}} f(z)\,dz = 1 - \frac{1}{2}(e^2 - e^{-2}).$$

Exercise 6. $f(z) = \overline{z}$, $\mathcal{C}$ is the straight line joining $z = 0$ to $z = 1 + j$.

Answer. In this case

$$\mathcal{C} = \begin{cases} x = t \\ y = t \end{cases} \quad 0 \le t \le 1.$$

So

$$\int_{\mathcal{C}} f(z)\,dz = \int_{t=0}^{1} (t - jt)(1 + j)\,dt = \int_{t=0}^{1} 2t\,dt = 1.$$

Exercise 7. $f(z) = (\overline{z})^{-1}$, $\mathcal{C}$ is the circle $x^2 + y^2 = 1$, traversed counterclockwise once.

Answer. Once again,

$$\mathcal{C} = \begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} \quad 0 \le \theta \le 2\pi.$$

So $x + jy$ on $\mathcal{C}$ is $\cos\theta + j\sin\theta = e^{j\theta}$, $x - jy = e^{-j\theta}$. Then

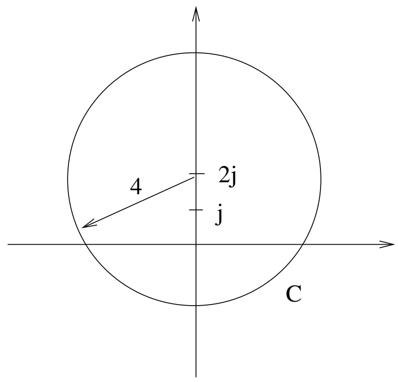
$$\int_{\mathcal{C}} \frac{1}{\overline{z}}\,dz = \int_{\theta=0}^{2\pi} \frac{1}{e^{-j\theta}} j e^{j\theta}\,d\theta = j\int_{0}^{2\pi} e^{2j\theta}\,d\theta = \left.\frac{e^{2j\theta}}{2}\right|_{0}^{2\pi} = 0.$$

Exercise 8. Evaluate $\oint_{\mathcal{C}} f(z)\,dz$ if $f(z) = e^z/(z - j)$ and $\mathcal{C}$ is the circle centered at $z = 0$ of radius $1/2$.

Answer. Since $z = j$ is not inside/on $\mathcal{C}$, $e^z/(z-j)$ is analytic in and on $\mathcal{C}$ and $\oint f(z)\,dz = 0$.

Exercise 9. Same as exercise 8 except $\mathcal{C}$ is the circle centered at $z = 2j$ of radius 4.

Answer.

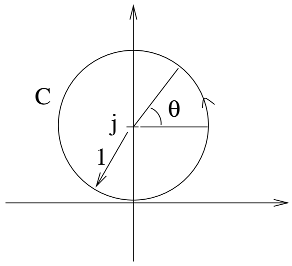


Now j is inside $\mathcal{C}$ so

$$\oint \frac{e^z}{z-j}\,dz = 2\pi j \left[e^z \Big|_j \right] = 2\pi j e^j.$$

Exercise 10. Evaluate $\oint 1/(\overline{z}+j)\,dz$ if $\mathcal{C}$ is the circle of radius 1 centered at j.

Answer. It would be tempting to say $\oint 1/(\overline{z}+j)\,dz = 2\pi j(0)$, but since $-j$ is not in/on $\mathcal{C}$, this is wrong since we have $\overline{z}$, *not* z! So the only way to deal with this, given that also $\overline{z}$ is not analytic, is to parametrize and carry out the calculations. Now $\mathcal{C} =$ circle of radius 1 centered at j.



Thus

$$\mathcal{C} = \begin{cases} x = 0 + \cos\theta \\ y = 1 + \sin\theta \qquad 0 \le \theta \le 2\pi, \end{cases}$$

i.e., on $\mathcal{C}$,

$$z = x + jy = j + e^{j\theta}, \qquad dz = je^{j\theta}\, d\theta$$

$$\overline{z} = -j + e^{-j\theta} \qquad \Longrightarrow \qquad \overline{z} + j = e^{-j\theta}.$$

So

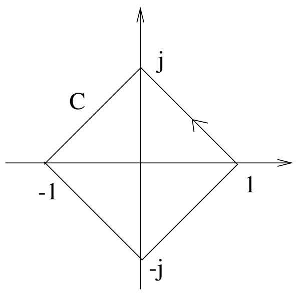
$$\oint_{\mathcal{C}} \frac{1}{\overline{z} + j}\, dz = \int_{\theta=0}^{2\pi} \frac{je^{j\theta}}{e^{-j\theta}}\, d\theta = j\int_0^{2\pi} e^{2j\theta}\, d\theta = j\left[\frac{e^{2j\theta}}{2j}\right]_0^{2\pi} = 0.$$

So we get 0 anyway!

Exercise 11. Evaluate $\oint \cos z/z\, dz$ if $\mathcal{C}$ is the path consisting of straight lines, joining the points $-j$ to 1 to j to -1.

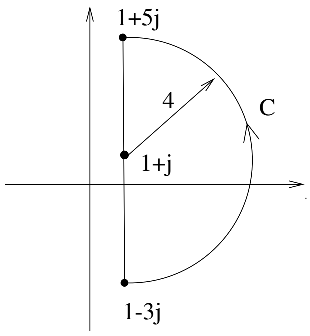
Answer.

$$\begin{aligned} \oint_{\mathcal{C}} \frac{\cos z}{z}\, dz &= 2\pi j[\cos 0] \quad \text{since 0 is inside } \mathcal{C} \\ &= 2\pi j. \end{aligned}$$



Exercise 12. Same as exercise 11 except $\mathcal{C}$ is the semicircle centered at $z = 1 + j$ of radius 4 with $\mathrm{Re}\,(z) \ge 1$ and the straight line from $1 + 5j$ to $1 - 3j$.

Answer.



In this case $z = 0$ is outside $\mathcal{C}$. Thus $\oint_{\mathcal{C}} \cos z/z\, dz = 0$ as $\cos z/z$ is analytic inside and on $\mathcal{C}$.

III.d *Taylor Series and Laurent Series*

So far we have discussed three situations: (1) $\int_{\mathcal{C}} f(z)\,dz$ with $\mathcal{C}$ just a path encountered in practice (i.e., maybe *not* a simple closed path); (2) $\oint_{\mathcal{C}} f(z)\,dz$ for $\mathcal{C}$ a simple closed path; (3) $\oint_{\mathcal{C}} f(s)/(s-z)\,ds$ for $\mathcal{C}$ a simple closed path, and z inside $\mathcal{C}$.

In case (1) (which *includes* simple closed paths), for general $f(z)$, we can only calculate $\int_{\mathcal{C}} f(z)\,dz$ by parametrizing the path and actually working out the integral. For case (2), if $f(z)$ is analytic inside and on $\mathcal{C}$, then $\oint_{\mathcal{C}} f(z)\,dz = 0$. This result also has an implication for case (1). If $\mathcal{C}$, C_1 are two paths with f analytic between and on C_1, C_2, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Indeed $C_1 + (-C_2)$ can be viewed as a simple closed path, so

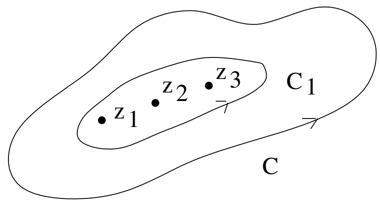
$$0 = \oint_{C_1+(-C_2)} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$$

and we can do the calculation with a simpler path. Finally, in case (3),

$$\oint_{\mathcal{C}} \frac{f(s)}{s-z}\,ds = [f(z)]2\pi j,$$

if again f is analytic inside and on $\mathcal{C}$.

Unfortunately for the applications we seek (inverting transforms and interpreting the answer) cases (1), (2), (3) are not enough. We shall wish to work out $\oint_{\mathcal{C}} f(z)\,dz$, with $\mathcal{C}$ a simple closed path, but with $f(z)$ complicated *and* blowing up at various points inside $\mathcal{C}$, say at $z_1, z_2, \ldots$. So we cannot usually use either cases (2) or (3) directly. Of course, once again one could try to replace $\mathcal{C}$ by a simpler path C_1, say a circle, such that f is analytic between and on C_1 and $\mathcal{C}$.



Then

$$\oint_{\mathcal{C}} f(z)\,dz = \oint_{C_1} f(z)\,dz,$$

and one could attempt to parametrize C_1 and actually carry out the calculations. Unfortunately, for most problems of practical interest, even this attempt does not work, since $f(z)$ will simply be too complicated for us to do the needed calculations on C_1. It seems that integrals for problems of practical interest cannot be solved, and yet there is a "sneaky" way (related to case (3)) by means of which we can find $\oint_{\mathcal{C}} f(z)\,dz$ relatively easily. This has to do with series expansions for $f(z)$, which we now consider.

III.d.1. Power Series.

We recall first a few results about power series, that is, series of type

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots$$

with $a_0, a_1, a_2, a_3, \ldots,$ complex *constants*. It is clear that as we begin to calculate $S_N(z) = \sum_{n=0}^{N} a_n(z-z_0)^n$ we may find that $|S_N(z)|$ blows up or oscillates wildly as $N \to \infty$. A trivial example of this is $\sum_{n=0}^{\infty}(z-1)^n$ with $z_0 = 1$; $a_1, a_2, \cdots = 1$. Then if $z = 2$, say,

$$S_N(2) = \sum_{n=0}^{N}(2-1)^n = \underbrace{1+1+1+\cdots+1}_{N+1 \text{ times}} = N+1.$$

So $|S_N(2)| = N+1 \to \infty$. We state in such a case that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *diverges.*

On the other hand, if $|z - z_0|$ is small enough, then S_N will actually approach a complex number L as $N \to \infty$. In this case we state that $\sum_0^\infty a_n(z-z_0)^n$ *converges* (to L) and write $L = \sum_{n=0}^\infty a_n(z - z_0)^n$.

One can spend a lot of time on these problems, but for us it will suffice to note a few facts.

Suppose we want to just see if a given power series converges without trying to evaluate the limit L. One of the most useful criteria is the *ratio test.* Look at the ratio of two *consecutive terms of the series:*

$$\frac{|a_{n+1}(z - z_0)^{n+1}|}{|a_n(z - z_0)^n|} = \left|\frac{a_{n+1}}{a_n}\right| |z - z_0|.$$

For *convergence*, we ask that z be such that this ratio, for n large, is less than 1. I.e.,

$$\lim_{n\to\infty} \left[\left|\frac{a_{n+1}}{a_n}\right| |z - z_0|\right] < 1$$

or

$$|z - z_0| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1.$$

Finally,

$$|z - z_0| < \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

So, if this estimate holds, the series *converges.* On the other hand if z is such that

$$|z - z_0| > \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|,$$

then the series *diverges.* If

$$|z - z_0| = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

the problem is delicate, and not within the scope of this course. We illustrate this with a useful example.

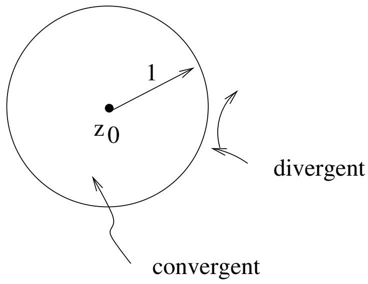
Example 1. Consider (for some given z_0) the series $\sum_{n=0}^{\infty}(z-z_0)^n$. Here all $a_n = 1$ and so we have *convergence* if

$$|z-z_0| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{1}\right| = 1$$

and *divergence* if

$$|z-z_0| > \lim_{n\to\infty}\left|\frac{1}{1}\right| = 1.$$

So if $|z-z_0| < 1$ the series *converges*, if $|z-z_0| > 1$, the series *diverges*. The radius of the circle centered at z_0 which divides the region of convergence from the region of divergence, is called the *radius of convergence*. In this case, it's 1.



Note that in general we do *not* have a clue from the ratio test as to what a convergent series actually converges to (i.e., what L is). But for $\sum_{n=0}^{\infty}(z-z_0)^n$ we can actually work out what L is. Set

$$S_N = \sum_{n=0}^{N}(z-z_0)^n = 1 + (z-z_0) + \cdots + (z-z_0)^N.$$

Then

$$(z-z_0)S_N = (z-z_0) + (z-z_0)^2 + \cdots + (z-z_0)^{N+1}$$

and so

$$S_N[1-(z-z_0)] = 1-(z-z_0)^{N+1}.$$

That is,

$$S_N = \frac{1-(z-z_0)^{N+1}}{1-(z-z_0)}.$$

Since $|z-z_0| < 1$, then

$$|(z-z_0)^{N+1}| = |z-z_0|^{N+1} \longrightarrow 0 \quad \text{as } N \to \infty$$

and so

$$S_N \longrightarrow \frac{1}{1-(z-z_0)} \quad \text{as } N \to \infty.$$

I.e.,

$$\sum_{n=0}^{\infty}(z-z_0)^n = \frac{1}{1-(z-z_0)}.$$

In summary,

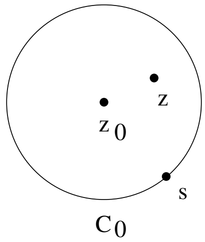
$$\sum_{n=0}^{\infty}(z-z_0)^n = \frac{1}{1-(z-z_0)} \quad \text{if} \quad |z-z_0| < 1.$$

But $\sum_{n=0}^{\infty}(z-z_0)^n$ does *not* exist if $|z-z_0| > 1$. Note that if $|z-z_0| > 1$, $1/(1-(z-z_0))$ *does* exist; it just does *not* equal $\sum_0^{\infty}(z-z_0)^n$. What happens if $|z-z_0| = 1$, is not clear from these arguments, and of not much interest to us.

III.d.2. Taylor and Laurent Series (Mathematics).

In this section we consider the purely mathematical aspects of the series we wish to use. Unfortunately, the Laurent series part as presented in this section cannot be effectively used in practice, since it's nonconstructive.

We now return to complex functions. We have first of all (as a warm up) *Taylor's Series.* Let f be analytic inside and on C_0, a circle of radius r_0 centered at z_0, and let z be a point inside C_0 as shown.



We have

$$f(z) = \frac{1}{2\pi j} \oint_C \frac{f(s)}{s - z}\, ds.$$

Now we write

$$\frac{1}{s-z} = \frac{1}{(s-z_0) - (z-z_0)} = \frac{1}{s-z_0}\left[\frac{1}{1 - \frac{z-z_0}{s-z_0}}\right].$$

But $|z - z_0| < |s - z_0| = r_0$ since z_0 is the center of the circle. That is,

$$\frac{1}{s-z} = \frac{1}{s-z_0}\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(s-z_0)^n} = \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(s-z_0)^{n+1}}.$$

We conclude

$$\begin{aligned} f(z) &= \frac{1}{2\pi j}\oint_C \frac{f(s)}{s-z}\, ds = \frac{1}{2\pi j}\oint_C f(s)\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(s-z_0)^{n+1}}\, ds \\ &= \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi j}\oint_C \frac{f(s)}{(s-z_0)^{n+1}}\, ds\right)}_{\frac{1}{n!}\frac{d^n f}{dz^n}(z_0) \text{ from the previous section}} (z-z_0)^n. \end{aligned}$$

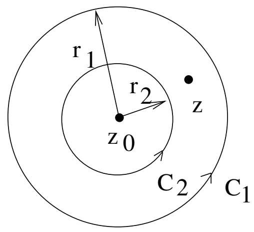
So

$$\boxed{f(z) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^n f}{dz^n}(z_0)(z-z_0)^n}.$$

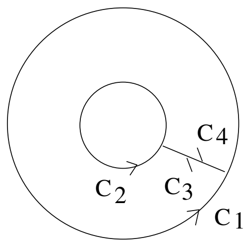
We have found the good old Taylor's Series! Unfortunately, this is not quite enough.

We need a variation of this series, called the *Laurent Series.*

Suppose f is analytic in the region *between* the two concentric circles C_1, C_2 as shown. Let z_0 be the center, and z a point in between the two circles.



Now repeating our earlier procedures, we cut the region between C_1, C_2 by introducing the paths C_3, C_4.



Then $\mathcal{C} = C_1 + C_3 + (-C_2) + C_4$ is a path which encloses a region in which f is analytic (observe C_2 is traversed backwards) and z is a point in this region. We have

$$f(z) = \frac{1}{2\pi j} \int_{\mathcal{C}} \frac{f(s)}{s-z}\, ds = \frac{1}{2\pi j} \left[\int_{C_1} + \int_{C_3} + \int_{-C_2} + \int_{C_4} \right].$$

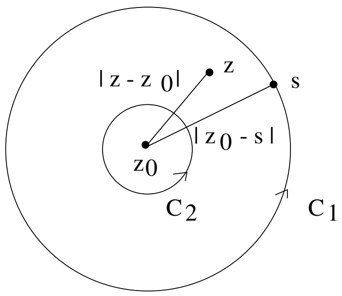
But

$$\int_{C_3} \frac{f(s)}{s-z}\, ds = - \int_{C_4} \frac{f(s)}{s-z}\, ds.$$

So

$$f(z) = \frac{1}{2\pi j} \oint_{C_1} \frac{f(s)}{s-z}\, ds - \frac{1}{2\pi j} \oint_{C_2} \frac{f(s)}{s-z}\, ds.$$

Let us focus first on $\oint_{C_1} f(s)/(s-z)\, ds$. Observe that s is a point on C_1, so $|z - z_0| < |z_0 - s|$



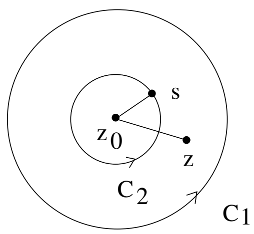
and

$$\frac{1}{s-z} = \frac{1}{(s-z_0)+(z_0-z)} = \frac{1}{s-z_0}\left[\frac{1}{1-\frac{(z-z_0)}{s-z_0}}\right] = \frac{1}{s-z_0}\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(s-z_0)^n}.$$

So

$$\oint_{C_1}\frac{f(s)}{s-z}\,ds = \sum_{n=0}^{\infty}\left[\oint_{C_1}\frac{f(s)}{(s-z_0)^{n+1}}\,ds\right](z-z_0)^n.$$

Note that $\oint_{C_1} f(s)/(s-z_0)^{n+1}ds$ may not be at all related to a derivative of f, since f may *not* have a derivative at z_0. We focus on the 2nd integral: $-\oint_{C_2} f(s)/(s-z)\,ds$. Now, s is on C_2, and so $|s-z_0| < |z-z_0|$.



We have

$$\frac{1}{s-z} = \frac{1}{(s-z_0)+(z_0-z)} = \frac{1}{z_0-z}\left[\frac{1}{\frac{s-z_0}{z_0-z}+1}\right] = \frac{1}{z-z_0}\left[\frac{-1}{1-\frac{(s-z_0)}{z-z_0}}\right].$$

So

$$-\oint_{C_2}\frac{f(s)}{s-z}\,ds = \oint_{C_2}\frac{f(s)}{(z-z_0)}\left[\frac{1}{1-\frac{(s-z_0)}{z-z_0}}\right]ds = \oint_{C_2}\frac{f(s)}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{s-z_0}{z-z_0}\right)^n ds$$

$$= \sum_{n=0}^{\infty} \left[\oint_{C_2} f(s)(s - z_0)^n \right] \frac{1}{(z - z_0)^{n+1}} \, ds.$$

In summary,

$$f(z) = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi j} \oint_{C_1} \frac{f(s)}{(s - z_0)^{n+1}} \, ds \right] (z - z_0)^n$$
$$+ \sum_{n=0}^{\infty} \left[\frac{1}{2\pi j} \oint_{C_2} f(s)(s - z_0)^n \, ds \right] \frac{1}{(z - z_0)^{n+1}}.$$

Observe that, for a given z_0, the integrals over C_1 and C_2 are *constants.* This is called the *Laurent Series for f.* In view of what was said at the beginning of this section about the difficulty of calculating integrals in practical situations, the Laurent Series is basically *never calculated in a practical problem from its definition.* The importance of this series comes from the *first term in the second sum (i.e., for $n = 0$).* In this case the term is

$$\frac{1}{2\pi j} \left[\frac{\oint_{C_2} f(s) \, ds}{s - z_0} \right].$$

So *if by some miracle we can find the Laurent Series for f, then $\oint_{C_2} f(s) \, ds$ can be found just by reading the coefficient of $1/(z - z_0)$! Even more usefully, if we can just find the coefficient of $1/(z - z_0)$—never mind the rest of the series—then we are done as far as finding $\oint_{C_2} f(s) \, ds$ is concerned.* How to achieve this goal is the topic of the next section.

One notational remark: Note that the Laurent series can be expressed after a summation index change as:

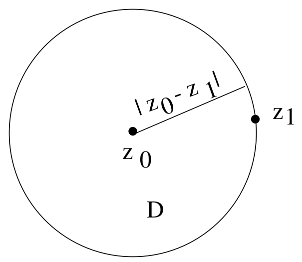
$$f(z) = \sum_{-\infty}^{\infty} c_n (z - z_0)^n$$

with

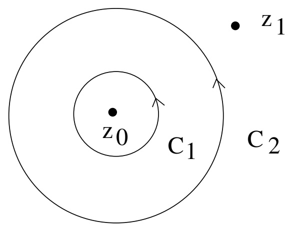
$$c_n = \frac{1}{2\pi j}\oint_{C_1} \frac{f(s)}{(s-z_0)^{n+1}}\, ds$$

for any $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. It is valid for all z between C_1 and C_2!

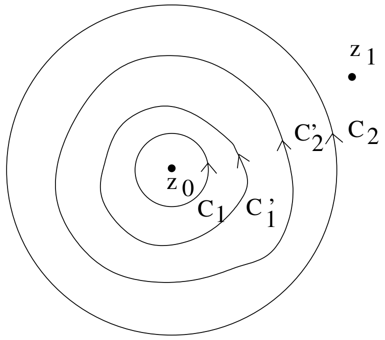
We conclude with the following *important remark*. Suppose—as is usually the case in practice—that f blows up at z_0 and z_1 and *nowhere else* in the disc D centered at z_0 of radius $|z_0 - z_1|$.



Let C_1, C_2 be two circles in D as shown.



We calculate (at least in theory) the Laurent series using our earlier formulas and the given C_1, C_2. The coefficients c_n in the series involve path integrals, some over C_1 and others over C_2. *So it would appear at first that the c_n depend on the specific C_1, C_2 chosen. This is* **not** *the case.* Suppose C_1', C_2' are two other paths, as shown, which may *not* be circles.



Then $\oint_{C_1} = \oint_{C_1'}$ and $\oint_{C_2} = \oint_{C_2'}$ since the functions being integrated have a derivative in the disc centered at z_0 of radius $|z - z_0|$, except at z_0 itself and on the rim of the disc. *So the c_n are always the same for any pair of nested paths, in the disc D, which also contain z_0.* That is to say, the dependence of the coefficients c_n on C_1, C_2 is fictitious, since any pair of paths C_1, C_2 inside D can be used. The resulting c_n will be the same. In other words, c_n *really only depend on z_0, z_1 and $f(z)$.*

III.d.3. Taylor and Laurent Series (Engineering).

The previous section was theoretical, and the part on the Laurent series is not of much practical use due to the inability to calculate the integrals involved.

From a practical point of view, the following considerations are more useful, and summarize what we did in the previous section. First, suppose $f(z)$ is analytic at z_0. Then we can write a Taylor series for f using the formula you have already seen in the past:

$$f(z) = \sum_{n=0}^{\infty} \left[\frac{d^n f}{dz^n}(z_0) \right] \frac{(z - z_0)^n}{n!}.$$

We can be sure that this series converges, i.e., actually equals f, for z near z_0. Indeed it will converge for all z inside a disc centered at z_0 which does not contain any singular points of $f(z)$.

On the other hand, suppose $f(z)$ has a singular point at z_0, but is analytic at all points *near* z_0. What we have in mind is a function that blows up at z_0 but is differentiable *near* z_0, e.g., $f(z) = z^2/(z^2+1)$ with z_0 either $+j$ or $-j$. We then can represent $f(z)$ in the Laurent series form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

for some coefficients c_n. These coefficients cannot be found from the definition of the Laurent series, since they involve integrals too complicated to be evaluated for $f(z)$ of practical interest. The c_n must be *evaluated in some other way*, and in general this is virtually impossible. For the $f(z)$ found in practice, this can however be done fairly easily, see the next section. We observe two things. First, the Laurent series will converge in the largest disc centered at z_0 (except at z_0 itself) which contains no other singular points.

Second, note that if

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

and C is a circle centered at z_0, then

$$\oint f(z)\,dz = \sum_{n=-\infty}^{\infty} c_n \oint_C (z - z_0)^n\,dz = c_{-1} 2\pi j$$

since $\oint_C (z - z_0)^n\,dz = 0$ except if $n = -1$, when it equals $2\pi j$ (by the earlier important example). Thus if we know c_{-1} (by some miracle), then we know $\oint_C f(z)\,dz$.

III.e *Calculation of the Laurent Series; Poles and Residues*

We begin this section by examining ways in which the Laurent series can actually be calculated. We then pass to the question of how to find just the coefficient of $1/(z-z_0)$. Finding the complete Laurent series is harder than finding just the coefficient of $1/(z-z_0)$.

As mentioned in the last section, a direct calculation of the Laurent series, by evaluating the integrals involved in the definition, is basically impossible. Instead we use Taylor and other series, as well as division, multiplication, etc. These steps will be illustrated in the examples that follow. We just have two important remarks.

(1) *When a series is obtained—on the way to finding the Laurent series—pay attention to where it converges.*

(2) We know the Laurent series is of type: $f(z) = \sum_{n=-\infty}^{n=\infty} c_n(z-z_0)^n$ with c_n constants. *It can be shown that any series of this type must be the Laurent series for f about z_0.*

Example 1. Let $f(z) = 1/z(2-z)$. Find the Laurent series for $f(z)$ valid about $z=0$ (i.e., $z_0 = 0$).

Answer. Note that the paths C_1, C_2 are not given here. We discussed this point in the last section. We seek a series of type $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ which converges near $z=0$. We don't even dream of using the definition of the series. Instead, note

$$\frac{1}{2-z} = \frac{1}{2}\frac{1}{1-\frac{z}{2}}.$$

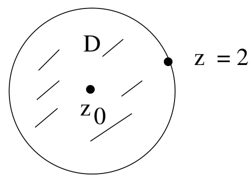
Now

$$\frac{1}{1-\frac{z}{2}} = \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n, \quad \text{converges if} \quad |z|<2.$$

So

$$\frac{1}{z(2-z)} = \frac{1}{z}\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\frac{z^{n-1}}{2^{n+1}}$$
$$= \frac{1}{z2} + \frac{1}{2^2} + \frac{z}{2^3} + \frac{z^2}{2^4} + \cdots .$$

We have thus found a series for $f(z) = 1/(z(2-z))$ of the right type. This is the Laurent series for f in the region $D : 0 < |z| < 2$.



Example 2. Same problem as Example 1, except the series is to be valid about $z = 2$.

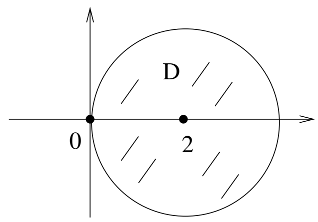
Answer. As a first step, we examine why the series found in Example 1 does *not* work. We got there

$$f(z) = \sum_{n=0}^{\infty}\frac{z^{n-1}}{2^{n+1}} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{z^{n-1}}{2^{n-1}} = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{n-1}.$$

So if z is any number with $|z| > 2$, then this series *diverges* ($|z/2| > 1$!). So $f(z)$ is O.K. for $z \neq 2$ but the series is *not, and we wish a series valid for some disc centered at* $z = 2$*!* The idea is to get a series so that if $|z-2|$ *is small* then it converges, and to do this we rearrange $f(z)$ as follows:

$$\frac{1}{z(z-2)} = \frac{1}{[(z-2)+2](z-2)} = \frac{1}{z-2}\left[\frac{1}{2-(2-z)}\right] = \frac{1}{(z-2)}\frac{1}{2}\left[\frac{1}{1-\frac{(2-z)}{2}}\right]$$
$$= \frac{1}{2(z-2)}\sum_{n=0}^{\infty}\frac{(2-z)^n}{2^n} = \sum_{n=0}^{\infty}\frac{(z-2)^{n-1}(-1)^n}{2^{n+1}}.$$

This series converges for $|z-2| < 2$, i.e., in the region shown.

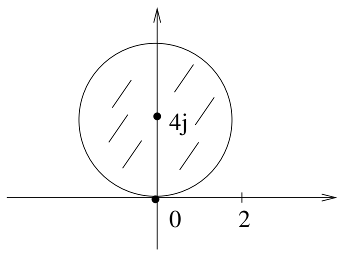


Remark 1. That the answer for Example 2 is so similar to that for Example 1—with $z-2$ in place of z—is basically a fluke.

Remark 2. Note that on the border of $|z-2| < 2$, we have the other singular point!

Example 3. Same as Examples 1 and 2, except now the Laurent series is to hold in a disc about $z = 4j$.

Answer. We need to find a series valid in a disc centered at $z = 4j$ of radius 4 (since the nearest singular point will be at $z = 0$, 4 units from $z = 4j$).



Since at $z = 4j$ the function is differentiable, we could calculate the Taylor series. This would be correct, but better to proceed as follows. We write $1/z(z-2)$ in powers of $z-4j$, i.e.,

$$\frac{1}{z(z-2)}=\frac{1}{[(z-4j)+4j]}\frac{1}{[(z-4j)+(4j-2)]}.$$

We could expand

$$\frac{1}{(z-4j)+4j}=\frac{1}{4j}\frac{1}{\frac{(z-4j)}{4j}+1}=\frac{1}{4j}\frac{1}{1-\frac{(-z+4j)}{4j}}$$

and

$$\frac{1}{z-4j+4j-2}=\frac{1}{4j-2}\frac{1}{1-\frac{(4j-z)}{4j-2}}$$

and multiply the series term by term. This is not recommended, since series multiplication term by term is a recipe for making mistakes. Series *addition* however is easy. So we use a *partial fraction expansion* (more on this later)

$$\frac{1}{z(z-2)}=\frac{A}{z}+\frac{B}{z-2},$$

so

$$1=A(z-2)+Bz \quad \text{for all } z, \quad \text{i.e., } A=-\frac{1}{2},\ B=\frac{1}{2}$$

and

$$\frac{1}{z(z-2)}=\frac{1}{2}\frac{1}{z-2}-\frac{1}{2}\frac{1}{z}.$$

Now

$$\frac{1}{z-2}=\frac{1}{(z-4j)+4j-2}=\frac{1}{4j-2}\frac{1}{1-\frac{4j-z}{4j-2}}=\frac{1}{4j-2}\sum_{n=0}^{\infty}\left(\frac{4j-z}{4j-2}\right)^n.$$

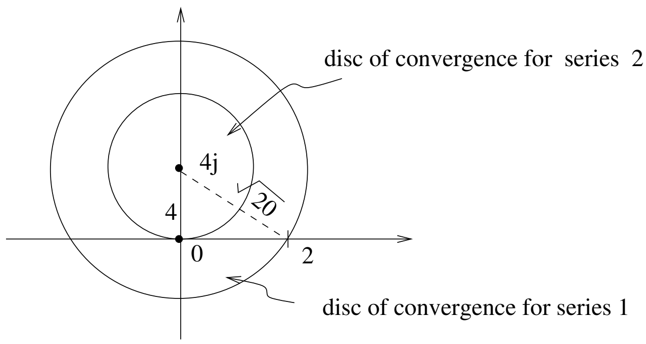
In the same way,

$$\frac{1}{z}=\frac{1}{(z-4j)+4j}=\frac{1}{4j}\frac{1}{1-(\frac{4j-z}{4j})}=\frac{1}{4j}\sum_{n=0}^{\infty}\left(\frac{4j-z}{4j}\right)^n.$$

So

$$\frac{1}{z(z-2)} = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(4j-z)^n}{(4j-2)^{n+1}} - \sum_{n=0}^{\infty}\frac{(4j-z)^n}{(4j)^{n+1}}\right)$$
$$= \sum_{n=0}^{\infty}(-4j+z)^n\frac{1}{2}\left[\frac{1}{(4j-2)^{n+1}} - \frac{1}{(4j)^{n+1}}\right](-1)^n.$$

Note that the first series converges for $|4j-z| < |4j-2| = \sqrt{20}$, the second for $|4j-z| < |4j| = 4$, so the sum converges for $|4j-z| < 4$. We can relate this to the location of the singular points: $1/(z-2)$ blows up at $z=2$, $1/z$ at $z=0$. So from $4j$ these points are $\sqrt{20}$ and 4 units distant respectively.



Example 4. Find a Laurent series for $e^{\frac{1}{z}}$ valid about $z=0$.

Answer. $e^{\frac{1}{z}}$ has only one singular point ($z=0$), so we expect a series valid for all $z \neq 0$. Now to do this, we actually use the Taylor series. We know

$$e^w = \sum_{n=0}^{\infty}\frac{w^n}{n!},$$

so put $w = 1/z$ to get

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\frac{1}{n!}.$$

Note that by the ratio test this converges for all $z \neq 0$, as $n!$ will ensure that the ratio of two consecutive terms is eventually (i.e., for n large enough) less than one, regardless of how small $|z|$ may be.

Example 5. Same as Example 4, except $f(z) = e^{\frac{1}{z}}/z^3$.

Answer. After Example 4 this is easy.

$$f(z) = \frac{1}{z^3}\left(\sum_{n=0}^{\infty} \frac{1}{z^n}\,\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{z^{n+3}}\,\frac{1}{n!}.$$

Do not let the above examples fool you. Finding a Laurent series can be really tough! Fortunately, for most practical problems we do *not* need to find the whole series, just the **residue**, i.e., the coefficient of $1/(z - z_0)$.

In Example 5, we found an example of a function whose Laurent series had terms of type $1/z^n$ with n arbitrarily large. This is not the situation usually found in practice. Instead, the "practical" functions have the property that the part of the Laurent series with negative $(z - z_0)$ powers *terminates* after a few terms, i.e., the series look like:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n + \frac{d_1}{(z - z_0)} + \frac{d_2}{(z - z_0)^2} + \cdots + \frac{d_m}{(z - z_0)^m}$$

with the d's$= 0$ after d_m. Some d's before d_m could also be zero but d_m itself is not. In such cases, f is said to have a pole of *order* m *at* z_0. If $m = 1$, the pole is *simple.*

As an example:

Example 6. Find the Laurent expansion of e^z/z^4 about $z = 0$.

Answer.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \Longrightarrow \qquad \frac{e^z}{z^4} = \sum_{n=0}^{\infty} \frac{z^{n-4}}{n!}.$$

That is,

$$\begin{aligned} \frac{e^z}{z^4} &= \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2}\frac{1}{2!} + \frac{1}{z3!} + \sum_{n=4}^{\infty} \frac{z^{n-4}}{n!} \\ &= \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{2z^2} + \frac{1}{6z} + \sum_{n=0}^{\infty} \frac{z^n}{(n+4)!}. \end{aligned}$$

We conclude that e^z/z^4 has a *pole of order* 4 *at* $z = 0$.

There is a way, given $f(z)$, to tell when it has a pole of order m at $z = z_0$. This happens when:

(a) $f(z)(z - z_0)^m\big|_{z=z_0}$ gives a finite number ($\neq$ zero)

(b) $f(z)(z - z_0)^p\big|_{z=z_0} = 0$ if $p > m$

(c) $f(z)(z - z_0)^q\big|_{z=z_0}$ blows up if $q < m$.

In practical situations, poles correspond to zeros of the transfer function, and are relatively easy to both spot and find their order. For example, in the previous case, we find

$$f(z) = \frac{e^z}{z^4},$$

so

$$\frac{e^z}{z^4} \cdot z^p\bigg|_{z=0} = e^z z^{p-4}\bigg|_{z=0} = \begin{cases} 0 & \text{if } p > 4 \\ 1 & \text{if } p = 4 \\ \infty & \text{if } p < 4 \end{cases}$$

and the function has a pole of order 4.

Now suppose f has a pole of order m at $z = z_0$. I.e., the Laurent series is

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n + \frac{d_1}{(z - z_0)} + \cdots + \frac{d_m}{(z - z_0)^m}.$$

Then

$$f(z)(z-z_0)^m = \sum_{n=0}^{\infty} c_n (z-z_0)^{n+m} + d_1(z-z_0)^{m-1} + \cdots + +d_{m-1}(z-z_0) + d_m.$$

We have

$$d_m = \lim_{z \to z_0} \left(f(z)(z-z_0)^m \right).$$

Next, differentiating both sides setting $z = z_0$ gives

$$d_{m-1} = \frac{d}{dz} \left[f(z)(z-z_0)^m \right] \Big|_{z=z_0}.$$

Differentiate again

$$d_{m-2} = \frac{1}{2} \frac{d^2}{dz^2} \left[f(z)(z-z_0)^m \right] \Big|_{z=z_0},$$

and so on. The way to remember this is that the right side is just the Taylor series for $f(z)(z-z_0)^m$ and the coefficients are given accordingly. In particular,

$$d_1 = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[f(z)(z-z_0)^m \right] \Big|_{z=z_0}$$

except, if $m = 1$, then

$$d_1 = \lim_{z \to z_0} \left(f(z)(z-z_0) \right).$$

So we can find d_1 (indeed as many coefficients of the Laurent series as we wish) reasonably easily if f just has a pole of order m at $z = z_0$. We pass to examples.

Example 7. Find the part of the Laurent series for $f(z) = 1/z(2-z)$ about $z = 0$ and $z = 2$, which involves negative exponents.

Answer. This is exactly the same function as in Examples 1 and 2. Now about

$z = 0$ we have

$$zf(z)\Big|_{z=0} = \frac{1}{2-z}\Big|_{z=0} = \frac{1}{2}$$

so $z = 0$ is a pole of order 1 (i.e., a simple pole) and its coefficient is $1/2$. About $z = 2$:

$$(z-2)f(z)\Big|_{z=2} = -\frac{1}{z}\Big|_{z=2} = -\frac{1}{2}$$

so $z = 2$ is also a simple pole, and the coefficient is $-1/2$. In the first case, the Laurent series starts with $\frac{1}{z} \cdot \frac{1}{2}$ and in the second case with $-\frac{1}{2} \cdot \frac{1}{(z-2)}$. In both cases, all other series terms involve nonnegative exponents of z and $z - 2$ respectively.

Example 8. Same as Example 7, except $f(z) = \cot z$ and $z_0 = 0$.

Answer. First of all note that $f(0)$ does not exist, so $z = 0$ is a singular point. We wish to multiply $f(z)$ by a power of z, say z^m, so that $z^m f(z)|_{z=0}$ is a nonzero number. Now

$$\cot z = \frac{\cos z}{\sin z} \quad \text{and} \quad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots,$$

so $\sin z/z \to 1$ as $z \to 0$, and the same is true of $z/\sin z$. In summary, $zf(z)|_{z=0} = \cos 0 = 1$, so $f(z)$ has a pole of order 1 at $z = 0$ and the coefficient of $1/z = 1$.

We conclude by recalling that the functions found in practice have the form

$$\frac{f(z)}{(z-z_0)^{n_0}(z-z_1)^{n_1}\cdots(z-z_m)^{n_m}}$$

with $f(z)$ differentiable. So the poles will be at $z_0, z_1, \ldots, z_m$ and these will have multiplicities $n_0, n_1, \ldots, n_m$, respectively except if f turns out to be also zero at one or more of $z_0, \ldots, z_m$. This is extremely rare in practice. If it happens, say at z_0, one just divides out $(z - z_0)$ from the top and bottom until the resulting $f(z)$

is not zero at z_0 or $(z - z_0)$ is no longer in the bottom. More examples in the next section.

Further Exercises:

Exercise 1. Find Laurent series expansion for $1/(z+j)$ valid in the regions: (a) $|z| < 1$, (b) $|z| > 1$, (c) $|z-2j| < 1$, (d) $|z+j| < 1$.

Answer. (a) Since $|z| < 1$, we write (as we wish to expand about $z = 0$)

$$\frac{1}{z+j} = \frac{1}{j}\,\frac{1}{1+(z/j)} = \frac{1}{j}\,\frac{1}{1-(-z/j)} = \frac{1}{j}\sum_{n=0}^{\infty}\left(-\frac{z}{j}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{j^{n+1}}.$$

This converges if $|z/j| < 1$, i.e., $|z| < |j| = 1$, as needed.

(b) Since $|z| > 1$,

$$\frac{1}{z+j} = \frac{1}{z}\,\frac{1}{1+j/z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n j^n}{z^n} = \sum_{n=0}^{\infty}\frac{(-1)^n j^n}{z^{n+1}}.$$

(c) Now $|z-2j| < 1$, so

$$\begin{aligned}\frac{1}{z+j} &= \frac{1}{(z-2j)+(2j+j)} = \frac{1}{(z-2j)+3j}\\ &= \frac{1}{3j}\,\frac{1}{[1-\frac{(2j-z)}{3j}]} = \frac{1}{3j}\sum_{n=0}^{\infty}\frac{(2j-z)^n}{(3j)^n}.\end{aligned}$$

(d) Finally, $1/(z+j)$ is *already* in the Laurent series expansion form for $f(z) = 1/(z+j)$ valid for $|z+j| < 1$.

Exercise 2. Find Laurent series representations for $\sin z/z^3$. For what z is this representation valid?

Answer. Now

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

Thus

$$\frac{\sin z}{z^3} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-2}}{(2n+1)!}, \qquad \text{valid at } z = 0.$$

Since there are no other singular points for $\sin z/(z^3)$ but $z = 0$, we conclude that this series holds for all $z \neq 0$. (We can see this also by the ratio test for convergence:

$$\left| \frac{z^{2(n+1)+1}}{z^{2n+1}} \right| < \lim_{n\to\infty} \left[\frac{\frac{1}{(2n+1)!}}{\frac{1}{(2(n+1)+1)!}} \right],$$

i.e., $|z^2| < \lim_{n\to\infty} (2n+2)(2n+3) = \infty$.)

Exercise 3. Determine the Laurent series expansion for $f(z) = 1/(z+1)(z+2)$ valid in the regions: (a) $|z+1| < 1$, (b) $|z+2| < 1$, (c) $|z| < 1$, (d) $|z| > 2$.

Answer. (a) Since $|z+1| < 1$, we have

$$\begin{aligned}
\frac{1}{(z+1)(z+2)} &= \frac{1}{(z+1)[(z+1)+1]} = \frac{1}{(z+1)[1-(-(z+1))]} \\
&= \frac{1}{z+1}\sum_{n=0}^{\infty}(-1)^n(z+1)^n = \sum_{n=0}^{\infty}(-1)^n(z+1)^{n-1}.
\end{aligned}$$

(b) Now $|z+2| < 1$, thus:

$$\begin{aligned}
\frac{1}{(z+1)(z+2)} &= \frac{1}{[(z+2)-1](z+2)} = \frac{-1}{z+2}\,\frac{1}{[1-(z+2)]} \\
&= \frac{-1}{z+2}\sum_{n=0}^{\infty}(z+2)^n = -\sum_{n=0}^{\infty}(z+2)^{n-1}.
\end{aligned}$$

(c) Next, $\dfrac{1}{(z+1)(z+2)} = \dfrac{A}{z+1} + \dfrac{B}{z+2}$ with $A(z+2) + B(z+1) = 1$. So

$$\left.\begin{aligned} A + B &= 0 \\ 2A + B &= 1 \end{aligned}\right\} \qquad \begin{aligned} A &= 1 \\ B &= -1 \end{aligned}$$

and $\dfrac{1}{(z+1)(z+2)} = \dfrac{1}{z+1} - \dfrac{1}{z+2}$. But

$$\frac{1}{z+1} = \sum_{n=0}^{\infty} (-1)^n z^n,$$
$$\frac{1}{z+2} = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{2^{n+1}}.$$

Finally,

$$\frac{1}{(z+1)(z+2)} = \sum_{n=0}^{\infty} (-1)^n z^n \left(1 - \frac{1}{2^{n+1}}\right).$$

(d) Now $|z| > 2$. We still have: $\dfrac{1}{(z+1)(z+2)} = \dfrac{1}{z+1} - \dfrac{1}{z+2}$ and

$$\frac{1}{z+1} = \frac{1}{z} \cdot \frac{1}{1+\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z}\right)^n,$$
$$\frac{1}{z+2} = \frac{1}{z} \cdot \frac{1}{1+\frac{2}{z}} = \frac{1}{z}\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n.$$

Finally, $\dfrac{1}{z+1} - \dfrac{1}{z+2} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(1-2^n)}{z^{n+1}}$.

Exercise 4. Determine the Laurent series expansion for $f(z) = z^4 e^{\frac{1}{z}}$ valid for $|z| > 0$.

Answer. Now $e^w = \displaystyle\sum_{n=0}^{\infty} \frac{w^n}{n!}$. Thus $e^{\frac{1}{z}} = \displaystyle\sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$. Finally,

$$z^4 e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{z^{4-n}}{n!}.$$

Exercise 5. Same as exercise 4 if $f(z) = z^3 \cos(1/z)$.

Answer. Observe that

$$\cos w = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \frac{w^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n)!}.$$

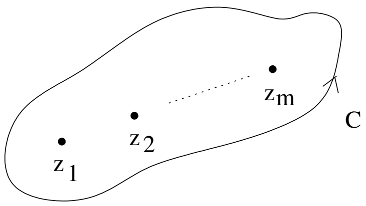
So

$$z^3 \cos\left(\frac{1}{z}\right) = z^3 \sum_{n=0}^{\infty} \frac{(-1)^n z^{-2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{3-2n}}{(2n)!}.$$

III.f *Summary and the Residue Theorem*

We first summarize the situation of the previous section and prepare for the practical problems that follow in the next sections.

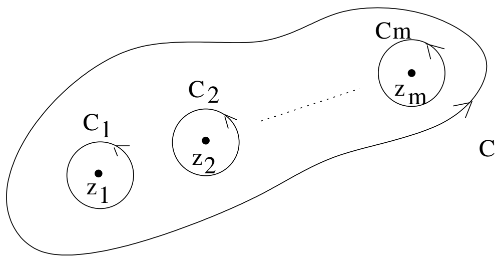
The main problem, as we shall see, is this: We are given a function $f(z)$ and a contour $\mathcal{C}$. Our problem is to evaluate $\oint_{\mathcal{C}} f(z)dz$. Since we know that $f(z)$ has poles at $z_1, z_2, \ldots, z_m$ inside $\mathcal{C}$, the integral will not $= 0$.



In practice, f will be given by the physical problem, and spotting $z_1, \ldots, z_m$ and their order is reasonably easy—see below. Since f is differentiable apart from the poles, we can, by introducing cuts, replace $\oint_{\mathcal{C}} f(z)dz$ by the sum

$$\oint_{C_1} f(z)\, dz + \oint_{C_2} f(z)\, dz + \cdots + \oint_{C_m} f(z)\, dz,$$

where $C_1, C_2, \ldots, C_m$ are small circles centered at $z_1, \ldots, z_m$, respectively. The radius of $C_1, \ldots, C_m$ is so small that $f(z)$ has no other singular point inside $C_1, \ldots, C_m$ besides $z_1, \ldots, z_m$.



So

$$\oint_{\mathcal{C}} f(z)\,dz = \oint_{C_1} f(z)\,dz + \cdots + \oint_{C_m} f(z)\,dz.$$

Consider $\oint_{C_1} f(z)\,dz$. Suppose we can find the residue of f at z_1: this means finding the coefficient of $1/(z-z_1)$ in the Laurent series for f, but we don't even dream of finding the whole series! Then

$$\oint_{C_1} f(z)\,dz = 2\pi j\,(\mathrm{Res}\,(z_1))\,,$$

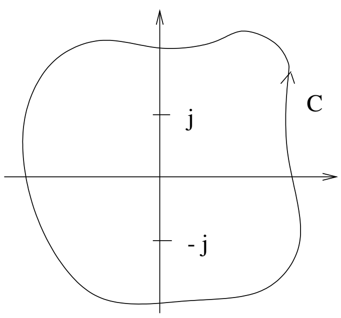
where $\mathrm{Res}\,(z_1)$ means the residue of f at z_1. We do the same for $\oint_{C_2}$, $\oint_{C_3}, \ldots$, etc. and get

$$\boxed{\oint_{\mathcal{C}} f(z)\,dz = 2\pi j\,(\mathrm{Res}\,(z_1) + \mathrm{Res}\,(z_2) + \cdots + \mathrm{Res}\,(z_m))}.$$

We now practice this important result.

Example 1. Find $\oint_{\mathcal{C}} \frac{(z-3j)}{(z+j)(z-j)}\,dz$ if $\mathcal{C}$ is a simple closed path enclosing both $z=j$ and $z=-j$.

Answer. Here $f(z) = (z-3j)/(z+j)(z-j)$ has two poles: at $z=j$ and at $z=-j$, and both poles are *inside* $\mathcal{C}$.



Now at $z = j$, we have

$$(z-j)\frac{(z-3j)}{(z+j)(z-j)}\bigg|_{z=j} = \frac{z-3j}{z+j}\bigg|_{z=j} = \frac{-2j}{2j} = -1.$$

Since this is not zero but had we multiplied by a higher power of $(t-j)$ we would have gotten zero, we have both that $z = j$ is a pole of order 1 (i.e., *simple*) and that the residue is -1.

In the same way, at $z = -j$,

$$(z+j)\frac{(z-3j)}{(z+j)(z-j)}\bigg|_{z=-j} = \frac{z-3j}{z-j}\bigg|_{z=-j} = \frac{-4j}{-2j} = 2.$$

So, again, $z = -j$ is a *simple* pole and the residue is 2.

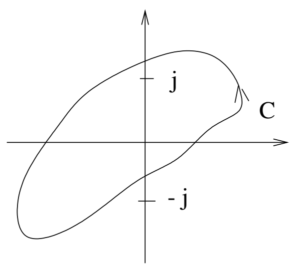
Finally,

$$\oint_{\mathcal{C}} \frac{z-3j}{(z+j)(z-j)}\,dz = 2\pi j(-1+2) = 2\pi j.$$

Example 2. Same as Example 1, except $\mathcal{C}$ now only encloses $z = j$ and not $z = -j$.

Answer. Now there is only one pole inside $\mathcal{C}$, namely $z = j$, and so

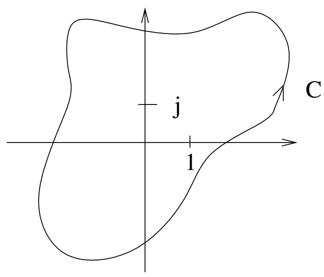
$$\oint_{\mathcal{C}} \frac{z-3j}{(z+j)(z-j)}\,dz = 2\pi j(-1).$$



Note that the fact $z = -j$ is no longer inside $\mathcal{C}$ has no effect whatsoever on the calculations we perform at $z = j$. The residue at $z = j$ is exactly the same as before.

Example 3. Find $\oint_{\mathcal{C}} \frac{\sin z}{(z-1)(z-j)} dz$ where $\mathcal{C}$ is a path enclosing both $z = 1$ and $z = j$.

Answer. Note that once again we do not specify $\mathcal{C}$ exactly! Since $\mathcal{C}$ encloses both $z = 1$ and $z = j$ *and* since $\sin z$ is always differentiable, $\sin z/(z-1)(z-j)$ has *two* poles, namely at $z = 1$ and $z = j$, both inside $\mathcal{C}$.



Thus

$$\oint_{\mathcal{C}} \frac{\sin z}{(z-1)(z-j)} dz = 2\pi j \left[(\text{Residue at } z = 1) + (\text{Residue at } z = j)\right].$$

Now at $z = 1$ we have

$$(z-1)\left[\frac{\sin z}{(z-1)(z-j)}\right]\Bigg|_{z=1} = \frac{\sin z}{z-j}\Bigg|_{z=1} = \frac{\sin(1)}{1-j} \neq 0.$$

Thus at $z = 1$ we have a pole of order 1 for the same reason as in the earlier examples, and

$$\text{Residue} = \frac{\sin(1)}{1-j}.$$

At $z = j$, we have

$$(z-j)\left[\frac{\sin z}{(z-1)(z-j)}\right]\Bigg|_{z=j} = \frac{\sin z}{z-1}\Bigg|_{z=j} = \frac{\sin(j)}{j-1}$$

and

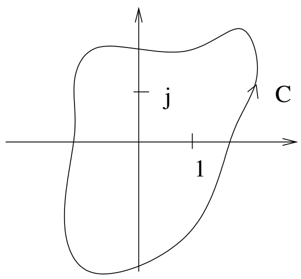
$$\text{Residue} = \frac{\sin(j)}{j-1} = \frac{e^{j^2} - e^{-j^2}}{2j(j-1)} = \frac{e^{-1} - e^{1}}{2j(j-1)}.$$

Finally,

$$\oint_C \frac{\sin z}{(z-1)(z-j)}\,dz = 2\pi j\left[\frac{\sin(1)}{1-j} + \frac{e^{-1} - e^{1}}{2j(j-1)}\right].$$

Example 4. Same as Example 3, except f is now $\sin z/(z-1)^2(z-j)$.

Answer. Note that now the pole at $z = 1$ is of order 2 (since $\sin(1)/(1-j) \neq 0$).



We need to calculate the residues and this is not as easy as the earlier examples! To practice, we do it in two ways. First, we expand $\sin z/(z-j)$ as a Taylor series about $z = 1$, or more precisely, we calculate the first few terms in this expansion. We have

$$\frac{\sin z}{z-j}\Bigg|_{z=1} = \frac{\sin(1)}{1-j}$$

$$\frac{d}{dz}\left(\frac{\sin z}{z-j}\right)\Bigg|_{z=1} = \frac{(\cos z)(z-j) - \sin z}{(z-j)^2}\Bigg|_{z=1} = \frac{\cos(1)(1-j) - \sin(1)}{(1-j)^2}.$$

These are the first two terms of the Taylor series for $\sin z/(z-j)$, i.e.,

$$\frac{\sin z}{z-j} = \frac{\sin(1)}{1-j} + \frac{\cos(1)(1-j)-\sin(1)}{(1-j)^2}(z-1) + \text{Junk}\cdot(z-1)^2.$$

If we divide by $(z-1)^2$, we find that the residue is

$$\frac{\cos(1)(1-j)-\sin(1)}{(1-j)^2},$$

exactly as given by the earlier formulas in the preceding section.

Now, for the fast way, since $z=1$ is a pole of order 2,

$$\frac{d}{dz}\left(\frac{\sin z}{z-j}\right)\bigg|_{z=1} = \frac{\cos(1)(1-j)-\sin(1)}{(1-j)^2}.$$

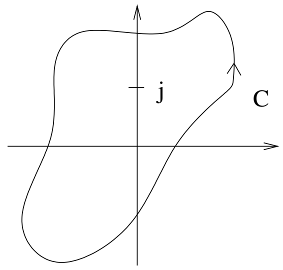
Next, $z=j$ is still a simple pole, so the residue there is $\sin(1)/(j-1)^2$, and we get

$$\oint_{\mathcal{C}} \frac{\sin z}{(z-1)^2(z-j)}\,dz = 2\pi j\left[\frac{\cos(1)(1-j)-\sin(1)}{(1-j)^2} + \frac{\sin(j)}{(j-1)^2}\right].$$

Example 5. Find $\oint_{\mathcal{C}} f(z)\,dz$ is $\mathcal{C}$ encloses the point $z=j$ and

$$f(z) = \frac{1}{(z-j)^3} + \frac{127}{(z-j)^2} + 100(z-j) + 10(z-j)^3 + e^z.$$

Answer. There is nothing to calculate in this example! $f(z)$ *is already given as a Laurent series about* $z=j$, except for e^z. But e^z is always differentiable and can be expanded in a *Taylor series* about $z=j$ and it does not have the term $1/(z-j)$ since all powers of $(z-j)$ will be positive. Equivalently, $f(z)$ contains the term $0/(z-j)$, so the residue is zero and $\oint_{\mathcal{C}} f(z)dz = 0$ even though j is inside $\mathcal{C}$!



Example 8. $\oint_{\mathcal{C}} \sin z/z\, dz$ where $z = 0$ is inside $\mathcal{C}$.

Answer. Note that at $z = 0$ we have $\sin z = 0$, so

$$z \cdot \frac{\sin z}{z}\bigg|_{z=0} = 0,$$

and clearly

$$z^m \cdot \frac{\sin z}{z}\bigg|_{z=0} = 0, \quad \text{for any} \quad m > 1.$$

So $\sin z/z$ does *not* blow up at $z = 0$, and $z = 0$ is *not* a pole, despite appearances. You can see this also from the expansion

$$\frac{\sin z}{z} = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\frac{1}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots .$$

This is actually the Laurent series, and there are *no* negative powers of z. So the residue = coeff. of $1/z = 0$ and

$$\oint_{\mathcal{C}} \frac{\sin z}{z}\, dz = 0.$$

This situation is called a *removable singularity:* it looks like a pole, but it is not since the top also vanishes at z_0 and the ratio stays bounded.

Further Exercises:

Calculate the residue at each singular point.

Exercise 1. $f(z) = 1/(z^2 + 3jz - 2)$.

Answer. $f(z) = 1/(z^2 + 3jz - 2)$. We factor the bottom:

$$z^2 + 3jz - 2 = 0 \quad \Longleftrightarrow \quad z = \frac{-3j \pm \sqrt{-9+8}}{2} = \frac{-3j \pm j}{2} = -j,\ -2j.$$

So $z^2 - 3jz - 2 = (z+j)(z+2j)$ and $f(z) = 1/(z+j)(z+2j)$. We have two singular points at $z = -j,\ -2j$. Note that each is a pole of order 1 and thus:

$$\begin{aligned} &\text{residue at } z = -j \text{ is} \quad \frac{1}{(-j+2j)} = \frac{1}{j}; \\ &\text{residue at } z = -2j \text{ is} \quad \frac{1}{(-2j+j)} = \frac{-1}{j}. \end{aligned}$$

Exercise 2. $f(z) = 1/(z^2 - 2jz - 1)$.

Answer. $f(z) = 1/(z^2 - 2jz - 1)$. Now

$$z^2 - 2jz - 1 = 0 \quad \Longleftrightarrow \quad z = \frac{2j \pm \sqrt{-4+4}}{2}.$$

We have $z^2 - 2jz - 1 = (z-j)^2$ and $f(z) = 1/(z-j)^2$. There is only one singular point, at $z = j$, and the residue there is

$$\frac{d}{dz}(f(z)(z-j)^2)\bigg|_{z=j} = \frac{d}{dz}(1)\bigg|_{z=j} = 0.$$

You could have seen this immediately, since $f(z) = 1/(z-j)^2$ is already written in a Laurent series about $z = j$, and the coefficient of $1/(z-j)$ is 0, since there is *no* $1/(z-j)$ term.

Exercise 3. $f(z) = \csc z$.

Answer. $f(z) = \csc z = 1/\sin z$. Note that $f(z)$ has a singular point whenever $\sin z = 0$, i.e., $(e^{jz} - e^{-jz})/(2j) = 0$ or $e^{2jz} = 1$ or $2jz = \log(1) = 2n\pi j$, $(n = 0, \pm 1, \pm 2, \ldots)$ or $z = n\pi$. Now what is the order of the pole at $z = n\pi$? Observe that we can expand $\sin z$ about $n\pi$ and get

$$\begin{aligned}
\sin z &= \sin(n\pi) + \frac{d}{dz}(\sin z)\bigg|_{z=n\pi}(z - n\pi) + \frac{1}{2!}\frac{d^2}{dz^2}(\sin z)\bigg|_{z=n\pi}(z - n\pi)^2 \\
&\qquad + \frac{1}{3!}\frac{d^3}{dz^3}(\sin z)\bigg|_{z=n\pi}(z - n\pi)^3 + \ldots \\
&= \cos(n\pi)(z - n\pi) + \frac{1}{3!}(-\cos(n\pi))(z - n\pi)^3 + \ldots \\
&= (-1)^n(z - n\pi) + \frac{1}{3!}(-1)^{n+1}(z - n\pi)^3 + \ldots .
\end{aligned}$$

Thus

$$f(z)(z - n\pi)\bigg|_{z=n\pi} = \frac{1}{(-1)^n + \frac{1}{3!}(-1)^{n+1}(z - n\pi)^2 + \ldots}\bigg|_{z=n\pi} = \frac{1}{(-1)^n}$$

and the pole is simple at each $n\pi$ with residue $1/(-1)^n$.

Exercise 4. $f(z) = \cos z/(e^z - 1)$.

Answer. $f(z) = \cos z/(e^z - 1)$. Now a singular point will arise whenever $e^z = 1$, i.e., $z = 2k\pi j$, $k = 0, \pm 1, \pm 2, \ldots$. Also,

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!}(z - z_0)^n \qquad \text{for any } z_0,$$

so

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!}(z - 2k\pi j)^n \quad \text{and} \quad e^z - 1 = (z - 2k\pi j) + \frac{(z - 2k\pi j)^2}{2!} + \ldots$$

and at $z = 2k\pi j$,

$$f(z)(z-2k\pi j)\bigg|_{z=2k\pi j} = \frac{\cos z}{1+\frac{(z-2k\pi j)^2}{2!}+\ldots}\bigg|_{z=2k\pi j} = \frac{\cos(2k\pi j)}{1}$$
$$= \frac{e^{j(2k\pi j)}+e^{-j(2k\pi j)}}{2} = \frac{e^{-2k\pi}+e^{2k\pi}}{2}$$

which is the residue.

Exercise 5. Calculate the part of the Laurent series for $z_0 = 0$ that involves negative powers of $(z - z_0)$ if $f = \sin z/[(z^2+1)z^3]$.

Answer. Now $f(z) = \sin z/[(z^2+1)z^3]$. It appears that $z = 0$ is a pole of order 3, but this is not so, since $\sin z = 0$ at $z = 0$ too. Then

$$\frac{(\sin z)z^2}{(z^2+1)z^3}\bigg|_{z=0} = \frac{z-\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots}{(z^2+1)z}\bigg|_{z=0} = \frac{1}{1} = 1$$

while

$$\frac{(\sin z)z^m}{(z^2+1)z^3}\bigg|_{z=0} = 0 \quad \text{if } m > 2.$$

So the pole of *order* 2 and the Laurent series for f has negative power part: $d_1/(z-0)^2 + d_2/(z-0)^2$ with

$$d_2 = \frac{(\sin z)z^2}{(z^2+1)z^3}\bigg|_{z=0} = 1, \qquad d_1 = \frac{d}{dz}\left[\frac{\sin z}{(z^2+1)z}\right]\bigg|_{z=0}.$$

Now

$$\frac{d}{dz}\left[\frac{\sin z}{z(z^2+1)}\right] = \frac{(\cos z)(z^3+z)-(\sin z)(3z^2+1)}{z^2(z^2+1)^2}$$
$$= \frac{z^3\cos z-(\sin z)3z^2}{z^2(z^2+1)^2} + \frac{z(\cos z)-\sin z}{z^2(z^2+1)^2}$$
$$= \frac{z\cos z-(\sin z)3}{(z^2+1)^2} + \frac{z(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots)-(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots)}{z^2(z^2+1)^2}.$$

Now let $z \to 0$. The first piece tends to $0/1^2 = 0$. The second piece is

$$\frac{-\frac{z^2}{2!} + \frac{z^3}{3!} + \text{higher powers of } z}{z^2(z^2+1)^2}.$$

So, as $z \to 0$, the second piece also tends to 0 and $d_1 = 0$. Thus the negative part of the series is just $1/z^2$.

Evaluate the following integrals.

Exercise 6. $\oint_{\mathcal{C}} ze^{\frac{1}{z}}\, dz$, if $\mathcal{C}$ is any simple closed path that contains $z = 0$.

Answer. $\oint_{\mathcal{C}} ze^{\frac{1}{z}}\, dz = 2\pi j$ (residue at $z = 0$). Now $e^{\frac{1}{z}} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!z^n}$, so

$$ze^{\frac{1}{z}} = z\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots\right) = z + 1 + \frac{1}{2!z} + \frac{1}{3!z^2} + \cdots.$$

Observe that $ze^{\frac{1}{z}}$ does not have a pole at $z = 0$, or, if you prefer, it has a pole of infinite order there. We conclude

$$\oint_{\mathcal{C}} ze^{\frac{1}{z}}\, dz = 2\pi j\left(\frac{1}{2!}\right) = \pi j.$$

Exercise 7. $\oint_{\mathcal{C}} \sin z/[z(z+j)^2]\, dz$, if $\mathcal{C}$ is the circle $|z| = 10$.

Answer. Since $|z| = 10$ has for interior points both $z = 0$ and $z = -j$, we consider both points. Note that $z = 0$ is *not* a pole (since $\sin z = 0$ at $z = 0$ too, and $\sin z = z - z^3/3! + z^5/5! + \cdots$). The only pole is $z = -j$, which is double. So

$$\frac{d}{dz}\left[\frac{\sin z}{z}\right]\Bigg|_{z=-j} = \frac{(\cos z)z - \sin z}{1}\Bigg|_{-j} = \cos(-j)(-j) - \sin(-j)$$

and

$$\oint_{\mathcal{C}} \frac{\sin z}{z(z+j)^2} = 2\pi j\left[\cos(-j)(-j) - \sin(-j)\right].$$

Exercise 8. (a) $\oint_{\mathcal{C}} \dfrac{e^z}{z^{10}}\, dz$, if (a) $\mathcal{C}$ is the circle $|z| = 1$, (b) $\mathcal{C}$ is the circle $|z + 4j| = 2$.

Answer. (a) $\oint_{\mathcal{C}} e^z/z^{10}\, dz = 2\pi j$ (residue at $z = 0$). Now e^z/z^{10} has a pole of order 10 at $z = 0$, so

$$\text{residue at } z = 0 = \frac{1}{9!}\frac{d^9}{dz^9}(e^z)\bigg|_{z=0} = \frac{1}{9!}e^0 = \frac{1}{9!}.$$

We conclude

$$\oint \frac{e^z}{z^{10}}\, dz = \frac{2\pi j}{9!}.$$

(b) $z = 0$ is not inside nor on $|z + 4j| = 2$. So e^z/z^{10} is analytic inside and on $\mathcal{C}$ and

$$\oint_{\mathcal{C}} \frac{e^z}{z^{10}}\, dz = 0.$$

III.g *An Application: Evaluation of Improper Real Integrals*

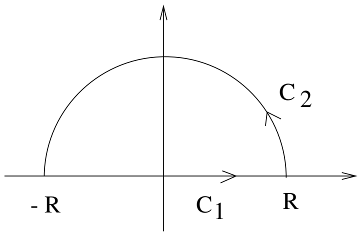
Complex integrals can be used to evaluate some real integrals. As a first example, consider integrals of the type $\int_{-\infty}^{\infty} f(x)dx$. We emphasize that x here denotes a real variable, and illustrate the concept by considering the following examples. The *basic idea is to form these "real" integrals into path integrals in the complex plane.*

Example 1. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$.

Answer. We recall that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \lim_{R\to\infty} \left[\int_{-R}^{R} \frac{1}{1+x^2}\, dx \right]$$

and we start by looking at $\int_{-R}^{R} \frac{1}{1+x^2} dx$. Now consider for a moment $f(z) = 1/(1+z^2)$ and suppose we wish to find $\oint_{\mathcal{C}} f(z)\, dz$ where $\mathcal{C} = C_1 + C_2$ as shown.



In particular, C_2 can be taken to be the semicircle of radius R. Now

$$\oint_{\mathcal{C}} f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$$

and note that we can parametrize C_1 by

$$C_1 = \begin{cases} x = x \\ y = 0. \end{cases}$$

So $z = x$, $dz = dx$ on C_1 and

$$\int_{C_1} f(z)\,dz = \int_{x=-R}^{R} f(x)\,dx = \int_{-R}^{R} \frac{1}{1+x^2}\,dx$$

and this is precisely what we wish to calculate! Now $\oint_{\mathcal{C}} f(z)\,dz$ can be done by residues, but unfortunately,

$$\oint_{C_1} f(z)\,dz = \int_{-R}^{R} \frac{1}{1+x^2}\,dx + \int_{C_2} f(z)\,dz$$

and we need to deal with $\int_{C_2} f(z)dz = \int_{C_2} \frac{1}{1+z^2}\,dz$. Observe that

$$C_2 = \begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases} \qquad 0 \le \theta \le \pi,$$

i.e., $z = Re^{j\theta}$. This enables us to *estimate* $\int_{C_2} \frac{1}{1+z^2}\,dz$:

$$\left| \int_{C_2} \frac{1}{1+z^2}\,dz \right| \le \int_{\theta=0}^{\pi} \left| \frac{1}{1+z^2} \right| \left| \frac{dz}{d\theta} \right| d\theta.$$

We have

$$\left| \frac{dz}{d\theta} \right| = |Rje^{j\theta}| = R$$

$$|1+z^2| \ge |z^2| - |1| = |z|^2 - 1 = R^2 - 1.$$

So

$$\left| \int_{C_2} \frac{1}{1+z^2}\,dz \right| \le \int_{\theta=0}^{\pi} \frac{1}{R^2-1} \cdot R\,d\theta = \frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty.$$

Remark. The above estimate is often not carried out in detail. It suffices to note that if $f(x) = P(x)/Q(x)$ with P, Q *polynomials* and the order of Q is at least two more than the order of $P(x)$, then the above estimate always goes through. I.e.,

$$\left| \int_{C_2} \frac{P(z)}{Q(z)}\,dz \right| \to 0 \quad \text{as} \quad R \to \infty.$$

In summary,

$$\oint_{\mathcal{C}} f(z)\,dz = \int_{-R}^{R} \frac{1}{1+x^2}\,dx + \int_{C_2} f(z)\,dz$$

and

$$\int_{C_2} f(z)\,dz \to 0 \quad \text{as} \quad |z| \text{ (i.e., } R) \to \infty.$$

But for R big, $\oint_{\mathcal{C}} f(z)dz$ **never changes**: Observe that $f(z)$ has 2 (simple) poles at $z = \pm j$. So

$$\oint_{\mathcal{C}} f(z)\,dz = 2\pi j(\operatorname{Res} z = j) = 2\pi j \cdot \frac{1}{2j} = \pi$$

for all such R, i.e.,

$$\pi = \int_{-R}^{R} \frac{1}{1+x^2}\,dx + \int_{C_2} f(z)\,dz.$$

So taking limit of both sides as $R \to \infty$ gives

$$\pi = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx + \lim_{R\to 0}\left[\int_{C_2} f(z)\,dz\right] = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

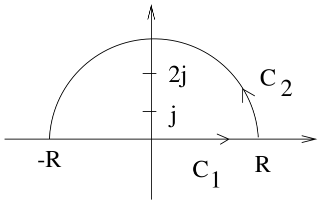
and we are done.

Example 2. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+4)}\,dx.$

Answer. We repeat the earlier process. Note first that the order of the bottom $= 4$, order of the top $= 1$ so we construct the path $\mathcal{C}$ just like before and know

$$\left|\int_{C_2} \frac{z}{(z^2+1)(z^2+4)}\,dz\right| \to 0 \quad \text{as} \quad R \to \infty.$$

All we need to calculate is $\oint_{\mathcal{C}} \frac{z}{(z^2+1)(z^2+4)}\,dz$ by means of residues.



Now the poles are $z = \pm j,\ \pm 2j$ (the zeros of the bottom) and these are all of order 1 (i.e., *simple*), but only two: j, $2j$ are inside $\mathcal{C}$. Thus

$$\begin{array}{ll} \textit{Residue at } j: & \dfrac{j}{2j(j^2+4)} = \dfrac{1}{2(3)} = \dfrac{1}{6} \\ \textit{Residue at } 2j: & \dfrac{2j}{((2j)^2+1)(4j)} = \dfrac{1}{2(-3)} = -\dfrac{1}{6}. \end{array}$$

So

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+4)}\, dx = 2\pi j \left(\frac{1}{6} - \frac{1}{6}\right) = 0.$$

We consider a different type of "real" integrals which can be done by complex path integrals. Specifically, we look at

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta.$$

We keep in mind that θ was used as a parameter when we evaluate a complex integral over the unit circle, so it seems reasonable to put $z = e^{j\theta}$, then

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{1}{2j}\left(z - \frac{1}{z}\right), \qquad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and

$$dz = je^{j\theta} d\,\theta = jz\, d\theta \quad \Longrightarrow \quad d\theta = \frac{dz}{jz}.$$

In summary:

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta = \oint_{\mathcal{C}} f\left(\frac{1}{2j}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right) \frac{dz}{jz}$$

where $\mathcal{C}$ is the unit circle.

Example 3. Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{\sin\theta + 2}\, d\theta$.

Answer. We put $z = e^{j\theta}$ and get

$$\int_0^{2\pi} \frac{1}{\sin\theta + 2}\, d\theta = \oint_{\mathcal{C}} \frac{1}{(z - \frac{1}{z})\frac{1}{2j} + 2} \frac{dz}{jz}$$

where $\mathcal{C}$ is the unit circle. Now

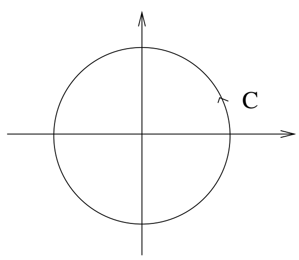
$$\oint_{\mathcal{C}} \frac{dz}{2jz + (z^2 - 1)\frac{1}{2}} = \oint_{\mathcal{C}} \frac{2\, dz}{z^2 - 1 + 4jz}.$$

The roots of the bottom are

$$z = \frac{-4j \pm \sqrt{-16 + 4}}{2} = (-2 \pm \sqrt{3})j,$$

so

$$z^2 + 4jz - 1 = (z + (2 + \sqrt{3})j)(z + (2 - \sqrt{3})j).$$



Observe that $(-2+\sqrt{3})j$ is inside $\mathcal{C}$, but $(-2-\sqrt{3})j$ is not, since

$$\begin{cases} |(-2+\sqrt{3})j| = 2-\sqrt{3} < 1, \\ |(-2-\sqrt{3})j| = 2+\sqrt{3} > 1. \end{cases}$$

We also note that the pole at $(-2+\sqrt{3})j$ is *simple*. Then

$$\oint_{\mathcal{C}} \frac{2}{z^2+4jz-1}\,dz = 2\pi j\left(\frac{2}{2\sqrt{3}j}\right) = \frac{2\pi}{\sqrt{3}}.$$

As a final comment, we recall that $f(x)$ is even iff $f(x) = f(-x)$. Suppose $f(x)$ is even, then

$$\int_0^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx.$$

We can thus evaluate $\int_0^{\infty} f(x)dx$ by evaluating $\int_{-\infty}^{\infty} f(x)dx$ and dividing by 2. As an example consider $\int_0^{\infty} \frac{1}{1+x^2}dx$. Here $f(x) = \frac{1}{1+x^2}$ for $f(x) = f(-x)$ and f is *even*. We thus have $\int_0^{\infty} \frac{1}{1+x^2}dx = \frac{1}{2}\cdot\pi$ from Example 1. Note also that $f(x)$ is *odd* if $f(x) = -f(-x)$, so

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\left[\int_{-R}^{R} f(x)\,dx\right]$$

but

$$\int_{-R}^{R} f(x)\,dx = \int_{-R}^{0} f(x)\,dx + \int_0^{R} f(x)\,dx = -\int_0^{R} f(x)\,dx + \int_0^{R} f(x)\,dx = 0,$$

and $\int_{-\infty}^{\infty} f(x)dx = 0$! Unfortunately, we can't do $\int_0^{\infty} f(x)dx$ if $f(x)$ is *odd*.

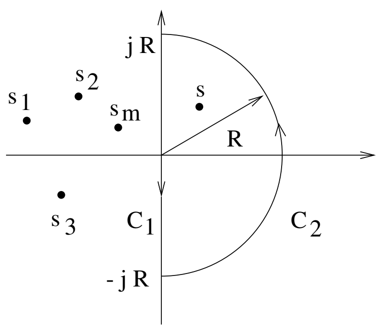
III.h *Inverting the Laplace Transform and Applications*

We now come one of the main reasons for all the work we have done: inverting the Laplace Transform. The basic idea—constructing suitable paths in the complex plane—is similar to the previous section. You may recall that given $y = f(t)$ (usually t is actually time), $L(f)(s)$—the Laplace Transform of f—is given by

$$L(f)(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

In elementary courses, s is thought of as being a real number, but in practical problems this is *not* the case and you should think of s as being complex. Indeed, $s = jw$ with $w =$ frequency is of particular significance. So we wish to think as $L(f)(s)$ being given for any complex s. The formulas you hopefully recall hold unchanged for s complex. In particular, $L(e^{at})(s) = 1/(s-a)$ if $\mathrm{Re}\,(s) > \mathrm{Re}\,(a)$.

Now let $f(s)$ be the transform of some function $h(t)$. We know $f(s)$ and wish to find $h(t)$, i.e., we wish to invert the transform. In practice, $f(s)$ will have some poles at some points $s_1, \ldots, s_m$ of the complex plane. As is often the case in practice we start by assuming $s_1, \ldots, s_m$ all lie in the left half plane. (We shall deal with the other case later.) We construct the path $\mathcal{C} = C_1 + C_2$ with C_2 a semicircle of radius R as shown.



Think of R as being large. If s is any point inside the semicircle (as shown), then since f is analytic inside and on $\mathcal{C}$ we have

$$f(s) = \frac{1}{2\pi j} \oint_{\mathcal{C}} \frac{f(z)}{z-s}\, dz.$$

Think of s as being temporarily fixed, and assume $|f(z)| \to 0$ as $|z| \to \infty$. This is the case for most "practical" functions but not for all. The latter case is more complicated and we do not deal with it in this course. Look at C_2: if z is on C_2 then $|f(z)| \to 0$ as $|z|$ (i.e., R) approaches ∞, and $1/|z-s|$ also goes to zero as $|z| \to \infty$. One can formally show that as a consequence $\int_{C_2} f(z)/(z-s)dz \to 0$ as $R \to \infty$, and a detailed proof is given in theoretical courses. So, since $f(s)$ never changes as $R \to \infty$, we get

$$\begin{aligned} f(s) &= \frac{1}{2\pi j} \oint_{\mathcal{C}} \frac{f(z)}{z-s}\, dz = \frac{1}{2\pi j} \left[\int_{C_1} \frac{f(z)}{z-s}\, dz + \int_{C_2} \frac{f(z)}{z-s}\, dz \right] \\ &\longrightarrow \frac{1}{2\pi j} \left[\int_{C_1^*} \frac{f(z)}{z-s}\, dz \right] \quad \text{as} \quad |z| \to \infty \end{aligned}$$

where C_1^* is the y-axis traversed from $+\infty$ to $-\infty$. In summary,

$$f(s) = \frac{1}{2\pi j} \int_{C_1^*} \frac{f(z)}{z-s}\, dz.$$

Note that C_1^* can be parametrized as

$$\begin{cases} x = 0 \\ y = -\ell \end{cases} \qquad -\infty \leq \ell \leq \infty.$$

So, on C_1^*, $z = -j\ell$, $-\infty \leq \ell \leq \infty$. Next, remember

$$L(e^{at}) = \frac{1}{s-a}, \quad \text{so} \quad L(e^{zt}) = \frac{1}{s-z}$$

and thus

$$f(s) = -\frac{1}{2\pi j} \int_{C_1^*} f(z) L(e^{zt})\, dz = L\left[-\frac{1}{2\pi j} \int_{C_1^*} f(z) e^{zt}\, dz \right].$$

On the other hand, $L(h(t)) = f(s)$, so

$$L(h(t)) = L\left[-\frac{1}{2\pi j} \int_{C_1^*} f(z) e^{zt}\, dz \right]$$

or

$$h(t) = -\frac{1}{2\pi j} \int_{C_1^*} f(z) e^{zt}\, dz.$$

You might be tempted, since we know the parametrization of C_1^* *and* since C_1^* is *not* a simple closed path, to try to actually evaluate the integral over C_1^* directly, i.e., by parametrizing and working out the integral for each t. *This would be fatal.* Instead, much as we did in the last section, we turn $\int_{C_1^*}$ into an integral over a simple closed path, which we can then evaluate by residues! Since $\mathcal{C} = C_1 + C_2$ *was* a simple closed path, you might think we would go back to $\mathcal{C}$ and pass to the limit as $R \to \infty$, but *usually not so.* Instead we start by looking at t, which as pointed out earlier, is almost always time. We are interested in what happens for $t > 0$, since in Laplace transform approaches, the clock starts at $t = 0$ (if it does not—say it starts at $t = a$ we can always move this to $t = 0$ by a shift in time). Look at $\int_{C_1}$, it is:

$$\int_{C_1} f(z) e^{zt}\, dz.$$

So over the old C_2 for the same function $f(z)e^{zt}$ we would have $|e^{zt}| = |e^{R(\cos\theta + j\sin\theta)t}|$ $= e^{R(\cos\theta)t}$ (note C_2 can be parametrized as $z = Re^{j\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$). This is really bad news if $t > 0$, since $e^{R(\cos\theta)t}$ blows up as $R \to \infty$.

In summary, for most $f(z)$ of practical interest,

$$\left| \int_{C_2} f(z) e^{zt}\, dz \right|$$

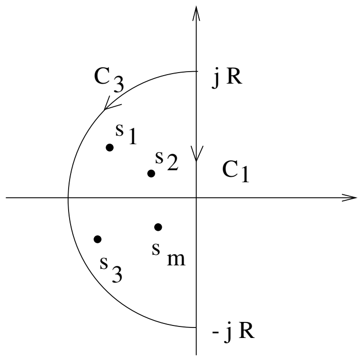
blows up as $R \to \infty$, due to $t > 0$. This is not always the case, as an example indicates. If $t < 0$ note that $\int_{C_2} f(z)e^{zt}\,dz \to 0$ as a general rule. Thus, *in this case*,

$$
\begin{aligned}
h(t) &= -\frac{1}{2\pi j}\int_{C_1^*} f(z)e^{zt}\,dz \\
&= \lim_{R\to\infty}\left[-\frac{1}{2\pi j}\int_{C_1} f(z)e^{zt}\,dz\right] \\
&= \lim_{R\to\infty}\left[-\frac{1}{2\pi j}\int_{C_1} f(z)e^{zt}\,dz - \frac{1}{2\pi j}\int_{C_2} f(z)e^{zt}\,dz\right] \\
&= \lim_{R\to\infty}\left[-\frac{1}{2\pi j}\int_{\mathcal{C}} f(z)e^{zt}\,dz\right].
\end{aligned}
$$

But $f(z)e^{zt}$ is differentiable inside and on $\mathcal{C}$ (the poles are on the left 1/2 plane, remember), and we have

$$\frac{1}{2\pi j}\int_{\mathcal{C}} f(z)e^{zt}\,dz = 0$$

and so $h(t) = 0$ for $t < 0$. True, but of not much practical interest! We want to know what $h(t)$ is for $t > 0$! To do this, we use the same idea, but use C_3: also a semicircle but on the *left half plane*.



Remember that R is big, so we may assume $C_4 = C_3 + (-C_1)$—which is a simple closed path—encloses all the poles at $s_1, s_2, \ldots, s_m$. Now since $t > 0$ and $z = Re^{j\theta}$ on C_3 with $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ then $\cos\theta < 0$ and so $|e^{zt}| = e^{R(\cos\theta)t} \to 0$ as $R \to \infty$! We can now conclude

$$\left| \int_{C_3} f(z)e^{zt}\, dz \right| \to 0 \quad \text{as} \quad |z| \to \infty.$$

So

$$\begin{aligned}
h(t) &= -\frac{1}{2\pi j} \int_{C_1^*} f(z)e^{zt}\, dz \\
&= \lim_{R\to\infty} \left[-\frac{1}{2\pi j} \int_{C_1} f(z)e^{zt}\, dz \right] \\
&= \lim_{R\to\infty} \left[\frac{1}{2\pi j} \int_{-C_1} f(z)e^{zt}\, dz + \frac{1}{2\pi j} \int_{C_3} f(z)e^{zt}\, dz \right] \\
&= \lim_{R\to\infty} \left[\frac{1}{2\pi j} \int_{C_4} f(z)e^{zt}\, dz \right].
\end{aligned}$$

But

$$\begin{aligned}
\frac{1}{2\pi j} \int_{C_4} f(z)e^{zt}\, dz &= \frac{1}{2\pi j} \cdot 2\pi j \{\text{Res. at } s_1 + \text{Res. at } s_2 + \cdots + \text{Res. at } s_m\} \\
&= \text{Res. at } s_1 + \text{Res. at } s_2 + \cdots + \text{Res. at } s_m.
\end{aligned}$$

This *never changes as R gets larger*, so

$$\begin{aligned}
h(t) &= \lim_{R\to\infty} \left[\frac{1}{2\pi j} \int_{C_4} f(z)e^{zt}\, dz \right] \\
&= \text{Res. at } s_1 + \text{Res. at } s_2 + \cdots + \text{Res. at } s_m.
\end{aligned}$$

We emphasize that the residues are those of $f(z)e^{zt}$, *not* of $f(z)$ but the *poles* will be those of $f(z)$, since e^{zt} is always differentiable, $\neq 0$, and cannot contribute "new" poles.

One final notational change. By tradition we write $f(s)e^{st}$ in place of $f(z)e^{zt}$. We pass to examples.

Example 1. Find the inverse Laplace transform of

$$f(s) = \frac{s}{(s+1)(s+2)}.$$

Answer. We use partial fraction expansions. Note that since s *is complex*, we can factor any polynomial into the product of linear factors $(s-s_1)^{\alpha_1}(s-s_2)^{\alpha_2}\cdots(s-s_m)^{\alpha_m}$ and with every factor of type $(s-s_1)^{\alpha_1}$ we associate the expansion:

$$\frac{A_1}{(s-s_1)} + \frac{A_2}{(s-s_1)^2} + \cdots + \frac{A_\alpha}{(s-s_1)^{\alpha_1}}.$$

In the present case,

$$f(s) = \frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

and thus

$$s = A(s+2) + B(s+1).$$

Choosing

$$\begin{cases} s=-1 \quad \text{gives} \quad -1 = A(1) \quad \text{or} \quad A=-1 \\ s=-2 \quad \text{gives} \quad -2 = B(-1) \quad \text{or } B = 2. \end{cases}$$

So

$$f(s) = -\frac{1}{s+1} + \frac{2}{s+2},$$

and it suffices to invert each piece separately. We choose C_4 just like before. Now $-e^{st}/(s+1)$ has a simple pole at $s=-1$, so the inverse of this part is the residue of $-e^{st}/(s+1)$ at $s=-1$, i.e., $-e^{(-1)t} = -e^{-t}$. In the same way, $2e^{st}/(s+2)$ has a simple pole at $s=-2$ and the residue is $2e^{-2t}$. The final answer is

$$-e^{-t} + 2e^{-2t}.$$

The previous method, using partial fractions, is useful since it decomposes the problem into simpler pieces. However, for a simple problem as given in Example 1, it is easier to just do it directly:

$$f(s)e^{st} = \frac{se^{st}}{(s+1)(s+2)}$$

has two simple poles at $s = -1$ and $s = -2$. The sum of the residues is

$$\frac{(-1)e^{-t}}{(-1+2)} + \frac{(-2)e^{-2t}}{(-2+1)} = -e^{-t} + 2e^{-2t},$$

just like before!

Important Remark. Suppose the problem were

$$\frac{s}{(2s+2)(s+2)}$$

(we just multiplied the bottom by two). Again the poles are at $s = -1$, $s = -2$, but the residue at $s = -1$ is *not* $(-1)e^{-t}/(-1+2)$. This is due to the fact that the residue is the Laurent series coefficient of $1/(s+1)$, *not* of $1/(2s+2)$. So, we must divide by two, i.e., write

$$\frac{s}{(2s+2)(s+2)} = \frac{s}{2(s+1)(s+2)}$$

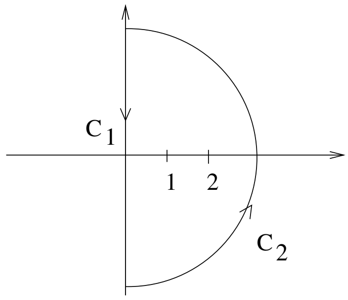
and the residue at $s = -1$ is

$$\frac{(-1)e^{-t}}{2(-1+2)}!$$

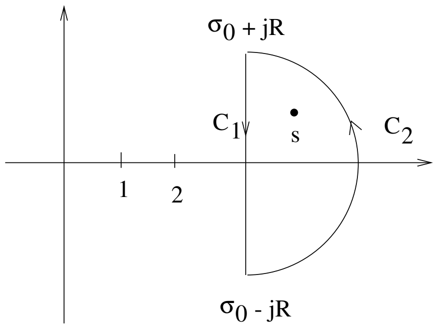
Example 2. Find the inverse Laplace transform of

$$f(s) = \frac{1}{(s-1)(s-2)}.$$

Answer. $f(s)$ and thus $f(s)e^{st}$ has poles at $s = 1$ and $s = 2$ so we can't use the same simple closed path as before with C_1 being the imaginary axis, because in such a case the poles would actually be on the right of C_1 as shown in the following picture.



So we move C_1 and C_2 to the *left* of $s = 2$ by choosing C_1 to lie on the line with real part $\sigma_0 > 2$ as shown in the picture below.

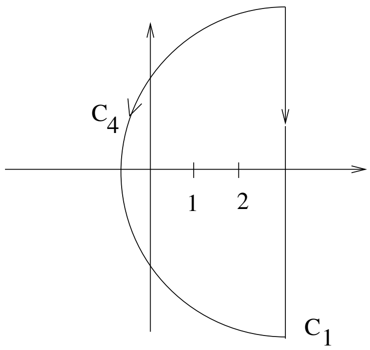


We repeat *exactly* what we did and conclude as before

$$h(t) = -\frac{1}{2\pi j}\int_{C_1^*} f(z)e^{zt}\,dz.$$

Again we introduce the other semicircle; but we face an apparent problem: on C_4, the real part of z is not always negative so it looks like we can't conclude that for $t > 0$,

$$\left|\int_{C_4} f(z)e^{zt}\,dz\right| \to 0 \quad \text{as } R \to \infty.$$



But this is not a real problem since for *most* z we have $\mathrm{Re}\,(z) < 0$ as $R \to \infty$. So, we just make the variable change $z = z' + \sigma_0$. Then

$$h(t) = \frac{1}{2\pi j} \int_{z=\sigma_0 - j\infty}^{\sigma_0 + j\infty} f(z) e^{zt} \, dz$$

becomes

$$\begin{aligned} h(t) &= \frac{1}{2\pi j} \int_{z'=-j\infty}^{j\infty} f(z' + \sigma_0) e^{(z'+\sigma_0)t} \, dz' \\ &= \frac{1}{2\pi j} e^{\sigma_0 t} \int_{z'=-j\infty}^{j\infty} f(z' + \sigma_0) e^{z't} \, dz'. \end{aligned}$$

Now we can repeat exactly what we did before! So

$$\begin{aligned} h(t) &= \sum \text{Residues of } \left[e^{\sigma_0 t} f(z' + \sigma_0) e^{z't} \right] \\ &= \sum \text{Residues of } \left[f(z' + \sigma_0) e^{(z'+\sigma_0)t} \right]. \end{aligned}$$

But notice that

$$\sum \text{Residues of } [f(z' + \sigma_0) e^{(z'+\sigma_0)t}] = \sum \text{Residues of } [f(z) e^{zt}]$$

since the poles of $f(z' + \sigma_0) e^{(z'+\sigma_0)t}$ have moved relative to those of $f(z)e^{zt}$ (by $-\sigma_0$) but the residue is the same!

In *summary,* we *still get in many practical cases that the inverse transform is* *the sum of the residues even if the poles are in the right* $1/2$ *plane.* You must be a bit careful with this as one of the examples below indicates. So, if $f(s) = 1/(s-1)(s-2)$, then $f(s)e^{st} = e^{st}/(s-1)(s-2)$ and the sum of the residues is

$$\frac{e^t}{(1-2)} + \frac{e^{2t}}{(2-1)} = -e^t + e^{2t}.$$

Note however the useful observation that when some of the poles are in the right $1/2$ *plane the inverse transform blows up as* $t \to \infty$.

Example 3. Find the inverse transform of: $f(s) = \dfrac{s}{s^2+1}$.

Answer. The poles are at $s = \pm j$, but based on the discussion given in Example 2 we do not worry about this. We get

$$f(s)e^{st} = \frac{se^{st}}{(s+j)(s-j)},$$

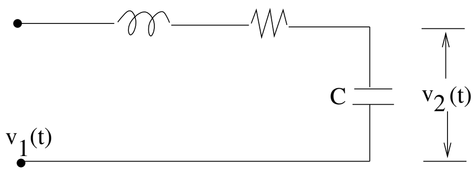
so the sum of the residues is

$$\frac{j}{2j}e^{jt} + \frac{(-j)}{(-2j)}e^{-jt} = \frac{e^{jt}+e^{-jt}}{2} = \cos t.$$

Note that the poles are on the imaginary axis, and the inverse transform oscillates but does *not* decay as $t \to \infty$.

We conclude with the following practical and important example.

Example 4. Consider the circuit shown.



Determine $v_2(t)$ if: (a) $v_1(t) = \sin wt$, (b) $v_1(t)$ is

$$\begin{cases} 0, & t < 1 \\ 1, & 1 \le t < \infty. \end{cases}$$

Assume that all the initial conditions are zero.

Answer. We have

$$L\frac{di}{dt} + Ri + \frac{Q}{C} = v_1(t), \qquad \frac{Q}{C} = v_2(t).$$

Since $Q = \int_0^t i(t)\,dt$, we let I denote the Laplace transform of i, V_1 the transform of $v_1(t)$ and V_2 the transform of $v_2(t)$, and take the transform of both sides of each equation to get:

$$\begin{aligned} sLI + RI + \frac{I}{sC} &= V_1 \\ \frac{I}{sC} &= V_2 \end{aligned}$$

and so

$$\frac{V_2}{V_1} = \frac{\frac{1}{sC}}{sL + R + \frac{1}{sC}} = \frac{1}{s^2LC + sRC + 1}.$$

Observe that the poles are at $s^2LC + sRC + 1 = 0$. So we find the roots of

$$s^2 + s\frac{R}{L} + \frac{1}{LC} = 0,$$

i.e.,

$$s = \frac{1}{2}\left[-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}\right].$$

Put

$$s_0 = -\frac{R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}, \qquad s_1 = -\frac{R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}},$$

and note that s_0, s_1 may be real, equal or complex depending on the sign of $\frac{R^2}{L^2} - \frac{4}{LC}$, but in every case they are in the left 1/2 plane. We rewrite

$$\frac{V_2}{V_1} = \frac{1}{LC(s-s_0)(s-s_1)}$$

and so

$$V_2 = \frac{V_1}{LC(s-s_0)(s-s_1)}.$$

We now pass to case (a): $v_1 = \sin wt$ and $V_1 = w/(s^2+w^2)$. In this case

$$V_2 = \frac{w}{LC(s-s_0)(s-s_1)(s+jw)(s-jw)}.$$

Suppose $s_0 \neq s_1$. Then there are four simple poles: s_0, s_1, $\pm jw$ and the residues of $V_2 e^{st}$ are:

$$\begin{aligned}
\text{at } s_0: &\quad \frac{w}{LC}\frac{e^{s_0 t}}{\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}(s_0^2+w^2)} \\
\text{at } s_1: &\quad \frac{w}{LC}\frac{e^{s_1 t}}{(-\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}})(s_1^2+w^2)} \\
\text{at } jw: &\quad \frac{w}{LC}\frac{e^{jwt}}{((jw)^2 + (jw)\frac{R}{L} + \frac{1}{LC})(2jw)} \\
\text{at } -jw: &\quad \frac{w}{LC}\frac{e^{-jwt}}{((-jw)^2 + (-jw)\frac{R}{L} + \frac{1}{LC})(-2jw)}.
\end{aligned}$$

So

$$\begin{aligned}
v_2(t) = \frac{w}{LC}\Bigg[&\frac{e^{s_0 t}}{\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}(s_0^2+w^2)} + \frac{e^{s_1 t}}{-\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}(s_1^2+w^2)} \\
&+ \frac{e^{jwt}}{(-w^2 + jw\frac{R}{L} + \frac{1}{LC})(2jw)} + \frac{e^{-jwt}}{(-w^2 - jw\frac{R}{L} + \frac{1}{LC})(-2jw)}\Bigg].
\end{aligned}$$

Next, suppose $s_0 = s_1 = -R/(2L)$. Then

$$V_2 = \frac{w}{LC(s-s_0)^2(s+jw)(s-jw)}.$$

There are now three poles: $\pm jw$ and s_0, with the latter of order two. The residue at s_0 of $V_2 e^{st}$ is thus

$$\frac{d}{ds}\left[\frac{we^{st}}{LC(s^2+1)}\right]\Bigg|_{s=s_0}$$

while the calculation of the residues at $\pm jw$ is unchanged. We observe that it may appear that $v_2(t)$ ends up being complex, due to the terms jwR/L and even more so if $(R^2/L^2) - 4/(LC) < 0$, so that

$$\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} = j\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}.$$

This is not the case, and *can be used as a check to see if mistakes have been made.* To see how the "j"'s must cancel, consider the two terms:

$$\frac{e^{jwt}}{[(\frac{1}{LC} - w^2) + jw\frac{R}{L}](2jw)} + \frac{e^{-jwt}}{[(\frac{1}{LC} - w^2) - jw\frac{R}{L}](-2jw)}.$$

Taking to a common denominator gives:

$$\begin{aligned}
&\frac{e^{jwt}[(\frac{1}{LC} - w^2) - \frac{jwR}{L}] - e^{-jwt}[(\frac{1}{LC} - w^2) + jw\frac{R}{L}]}{(2jw)[(\frac{1}{LC} - w^2)^2 + w^2\frac{R^2}{L^2}]} \\
&\quad = \frac{1}{[(\frac{1}{LC} - w^2)^2 + w^2\frac{R^2}{L^2}]w}\left\{\left(\frac{1}{LC} - w^2\right)\left(\frac{e^{jwt} - e^{-jwt}}{2j}\right) - \frac{wR}{L}\left(\frac{e^{jwt} + e^{-jwt}}{2}\right)\right\} \\
&\quad = \frac{1}{w[(\frac{1}{LC} - w^2)^2 + w^2\frac{R^2}{L^2}]}\left\{\left(\frac{1}{LC} - w^2\right)\sin wt - \frac{wR}{L}\cos wt\right\}.
\end{aligned}$$

The other two terms combine accordingly. We now pass to case (b), where

$$v_1(t) = \begin{cases} 0, & t < 1 \\ 1, & 1 \le t < \infty \end{cases}$$

so $v_1(t) = u(t-1)$ (where u is the Heaviside function) and $V_1 = e^{-s}/s$, leading to

$$V_2 = \frac{e^{-s}}{LC(s-s_0)(s-s_1)s}$$

and thus

$$V_2 e^{st} = \frac{e^{st}e^{-s}}{LC(s-s_0)(s-s_1)s} = \frac{e^{s(t-1)}}{LC(s-s_0)(s-s_1)s}.$$

Note that there are 3 poles: 0, s_0, s_1 *but there is a problem now: if* $t < 1$ *then* $t-1 < 0$ *and so* $\mathrm{Re}\,[s(t-1)] > 0$ *for* $\mathrm{Re}\, s < 0$*. On the other hand if* $t > 1$ *then* $\mathrm{Re}\,[s(t-1)] < 0$ *for* $\mathrm{Re}\, s < 0$. In the second case ($t > 1$), we can use the previous result. If $t > 1$,

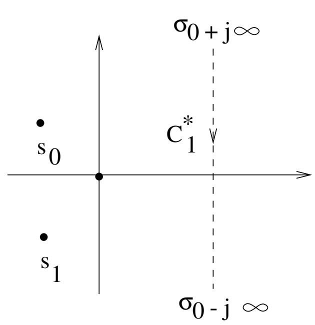
$$v_2(t) = \frac{e^0}{LC(-s_0)(-s_1)} + \frac{e^{s_0(t-1)}}{LC(s_0-s_1)s_0} + \frac{e^{s_1(t-1)}}{LC(s_1-s_0)s_1}$$

where we have assumed $s_0 \neq s_1$. If $s_0 = s_1$ we need to find the residue by taking derivatives. So, in this case, the answer is obtained just like before. *But what do we do if* $t < 1$*!*

Remember the key formula:

$$h(t) = -\frac{1}{2\pi j}\int_{C_1^*} f(z)e^{zt}\,dz$$

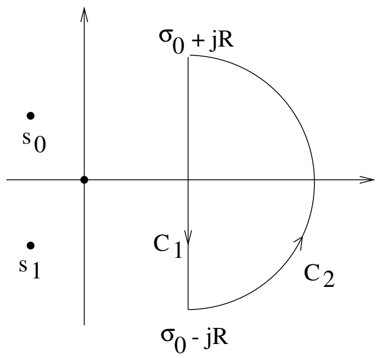
with C_1* the straight line from $\sigma_0 + j\infty$ to $\sigma_0 - j\infty$.



Here

$$-\frac{1}{2\pi j}\int_{C_1^*} f(z)e^{zt}dz = -\frac{1}{2\pi j}\int_{C_1^*} \frac{e^{z(t-1)}}{LC(z-z_0)(z-z_1)z}\, dz$$

where we have gone back in notation from s to z. Since $t-1<0$, we do the obvious; instead of a semicircle to the left of C_1^*, we *construct one to the right since then* $|e^{z(t-1)}| \to 0$ *as* $R \to \infty$*!*



Then

$$\begin{aligned}
h(t) &= -\frac{1}{2\pi j}\int_{C_1^*} f(z)e^{zt}\, dz = -\frac{1}{2\pi j}\lim_{R\to\infty}\int_{C_1} f(z)e^{zt}\, dz \\
&= -\frac{1}{2\pi j}\lim_{R\to\infty}\int_{C_1} f(z)e^{zt}\, dz - \frac{1}{2\pi j}\lim_{R\to\infty}\int_{C_2} f(z)e^{zt}\, dz \\
&= -\frac{1}{2\pi j}\lim_{R\to\infty}\oint_{C_1+C_2} f(z)e^{zt}\, dz.
\end{aligned}$$

But $f(z)e^{zt}$ is differentiable inside and on $C_1 + C_2$, so

$$\oint_{C_1+C_2} f(z)e^{zt}\, dz = 0\,!$$

In summary, we have found

$$\begin{aligned}
v_1(t) &= 0, \qquad \text{if} \quad t<1 \\
v_2(t) &= \frac{1}{LC}\left[\frac{1}{s_0 s_1} + \frac{e^{s_0(t-1)}}{(s_0-s_1)s_0} + \frac{e^{s_1(t-1)}}{(s_1-s_0)s_1}\right] \quad \text{if } t>1
\end{aligned}$$

where we have assumed $s_0 \neq s_1$. Note that this is as it should be, since $v_1(t) = 0$ if $t < 1$ and there are not initial voltages/currents. *Thus $v_2(t)$ should be zero if $t < 1$ and this is what we have found.*

Further Exercises:

Evaluate the following integrals.

Exercise 1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{(3\cos\theta+5)^2}.$

Answer. To evaluate $\int_0^{2\pi} d\theta/(3\cos\theta+5)^2$ put

$$\begin{cases} z = e^{j\theta}, & \cos\theta = \dfrac{1}{2}\left[z+\dfrac{1}{z}\right], \\ dz = jz\,d\theta, & C = \text{unit circle.} \end{cases}$$

Then

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{(3\cos\theta+5)^2} &= \oint_{\mathcal{C}} \frac{1}{[\frac{3}{2}(z+\frac{1}{z})+5]^2}\frac{dz}{jz} = \frac{1}{j}\oint_{\mathcal{C}} \frac{z\,dz}{[\frac{3}{2}(z^2+1)+5z]^2} \\ &= \frac{4}{9j}\oint_{\mathcal{C}} \frac{z\,dz}{[z^2+1+\frac{10}{3}z]^2}. \end{aligned}$$

Next, note that

$$z^2 + \frac{10}{3}z + 1 = 0 \quad \text{iff} \quad z = \frac{-\frac{10}{3} \pm \sqrt{\frac{100}{9} - 4}}{2},$$

i.e.,

$$z = -\frac{5}{3} \pm \sqrt{\frac{25}{9} - 1} = -\frac{5}{3} \pm \frac{4}{3} = -3, \, -\frac{1}{3}.$$

So $\left[z^2+1+\dfrac{10z}{3}\right]^2 = \left[(z+3)\left(z+\dfrac{1}{3}\right)\right]^2$ and

$$\frac{4}{9j}\oint_{\mathcal{C}} \frac{z\,dz}{[z^2+1+\frac{10}{3}z]^2} = \frac{4}{9j}\oint_{\mathcal{C}} \frac{z\,dz}{(z+3)^2(z+\frac{1}{3})^2}.$$

The only pole inside $\mathcal{C}$ is at $z=-1/3$. It is a pole of order 2, thus

$$\begin{aligned} &\text{Residue of } \frac{z}{(z+3)^2(z+\frac{1}{3})^2} \text{ at } z = -\frac{1}{3} \text{ is } \frac{d}{dz}\left[\frac{z}{(z+3)^2}\right]_{z=-\frac{1}{3}} \\ &\quad = \left.\frac{(z+3)^2 - z(2)(z+3)}{(z+3)^4}\right|_{z=-\frac{1}{3}} = \left.\frac{3-z}{(z+3)^2}\right|_{z=-\frac{1}{3}} = \frac{3+\frac{1}{3}}{(3-\frac{1}{3})^3} = \frac{10}{8}\cdot\frac{9}{8^2}. \end{aligned}$$

We conclude

$$\int_0^{2\pi} = 2\pi j \left[\frac{4}{9j} \cdot \frac{10}{8} \cdot \frac{9}{8^2}\right] = \frac{5\pi}{32}.$$

Exercise 2. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta}.$

Answer. The transformation to the z-plane is the same as the previous problem, except $\sin\theta = (1/(2j))[z - 1/z]$. So

$$\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta} = \oint_{\mathcal{C}} \frac{1}{1+\frac{1}{(2j)^2}(z-\frac{1}{z})^2} \frac{dz}{jz} = \frac{1}{j}\oint_{\mathcal{C}} \frac{z}{z^2 - \frac{1}{4}(z^2-1)^2}\, dz.$$

(The idea is to multiply top and bottom by z in order to remove the "$1/z$" term.)

Note next that

$$z^2 - \frac{1}{4}(z^2-1)^2 = z^2 - \frac{1}{4}(z^4 - 2z^2 + 1) = -\frac{1}{4}(z^4 - 6z^2 + 1).$$

Put $w = z^2$. Then $z^4 - 6z^2 + 1 = w^2 - 6w + 1$ and $w^2 - 6w + 1 = 0$ iff $w = \frac{1}{2}(6 \pm \sqrt{36-4}) = 3 \pm \sqrt{8}$, i.e., $w^2 - 6w + 1 = (w - 3 + \sqrt{8})(w - 3 - \sqrt{8})$. We thus have

$$\begin{aligned} z^4 - 6z^2 + 1 &= (z^2 - 3 + \sqrt{8})(z^2 - 3 - \sqrt{8}) \\ &= (z + (3-\sqrt{8})^{\frac{1}{2}})(z - (3-\sqrt{8})^{\frac{1}{2}})(z + (3+\sqrt{8})^{\frac{1}{2}})(z - (3+\sqrt{8})^{\frac{1}{2}}). \end{aligned}$$

To save writing, put $s_0 = (3-\sqrt{8})^{\frac{1}{2}}$, $s_1 = (3+\sqrt{8})^{\frac{1}{2}}$. Thus

$$\frac{1}{j}\oint_{\mathcal{C}} \frac{z}{z^2 - \frac{1}{4}(z^2-1)^2}\, dz = -\frac{1}{4^{-1}j}\oint_{\mathcal{C}} \frac{z\, dz}{(z-s_0)(z+s_0)(z-s_1)(z+s_1)}.$$

Note that $\pm s_1$ are outside $\mathcal{C}$, and that the (simple) poles at $\pm s_0$ are inside $\mathcal{C}$. Thus

$$\frac{1}{j}\oint_{\mathcal{C}} \frac{z}{z^2 - \frac{1}{4}(z^2-1)^2}\, dz = -\frac{2\pi j}{4^{-1}j}\left[\frac{s_0}{2s_0(s_0^2 - s_1^2)} - \frac{s_0}{-2s_0(s_0^2 - s_1^2)}\right] = -16\pi\left[\frac{s_0}{2s_0(s_0^2 - s_1^2)}\right]$$

$$= 16 \cdot \frac{\pi}{2}\left[\frac{1}{(3+\sqrt{8})-(3-\sqrt{8})}\right] = \frac{\pi}{2}(16)\frac{1}{2\sqrt{8}} = \frac{4\pi}{\sqrt{8}}.$$

Exercise 3. $\displaystyle\int_{-\infty}^{\infty} \frac{x^4}{(x^2+1)^3}\,dx.$

Answer. Observe that the order of the bottom is 6, the order of the top is 4, thus

$$\int_{-\infty}^{\infty} \frac{x^4}{(x^2+1)^3}\,dx = 2\pi j(\text{sum of residues of } \frac{z^4}{(z^2+1)^3} \text{ in the upper 1/2 plane}).$$

Now $z^4/(z^2+1)^3 = z^4/[(z+j)^3(z-j)^3]$. Consequently there is only the pole (of order 3) $z = j$ in the upper 1/2 plane and the residue is

$$\frac{1}{2}\frac{d^2}{dz^2}\left[\frac{z^4}{(z+j)^3}\right]\Bigg|_{z=j}.$$

Now

$$\frac{d}{dz}\left[\frac{z^4}{(z+j)^3}\right] = \frac{4z^3(z+j)^3 - z^4 3(z+j)^2}{(z+j)^6} = \frac{4z^3(z+j) - 3z^4}{(z+j)^4} = \frac{z^4+4jz^3}{(z+j)^4}.$$

and thus

$$\begin{aligned}\frac{d^2}{dz^2}\left[\frac{z^4}{(z+j)^3}\right] &= \frac{(4z^3+12jz^2)(z+j)^4 - (z^4+4jz^3)4(z+j)^3}{(z+j)^8}\\ &= \frac{(4z^3+12jz^2)(z+j) - (z^4+4jz^3)4}{(z+j)^5},\end{aligned}$$

and as $z = j$ this becomes

$$\frac{(-4j-12j)(2j)-(1+4)4}{2^5 j} = \frac{32-20}{2^5 j}.$$

In conclusion,

$$\int_{-\infty}^{\infty} \frac{x^4}{(x^2+1)^3}\,dx = 2\pi j\left[\frac{12}{32j}\right] = \frac{3\pi}{4}.$$

Exercise 4. $\displaystyle\int_{0}^{\infty} \frac{dx}{x^2+1}.$

Answer. We have $\int_0^\infty (dx)/(x^2+1)$, and observe that $1/(x^2+1)$ is even. We conclude that

$$\int_0^\infty \frac{dx}{x^2+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{x^2+1} = \frac{1}{2}(2\pi j)(\text{sum of residues of } \frac{1}{z^2+1} \text{ in the upper } \frac{1}{2} \text{ plane}).$$

We note $1/(z^2+1) = 1/[(z+j)(z-j)]$, and thus

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2}\cdot 2\pi j\left(\frac{1}{2j}\right) = \frac{\pi}{2}.$$

Exercise 5. $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sin^2\theta + 4\cos^2\theta}.$

Answer. We need to evaluate $\int_0^{2\pi}(d\theta)/(\sin^2\theta + 4\cos^2\theta)$ and the process is the same as the earlier questions. Note first that $\sin^2\theta + 4\cos^2\theta = 1 + 3\cos^2\theta$ and so

$$\int_0^{2\pi} \frac{d\theta}{\sin^2\theta + 4\cos^2\theta} = \int_0^{2\pi}\frac{d\theta}{1+3\cos^2\theta} = \oint_{\mathcal{C}} \frac{1}{1+\frac{3}{4}(z+\frac{1}{z})^2}\frac{dz}{jz} = \frac{4}{j}\oint_{\mathcal{C}}\frac{z}{4z^2+3(z^2+1)^2}\,dz.$$

But $4z^2+3(z^2+1)^2 = 3z^4+10z^2+3$ and we have

$$z^2 = \frac{-10\pm\sqrt{100-4(3)(3)}}{6} = \frac{-10\pm 8}{6} = -3,\ -\frac{1}{3}.$$

Thus $z = \pm\sqrt{3}\,j,\ \pm(1/\sqrt{3})j$. Again put $s_0 = \sqrt{3}j$, $s_1 = (1/\sqrt{3})j$. Then

$$\frac{4}{j}\oint_{\mathcal{C}}\frac{z\,dz}{4z^2+3(z^2+1)^2} = \frac{4}{3j}\oint_{\mathcal{C}}\frac{z}{(z-s_0)(z+s_0)(z-s_1)(z+s_1)}\,dz.$$

Since $\pm s_0$ is outside $\mathcal{C}$, we find

$$\begin{aligned}\frac{4}{j}\oint_{\mathcal{C}}\frac{z\,dz}{4z^2+3(z^2+1)^2} &= \frac{4}{3j}\cdot 2\pi j\left[\frac{s_1}{(s_1^2-s_0^2)(2s_1)} - \frac{s_1}{(s_1^2-s_0^2)(-2s_1)}\right] = \frac{8\pi}{3}\cdot\frac{1}{s_1^2-s_0^2}\\ &= \frac{8\pi}{3}\cdot\frac{1}{(-\frac{1}{3})-(-3)} = \frac{8\pi}{3}\cdot\frac{1}{3-\frac{1}{3}} = \frac{8\pi}{8} = \pi.\end{aligned}$$

IV. Fourier Series

IV.a *Basic Ideas*

Fourier Series furnish a means of writing arbitrary functions, found in practice, in terms of an infinite sum of "nice" functions (in the simplest case: sines and cosines). Thus a wide variety of inputs to a system can be handled if the response to sine and cosine inputs is known.

Not only that, but there exist other families of functions that work besides sines and cosines (see below) and some even work for problems in higher dimensions, i.e., for distributed parameter questions. To begin, we note the following relationships. Suppose $n \neq m$, but both are positive integers, then

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin nt \sin mt \, dt &= \int_{-\pi}^{\pi} \frac{(e^{jnt} - e^{-jnt})(e^{jmt} - e^{-jmt})}{-4} \, dt \\
&= \int_{-\pi}^{\pi} \frac{e^{j(n+m)t} - e^{j(m-n)t} - e^{-j(m-n)t} + e^{-j(n+m)t}}{-4} \, dt \\
&= \int_{-\pi}^{\pi} \frac{-\cos(n+m)t + \cos(m-n)t}{2} \, dt \\
&= \frac{1}{2} \left[\frac{-\sin(n+m)t}{n+m} + \frac{\sin(m-n)t}{m-n} \right]_{-\pi}^{\pi} = 0.
\end{aligned}$$

Exactly the same type of calculation shows:

$$\begin{aligned}
&\int_{-\pi}^{\pi} \sin nt \cos mt \, dt = 0 \qquad \text{(even if } n = m) \\
&\int_{-\pi}^{\pi} \cos nt \cos mt \, dt = 0 \qquad \text{(if } m \neq n).
\end{aligned}$$

On the other hand, if $m = n$ then

$$\int_{-\pi}^{\pi} \cos nt \cos mt \, dt = \int_{-\pi}^{\pi} \cos^2 nt \, dt = \int_{-\pi}^{\pi} \frac{1 + \cos 2nt}{2} \, dt = \frac{1}{2} \cdot 2\pi = \pi,$$

and

$$\int_{-\pi}^{\pi} \sin nt \sin mt \, dt = \int_{-\pi}^{\pi} \sin^2 nt \, dt = \int_{-\pi}^{\pi} \frac{1 - \cos 2nt}{2} \, dt = \pi.$$

Exactly the same results hold if we integrate $\int_d^{d+2\pi}$ instead of $\int_{-\pi}^{\pi}$. In practice, $d = 0$ or $d = -\pi$ (what we chose).

Given an arbitrary function f (such that $|f(t)|$ can be integrated—this is a condition almost always satisfied in practice) we represent f as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \tag{1}$$

where

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt, \\ a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt. \end{aligned}$$

Note that a_0, a_n, b_n are *constants*. To see how the coefficients are arrived at, note the following: Choose a value of n, say n_0. Then multiply both sides of (1) by $\sin n_0 t$ and integrate from $-\pi$ to π. We have

$$\begin{aligned} \int_{-\pi}^{\pi} f(t) \sin n_0 t \, dt = \frac{a_0}{2} \int_{-\pi}^{\pi} \sin n_0 t \, dt + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nt \sin n_0 t \, dt \\ + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nt \sin n_0 t \, dt. \end{aligned}$$

Now $\int_{-\pi}^{\pi} \cos nt \sin n_0 t \, dt = 0$ for any $n = 1, 2, \ldots$, while $\int_{-\pi}^{\pi} \sin nt \sin n_0 t \, dt$ will also be zero except when $n = n_0$. Then we get π, and so

$$\int_{-\pi}^{\pi} f(t) \sin n_0 t \, dt = b_{n_0} \pi$$

or

$$b_{n_0} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin n_0 t \, dt.$$

In the same way, if we multiply by $\cos n_0 t$ on both sides of (1) and then integrate from $-\pi$ to π we get the formula for a_{n_0}. If you multiply by 1 and integrate, we get

$$\int_{-\pi}^{\pi} f(t) \, dt = \frac{a_0}{2} \cdot 2\pi.$$

So $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt$ as given. Note that this requires that the series start with "$a_0/2$", not "a_0".

The sum of the series on the right hand side of (1) *only represents* f *in the interval* $-\pi < t < \pi$ (*except where* $f(t)$ *has a jump*) unless f is *periodic* of period 2π (i.e., $f(t + 2\pi) = f(t)$ for every t) in which case the sum of the series represents f for *every* t (*except where* $f(t)$ *has a jump*). If f is *not* periodic then the sum of the series still represents a periodic function since it is the sum of functions of period 2π (namely $\sin nt$ and $\cos nt$). See the following examples for clarification of the above satements. The theoretical situation is summarized by the following theorem.

Dirichlet's Theorem. *For* $-\pi \le t < \pi$ *suppose* $f(t)$ *is bounded, has a finite number of maximums and minimums and only a finite number of discontinuities (conditions satisfied in practice!). Let* $f(t)$ *be defined for other values of* t *by* $f(t) = f(t + 2\pi)$. *Then the sum of the Fourier series for* $f(t)$ *converges to:*

$$\frac{1}{2}[f(t+) + f(t-)] \tag{2}$$

at every t *where*

$$f(t_0+) = \lim_{t \to t_0} f(t) \quad \textit{as } t \to t_0 \textit{ from the right side only,}$$

$$f(t_0-) = \lim_{t \to t_0} f(t) \quad \textit{as } t \to t_0 \textit{ from the left side only.}$$

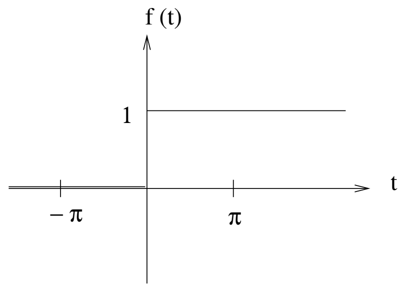
The conditions on f of Dirichlet's Theorem are almost always satisfied in practice. *Equation (2) means that if* f *has a jump at* t *then the Fourier series converges to the middle of the jump; if the function is smooth at* t*, then the Fourier series converges to* $f(t)$.

Example 1. Write a Fourier series for $f(t)$ if

$$f(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

and sketch its sum.

Answer.



We first calculate the coefficients:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi} dt = 1,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_0^{\pi}\cos nt\,dt = \left.\frac{\sin nt}{n\pi}\right|_0^{\pi} = 0,$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}\sin nt\,dt = \frac{1}{\pi}\left[-\frac{\cos nt}{n}\right]_0^{\pi} = \frac{1}{n\pi}[1 - \cos n\pi].$$

Therefore,

$$\begin{aligned} b_1 &= \frac{1}{\pi}[1 - \cos\pi] = \frac{2}{\pi}, \\ b_2 &= \frac{1}{2\pi}[1 - \cos 2\pi] = 0, \\ b_3 &= \frac{1}{3\pi}[1 - \cos 3\pi] = \frac{2}{3\pi}. \end{aligned}$$

Therefore,

$$b_n = \begin{cases} \dfrac{2}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

Therefore the Fourier series for f is

$$\frac{1}{2} + \sum_{n \text{ odd}} \frac{2}{n\pi} \sin nt,$$

i.e.,

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin t + \frac{2}{3\pi}\sin 3t + \frac{2}{5\pi}\sin 5t + \cdots.$$

The following trick is very useful for writing the sum. Let $n = 2m - 1$. Then

$$\begin{aligned} &\text{when } m = 1 &&\Longrightarrow && n = 1 \\ &\text{when } m = 2 &&\Longrightarrow && n = 3 \\ &\text{when } m = 3 &&\Longrightarrow && n = 5, \end{aligned}$$

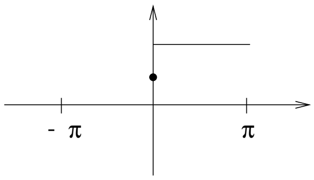
that is, as m runs over *all* the integers > 0, n runs over all the *odd* ones > 0. Therefore,

$$\sum_{n \text{ odd}} \frac{2}{n\pi} \sin nt = \sum_{m=1}^{\infty} \frac{2}{(2m-1)\pi} \sin(2m-1)t \qquad (\text{let } n = 2m - 1)$$

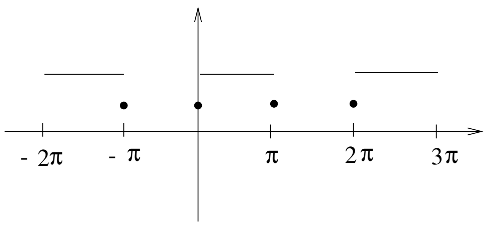
and we finally write

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \sin(2n-1)t.$$

(Other substitutions are also useful.) Passing to the graph of the sum, by the Dirichlet's Theorem,



The graph of the sum of the series for $-3\pi < t < 3\pi$ is because of periodicity:



Note that the series repeats outside $-\pi < t < \pi$ what it did *inside* the interval. Outside $-\pi < t < \pi$ the series therefore does *not* represent f.

Example 2. Find the Fourier series for $f(t) = \begin{cases} 0, & -\pi < t < 0 \\ \sin t, & 0 \le t < \pi \end{cases}$.

Answer.

$$a_0 = \frac{1}{\pi} \int_0^{\pi} \sin t \, dt = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} \sin t \cos nt \, dt.$$

To evaluate a_n it is again useful to recall

$$\sin t = \frac{e^{jt} - e^{-jt}}{2j}, \qquad \cos t = \frac{e^{jt} + e^{-jt}}{2}.$$

We find

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{\pi} \left(\frac{e^{jt} - e^{-jt}}{2j}\right)\left(\frac{e^{jnt} + e^{-jnt}}{2}\right) dt \\
&= \frac{1}{4\pi j}\int_0^{\pi} \left[e^{j(n+1)t} + e^{j(1-n)t} - e^{-j(1-n)t} - e^{-j(n+1)t}\right] dt \\
&= \frac{1}{4\pi j}\int_0^{\pi} \left[2j\sin(n+1)t + 2j\sin(1-n)t\right] dt \\
&= \frac{1}{2\pi}\int_0^{\pi} [\sin(n+1)t - \sin(n-1)t]\, dt \\
&= \frac{1}{2\pi}\left[\frac{-\cos(n+1)t}{n+1} + \frac{\cos(n-1)t}{n-1}\right]_0^{\pi} \\
&= \frac{1}{2\pi}\left[\frac{(\cos(n-1)\pi) - 1}{n-1} - \frac{(\cos(n+1)\pi) - 1}{n+1}\right] \\
&= \frac{1}{2\pi}[(\cos(n-1)\pi) - 1]\frac{2}{n^2-1} = \frac{1}{\pi(n^2-1)}[(\cos(n-1)\pi) - 1].
\end{aligned}$$

Hence

$$a_n = \begin{cases} 0, & n \text{ odd} \\ \dfrac{-2}{\pi(n^2-1)}, & n \text{ even.} \end{cases}$$

Note that if $n = 1$, this formula will not work and a_1 needs to be calculated separately. Similarly,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{\pi} \sin t \sin nt\, dt = \frac{1}{\pi}\int_0^{\pi} \left(\frac{e^{jt} - e^{-jt}}{2j}\right)\left(\frac{e^{jnt} - e^{-jnt}}{2j}\right) dt \\
&= -\frac{1}{4\pi}\int_0^{\pi} \left[e^{j(n+1)t} - e^{-j(n-1)t} - e^{j(n-1)t} + e^{j(n+1)t}\right] dt \\
&= -\frac{1}{4\pi}\int_0^{\pi} 2\left[\cos(n+1)t - \cos(n-1)t\right] dt
\end{aligned}$$

$$= -\frac{1}{2\pi}\left[\frac{\sin(n+1)t}{n+1} - \frac{\sin(n-1)t}{n-1}\right]_0^\pi = -\frac{1}{2\pi}\left[\frac{\sin(n+1)\pi}{n+1} - \frac{\sin(n-1)\pi}{n-1}\right] = 0.$$

But if $n = 1$, this does not work, since we divide by $n-1$! We thus need to calculate b_1 separately:

$$\begin{aligned} b_1 &= \frac{1}{\pi}\int_0^\pi \sin t \sin t\, dt = \frac{1}{\pi}\int_0^\pi \frac{1-\cos 2t}{2}\, dt \\ &= \frac{1}{2\pi}\left[t - \frac{\sin 2t}{2}\right]_0^\pi = \frac{1}{2}. \end{aligned}$$

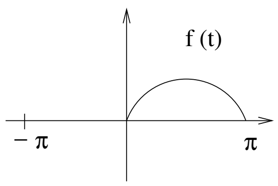
and $a_1 = 1/(2\pi)\int_0^\pi \sin 2t\, dt = 0$. Therefore,

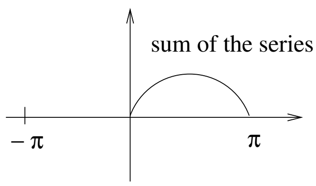
$$f(t) = \frac{1}{\pi} + \frac{1}{\pi}\sum_{n \text{ even}} \left(\frac{-2}{\pi}\right)\frac{1}{n^2-1}\cos nt + \frac{1}{2}\sin t.$$

In this example, it is useful to introduce the change $n = 2m$. Then

$$f(t) = \frac{1}{\pi} + \sum_{m=1}^{\infty} \frac{-2}{m}\,\frac{\cos 2mt}{4m^2-1} + \frac{1}{2}\sin t.$$

Comparing the various graphs:





Note that since there are no jumps in $f(t)$, the Fourier series agrees with $f(t)$ *inside* the interval $-\pi < t < \pi$.

Example 3. Let

$$f(t) = \begin{cases} -t, & \text{for } -\pi < t < 0 \\ 0, & \text{for } 0 < t < \pi \end{cases}.$$

Find the Fourier series for $f(t)$ and sketch a graph of the sum for $-2\pi < t < 2\pi$.

Answer.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{0} -t\,dt = \frac{1}{\pi}\cdot\frac{\pi^2}{2} = \frac{\pi}{2},\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -t\cos nt\,dt\\
&= \frac{1}{\pi}\left[\frac{-t\sin nt}{n}\Big|_{-\pi}^{0} + \int_{-\pi}^{0}\frac{\sin nt}{n}\,dt\right] \qquad \text{(integration by parts)}\\
&= \frac{1}{n^2\pi}[(\cos n\pi) - 1].
\end{aligned}
$$

Therefore,

$$
a_1 = \frac{-2}{\pi},\ a_2 = 0,\ a_3 = \frac{-2}{9\pi},\ldots .
$$

Or

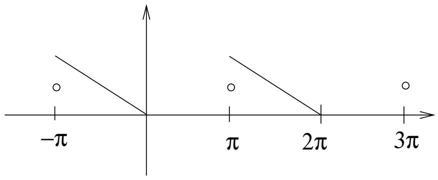
$$
a_n = \begin{cases} 0, & n \text{ even}\\ \dfrac{-2}{n^2\pi}, & n \text{ odd,}\end{cases}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{0} -t\sin nt\,dt = \frac{1}{\pi}\left[\frac{t\cos nt}{n}\right]_{-\pi}^{0} - \frac{1}{\pi}\int_{-\pi}^{0}\frac{\cos nt}{n}\,dt\\
&= \frac{\cos n\pi}{n} = \frac{(-1)^n}{n}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f(t) &= \frac{\pi}{4} + \sum_{n \text{ odd}}\left(\frac{-2}{\pi n^2}\cos nt\right) + \sum_{n=1}^{\infty}\frac{1}{n}(-1)^n\sin nt\\
&= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)t + \sum_{n=1}^{\infty}\frac{1}{n}(-1)^n\sin nt.
\end{aligned}
$$

Graph of the series:



Graph of the series

Note that at $t = \pi$, the sum of the Fourier series is $\pi/2$!

IV.b *Even and Odd Functions*

The concepts of even and odd functions are important for two reasons. The first is the *rapid* calculation of Fourier series. The second reason will be mentioned below. We call a function $f(t)$ *even* if $f(t) = f(-t)$.

Example 1. $f(t) = t^2$, $f(t) = \cos t$, $f(t) = \cos nt$ are *even* functions.

A function $f(t)$ is *odd* if $f(t) = -f(-t)$.

Example 2. $f(t) = t$, $f(t) = \sin t$, $f(t) = \sin nt$ are *odd* functions.

We note that the product of two even or of two odd functions is an even function whereas the product of an even and an odd function is odd. That is,

$$\text{(even)(even)=even;} \quad \text{(odd)(odd)=even;} \quad \text{(even)(odd)=odd.}$$

We also note the following properties of these functions:

(i) Let $f(t)$ be an *odd* function. Then $\int_{-\pi}^{\pi} f(t)\,dt = 0$.

To see this, note that

$$\begin{aligned}
\int_{-\pi}^{\pi} f(t)\,dt &= \int_{-\pi}^{0} f(t)\,dt + \int_{0}^{\pi} f(t)\,dt \\
&= \int_{\pi}^{0} f(-y)(-dy) + \int_{0}^{\pi} f(t)\,dt \\
&= \int_{\pi}^{0} f(y)\,dy + \int_{0}^{\pi} f(t)\,dt \\
&= -\int_{0}^{\pi} f(t)\,dt + \int_{0}^{\pi} f(t)\,dt = 0
\end{aligned}$$

when, in the second step, we made the substitution $y = -t$.

(ii) Let $f(t)$ be an *even* function. Then $\int_{-\pi}^{\pi} f(t)\,dt = 2\int_{0}^{\pi} f(t)\,dt$.

Since

$$\begin{aligned}
\int_{-\pi}^{\pi} f(t)\,dt &= \int_{-\pi}^{0} f(t)\,dt + \int_{0}^{\pi} f(t)\,dt \\
&= \int_{\pi}^{0} f(-y)(-dy) + \int_{0}^{\pi} f(t)\,dt \\
&= -\int_{\pi}^{0} f(y)\,dy + \int_{0}^{\pi} f(t)\,dt \\
&= \int_{0}^{\pi} f(y)\,dy + \int_{0}^{\pi} f(t)\,dt = 2\int_{0}^{\pi} f(t)\,dt.
\end{aligned}$$

Using the above remarks, we can then state the following: If f is an *even* function on $-\pi < t < \pi$, then the Fourier series for f has *cosine terms only* and the coefficients are given by:

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_{0}^{\pi} f(t)\cos nt\,dt, \qquad n = 1, 2, \dots, \\
a_0 &= \frac{2}{\pi}\int_{0}^{\pi} f(t)\,dt.
\end{aligned}$$

If f is an *odd* function on $-\pi < t < \pi$, then the Fourier series for f has *sine terms only* (since $a_n = 0$) and the coefficients are given by:

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(t)\sin nt\,dt.$$

If f is *even*, then $f(t)\sin nt$ is *odd* ((even)(odd)) and therefore, by (i), $b_n = 0$. Identically, $f(t)\cos nt$ is *even* and therefore, by (ii),

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{2}{\pi}\int_{0}^{\pi} f(t)\cos nt\,dt.$$

Example 3. $f(t) = t$ for all t. Then f is *odd* and we *immediately* conclude that

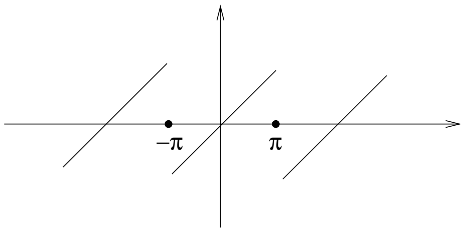
$a_n = 0$, and

$$b_n = \frac{2}{\pi}\int_0^\pi t\sin nt\,dt \overset{\text{parts}}{=} \frac{2}{\pi}\left\{\left[\frac{-t\cos nt}{n}\right]_0^\pi + \int_0^\pi \frac{\cos nt}{n}\,dt\right\}$$
$$= -\frac{2}{n}\cos n\pi = \frac{2}{n}(-1)^{n+1}.$$

Therefore,

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nt.$$

The graph of the sum is:



Example 4. $f(t) = |t|$. Then $f(t)$ is *even* and we conclude $b_n = 0$ and

$$a_n = \frac{2}{\pi}\int_0^\pi |t|\cos nt\,dt = \frac{2}{\pi}\int_0^\pi t\cos nt\,dt = \frac{2}{\pi n^2}\left[\cos n\pi - 1\right]$$
$$= \begin{cases} 0, & n \text{ even,} \\ -\dfrac{4}{\pi n^2}, & n \text{ odd,} \end{cases}$$
$$a_0 = \frac{2}{\pi}\int_0^\pi t\,dt = \pi.$$

Therefore,

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)t.$$

Note that we can also obtain useful identities. For example,

$$f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}, \quad \text{so} \quad \frac{\pi}{2} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

If you are in doubt whether a given function is even or odd (or neither) you may be able to decide by looking at its graph. In any case, if still not sure, you can calculate the Fourier series the long way by calculating *both* the a_n and b_n without using the short cut.

There is another way in which even and odd functions enter in the discussion. Often we are interested in representing a function $f(t)$ not for $-\pi < t < \pi$ but rather for only $0 < t < \pi$. This is the case if, for example, $f(t)$ is only defined in the interval $0 < t < \pi$. We can then *extend backwards* the function $f(t)$ so that it becomes defined for $-\pi < t < \pi$ as *either an even or an odd function.* (That is, we disregard what f is really doing for $-\pi < t < 0$ and define it for convenience.) We can then write the Fourier series for the *extended* function (which we can do quickly since the extended function is even or odd) but *such a series will, in general, represent the original function only for* $0 < t < \pi$. If we extend f as an *odd* function we obtain a Fourier series for f in terms of sines only, i.e., a *Fourier sine series.* Analogously if f is extended as an *even* function we obtain a Fourier series for f in terms of cosines only, i.e., a *Fourier cosine series.* Both series will represent f for $0 < t < \pi$.

Example 5. Let $f(t) = t$ for $0 < t < \pi$. Obtain a Fourier sine series and a Fourier cosine series for f and sketch their sum.

Answer. A Fourier cosine series is obtained by extending f as an *even* function. For this extension we obtain the Fourier series as follows:

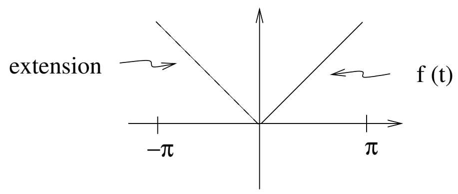
$$b_n = 0 \qquad \text{(by definition)}$$

while

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi t\cos nt\,dt = \frac{2}{\pi}\left[\frac{t\sin nt}{n}\bigg|_0^\pi - \int_0^\pi \frac{\sin nt}{n}\,dt\right] \\
&= \frac{2}{\pi}\left[\frac{\pi\sin n\pi}{n} + \frac{\cos nt}{n^2}\bigg|_0^\pi\right] \\
&= \frac{2}{\pi n^2}[1-\cos n\pi] = \begin{cases} 0, & n \text{ even} \\ \dfrac{2(2)}{\pi n^2}, & n \text{ odd,} \end{cases} \\
a_0 &= \frac{2}{\pi}\int_0^\pi t\,dt = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi,
\end{aligned}
$$

so that

$$
f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)^2}\cos(2n-1)t.
$$



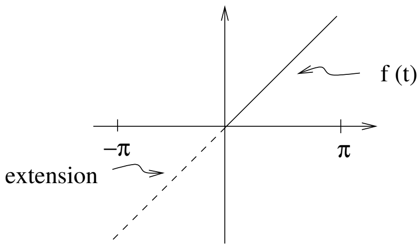
Not surprisingly, this is the same as Example 4.

To obtain a Fourier sine series we extend f as an *odd* function. For this extension we obtain the series by finding

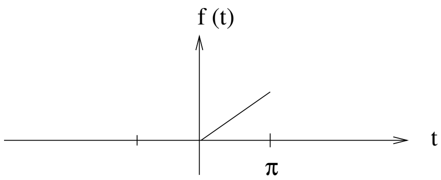
$$
\begin{aligned}
a_n &= 0 \qquad \text{(by definition)} \\
b_n &= \frac{2}{\pi}\int_0^\pi t\sin nt\,dt = \frac{2}{\pi}\left[\frac{-t\cos nt}{n}\bigg|_0^\pi + \int_0^\pi \frac{\cos nt}{n}\,dt\right] \\
&= \frac{2}{\pi}\left[\frac{-t\cos nt}{n}\bigg|_0^\pi + \frac{\sin nt}{n^2}\bigg|_0^\pi\right] \\
&= \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] = -\frac{2}{n}\begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases} = -\frac{2}{n}(-1)^n = \frac{2(-1)^{n+1}}{n}.
\end{aligned}
$$

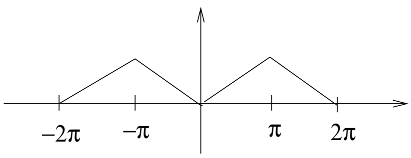
Thus,

$$f = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt.$$

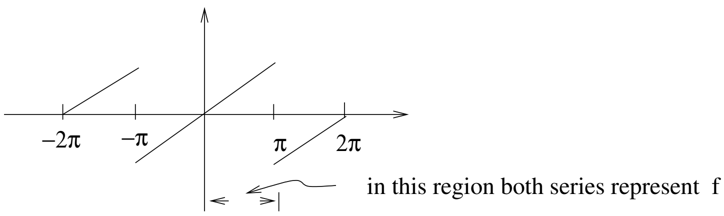


Both the above series represent f in the interval $0 < t < \pi$ although their behavior outside this interval is very different. The graphs of the functions involved are as follows:





Fourier Cosine Series



Fourier Sine Series

Example 6. $f(t) = \begin{cases} \pi, & 0 < t < \pi/2 \\ 0, & \pi/2 < t < \pi \end{cases}$.

The Fourier *sine* series for f is found by using

$$a_n = 0,$$
$$b_n = 2 \int_0^{\pi/2} \sin nt \, dt = \frac{2}{n} \left(1 - \cos \frac{n}{2} \right).$$

Therefore

$$b_n = \begin{cases} \frac{2}{n}, & n \text{ odd} \\ 0, & n \text{ a multiple of } 4 \\ \frac{4}{n}, & n \text{ even, not a multiple of } 4, \end{cases}$$

and we obtain the series

$$\sum_{n=1}^{\infty} \frac{2}{2n-1} \sin(2n-1)t + \sum_{n=1}^{\infty} \frac{4}{2(2n-1)} \sin 2(2n-1)t.$$

Such a series represents in $-\pi < t < \pi$ the function which is the *odd* extension of f.

The Fourier *cosine* series for f is found by using

$$b_n = 0, \qquad a_0 = \frac{2}{\pi} \int_0^{\pi/2} \pi \, dt = \pi,$$
$$a_n = 2 \int_0^{\pi/2} \cos nt \, dt = \frac{2}{n} \sin \frac{n\pi}{2} = \begin{cases} 0, & n \text{ even} \\ \frac{2}{n}(-1)^{\frac{n-1}{2}}, & n \text{ odd} \end{cases}$$

and we obtain the series

$$\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{2n-1} \cos(2n-1)t.$$

This represents the function which is the *even* extension of f. Both series represent f in the interval $0 < t < \pi$ (only!).

IV.c *Extension of the Interval*

So far, the Fourier series we have obtained are representatives for $f(t)$ for $-\pi < t < \pi$. We now consider the problem of representing f in the interval $-T < t < T$ where T denotes an arbitrary positive number. Let $f(t)$ be a given function in the interval $-T < t < T$. Set $g(w) = f(Tw/\pi)$ where w ranges over the interval $-\pi < w < \pi$. We can write a Fourier series for g as follows:

$$f\left(\frac{Tw}{\pi}\right) = g(w) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nw + b_n \sin nw$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(w)\,dw \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(w)\cos nw\,dw, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(w)\sin nw\,dw. \end{aligned}$$

Let $t = Tw/\pi$ (note that as w ranges over $-\pi < w < \pi$, t ranges over $-T < t < T$). The above expression becomes:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{T} + b_n \sin\frac{n\pi t}{T}$$

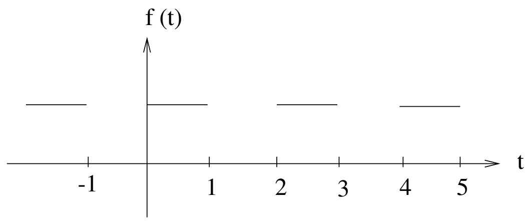
where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(w)\,dw = \frac{1}{T}\int_{-T}^{T} f(t)\,dt \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(w)\cos nw\,dw = \frac{1}{T}\int_{-T}^{T} f(t)\cos\frac{n\pi t}{T}\,dt \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(w)\sin nw\,dw = \frac{1}{T}\int_{-T}^{T} f(t)\sin\frac{n\pi t}{T}\,dt. \end{aligned}$$

The series we have obtained represents f in the interval $-T < t < T$. Outside this interval this series *repeats* and hence it will *not* represent f *unless* f *is periodic with period* $2T$.

Important Remark. The period is $2T$, *not* T.

Example 1. $f(t) = \begin{cases} 1, & 2n < t < 2n+1, \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots \\ 0, & \text{otherwise} \end{cases}$. Find the Fourier series for f.



Answer. Note that $f(t)$ has period 2. If we expand f in terms of $\sin nt$, $\cos nt$, then we obtain a representation for f valid *only* between $-\pi$ and π. We instead expand f in terms of $\sin n\pi t$ and $\cos n\pi t$ ($T = 1$ in this case) to get a representation for f good "almost" everywhere (i.e., except at the jumps!). The coefficients are as follows:

$$a_0 = \frac{1}{1}\int_{-T}^{T} f(t)\,dt = \int_0^1 1\,dt = 1,$$

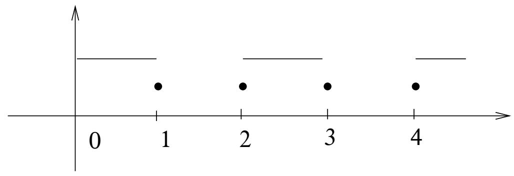
$$a_n = \frac{1}{1}\int_0^1 \cos n\pi t\,dt = \left.\frac{\sin n\pi t}{n\pi}\right|_0^1 = 0,$$

$$b_n = \frac{1}{1}\int_0^1 \sin n\pi t\,dt = -\left.\frac{\cos n\pi t}{n\pi}\right|_0^1 = \frac{1}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 0, & n \text{ even} \\ \dfrac{2}{n\pi}, & n \text{ odd,} \end{cases}$$

and thus,

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin(2n-1)\pi t.$$

The graph of the sum of the Fourier series is:



Remark. If $f(t)$ is *odd* in the interval $-T < t < T$, then the Fourier series has sine terms *only*, i.e.,

$$f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{T}$$

where

$$b_n = \frac{2}{T} \int_0^T f(t) \sin \frac{n\pi t}{T}\, dt.$$

Analogously if $f(t)$ is *even*, the Fourier series has cosine terms only, i.e.,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{T}$$

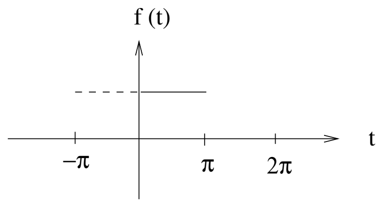
where

$$a_0 = \frac{2}{T} \int_0^T f(t)\, dt, \qquad a_n = \frac{2}{T} \int_0^T f(t) \cos \frac{n\pi t}{T}\, dt.$$

If $f(t)$ is defined only on $0 < t < T$ or if we are interested in representing f only on this interval, we can extend f to the whole of $-T < t < T$ either as an *even* function (and obtain a cosine series for f) or as an *odd* function (and obtain a sine series for f).

Example 2. Find the cosine series for $f(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$.

Answer. We extend f as an *even* function, and we note that $T = 2\pi$ and $b_n = 0$.



$$a_0 = \frac{1}{\pi}\int_0^{\pi} 1\,dt = 1$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos\frac{nt}{2}\,dt = \frac{1\cdot 2}{n\pi}\sin\frac{nt}{2}\Big|_0^{\pi} = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$= \begin{cases} 0, & n \text{ even} \\ \dfrac{2}{n\pi}, & n = 1, 5, 9, 13 \\ \dfrac{-2}{n\pi}, & n = 3, 7, 11 \end{cases} = \begin{cases} 0, & n \text{ even} \\ \dfrac{2}{n\pi}(-1)^{\frac{n-1}{2}}, & n \text{ odd.} \end{cases}$$

Thus,

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi}(-1)^{n-1}\cos\frac{(2n-1)t}{2}.$$

Note that this function represents f in the interval $0 < t < 2\pi$ only.

Example 3. Let $f(t) = 1 - t$ for $0 < t < 1$. Find both a sine series and a cosine series for f.

Answer. To find the sine series, we extend f as an *odd* function. Then,

$$a_n = 0,$$

$$b_n = \frac{2}{1}\int_0^1 (1-t)\sin n\pi t\,dt$$

$$= -2(1-t)\frac{\cos n\pi t}{n\pi}\Big|_0^1 + 2\int_0^1 \frac{\cos n\pi t}{n\pi}(-1)\,dt = \frac{2}{n\pi}.$$

Thus we have the series

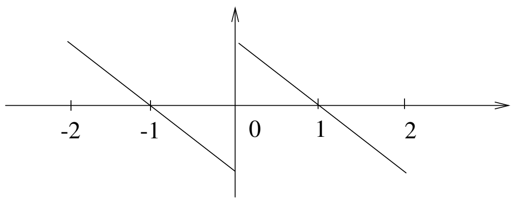
$$\sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi t.$$

To find the cosine series we extend f as an even function. Then

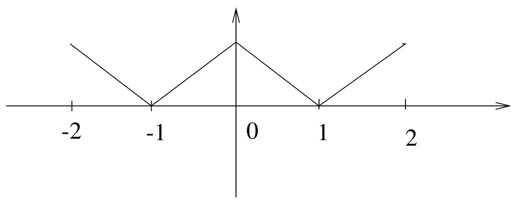
$$\begin{aligned} b_n &= 0, \\ a_0 &= \frac{2}{1} \int_0^1 (1-t)\,dt = 1, \\ a_n &= \frac{2}{1} \int_0^1 (1-t) \cos n\pi t\,dt = \frac{2}{n\pi} \int_0^1 \sin n\pi t\,dt \\ &= \frac{2}{n\pi} \left[\frac{-\cos n\pi t}{n\pi} \right]_0^1 = \frac{2}{(n\pi)^2} \left[1 - (-1)^n \right]. \end{aligned}$$

We have the series:

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{[(2n-1)\pi]^2} \cos(2n-1)\pi t.$$



Sine series



Cosine series

Further Exercises:

Find the Fourier series for the following function. Sketch the sum of the series you obtain.

Exercise 1. $f(t) = \begin{cases} 0, & -\infty < t < 0 \\ t, & 0 \le t < \infty \end{cases}.$

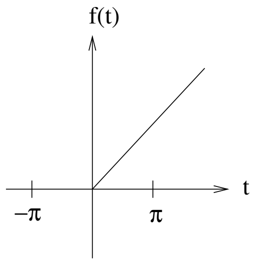
Answer. We calculate the coefficients:

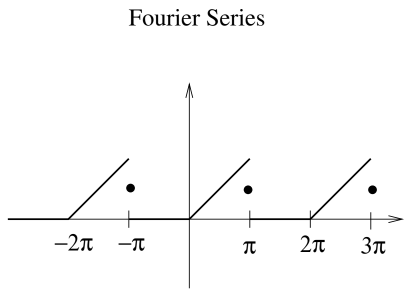
$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_0^{\pi} t\sin nt\,dt \\
&= \frac{1}{\pi}\left[\left.\frac{-t\cos nt}{n}\right|_0^{\pi} + \int_0^{\pi}\frac{\cos nt}{n}\,dt\right] = \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] \\
&= -\frac{\cos n\pi}{n} = \frac{(-1)^{n+1}}{n}, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_0^{\pi} t\cos nt\,dt \\
&= \frac{1}{\pi}\left[\left.\frac{t\sin nt}{n}\right|_0^{\pi} - \int_0^{\pi}\frac{\sin nt}{n}\,dt\right] \\
&= \frac{1}{\pi}\left.\frac{\cos nt}{n^2}\right|_0^{\pi} = \frac{1}{\pi n^2}(\cos n\pi - 1) = \frac{1}{\pi n^2}\cdot\begin{cases} 0, & n \text{ even} \\ -2, & n \text{ odd} \end{cases}, \\
a_0 &= \frac{1}{\pi}\int_0^{\pi} t\,dt = \frac{\pi}{2}.
\end{aligned}$$

We conclude that for $-\pi \le t \le \pi$,

$$f(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt + \sum_{n=1}^{\infty}\frac{(-2)}{\pi(2n-1)^2}\cos(2n-1)t.$$

The graphs are as follows:





Note that outside of $-\pi \le t \le \pi$, the series is very different from $f(t)$.

Exercise 2. $f(t) = \begin{cases} 0, & -\pi < t < 0 \\ t, & 0 \le t < \pi \end{cases}$, $f(t)$ periodic outside the interval $-\pi < t < \pi$.

Answer. In this case, f is periodic *outside* $-\pi \le t \le \pi$ and the same as the previous f *inside* $-\pi \le t \le \pi$. The Fourier series is thus identical in this case to what it was for Exercise 1. The only difference is that now *outside* of $-\pi \le t \le \pi$, the series *still agrees with* f (except at the jumps).

Exercise 3. $f(t) = \begin{cases} 1, & -\pi < t < 0 \\ -1, & 0 < t < \pi \end{cases}$, $f(t)$ periodic outside the interval $-\pi < t < \pi$.

Answer. Since $f(t) = \begin{cases} 1, & -\pi < t < 0 \\ -1, & 0 < t < \pi \end{cases}$, we note that it is *odd.* This saves having to calculate a_n, since all $a_n = 0$. If you do not notice this, you should still obtain all $a_n = 0$ by actual calculation.

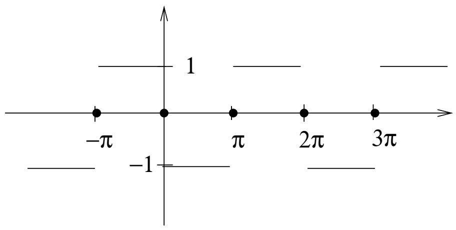
Furthermore,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\int_0^{\pi}(-1)\sin nt\,dt \\ &= \frac{2}{\pi}\left[\frac{\cos nt}{n}\right]_0^{\pi} = \frac{2}{n\pi}(\cos n\pi - 1) = \frac{2}{n\pi}\cdot\begin{cases} 0, & n \text{ even} \\ -2, & n \text{ odd} \end{cases}. \end{aligned}$$

Thus

$$f(t) = \sum_{n=1}^{\infty} \frac{-4}{(2n-1)\pi} \sin[(2n-1)t]$$

and the sum of the series has a graph:



Exercise 4. $f(t) = \begin{cases} 0, & -\pi < t < -\pi/2 \\ -1, & -\pi/2 \le t \le \pi/2 \\ 0, & \pi/2 < t < \pi \end{cases}$, $f(t)$ periodic outside $-\pi < t < \pi$.

Answer. In this case, $f(t)$ is *even*. Now all $b_n = 0$ (as can also be seen by direct calculation) and

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt \, dt = -\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos nt \, dt$$

$$= -\frac{2}{\pi} \left[\frac{\sin nt}{n} \right]_0^{\frac{\pi}{2}} = -\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

Now

$\sin(n\pi/2)$	1	0	−1	0	1	0	−1
n	1	2	3	4	5	6	7

Thus

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n \text{ even}, \\ -(-1)^{\frac{n+1}{2}}, & n \text{ odd}, \end{cases}$$

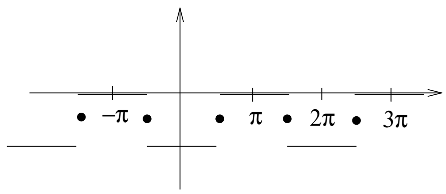
and

$$a_0 = \frac{-2}{\pi} \int_0^{\frac{\pi}{2}} 1 \, dt = -1.$$

Thus

$$f(t) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{(2n-1)\pi} \cos[(2n-1)t]$$

and the sum of the series has a graph given by



Exercise 5. Find the Fourier sine series and the Fourier cosine series for f if $f(t) = 1 - t$, $0 \leq t \leq \pi$. Sketch the sum of the two series.

Answer. We first find the Fourier sine series. Now, in this case $a_n = 0$,

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt \, dt = \frac{2}{\pi} \int_0^{\pi} (1-t) \sin nt \, dt \\ &= \frac{2}{\pi} \left[\frac{-(1-t)\cos nt}{n} \bigg|_0^{\pi} + \int_0^{\pi} (-1) \frac{\cos nt}{n} \, dt \right] \\ &= \frac{2}{\pi n} (1 - (1-\pi)\cos n\pi) = \frac{2}{n\pi} \left[1 + (\pi - 1)(-1)^n \right]. \end{aligned}$$

So the series is

$$\sum_{n=1}^{\infty} \frac{2}{n\pi} \left[1 + (\pi - 1)(-1)^n \right] \sin nt.$$

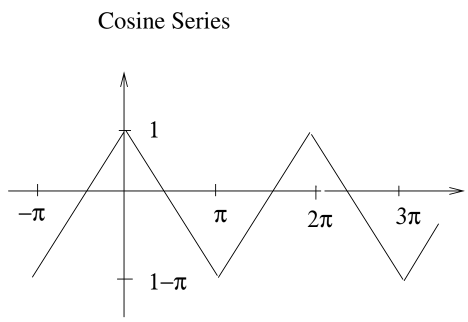
Next for the Fourier cosine series: now $b_n = 0$ and

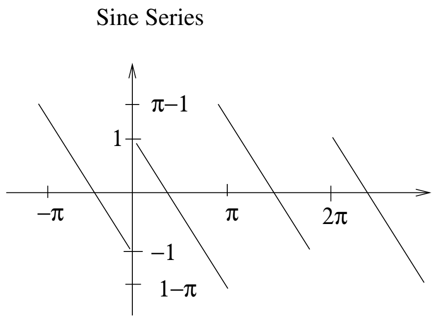
$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} (1-t) \cos nt \, dt = \frac{2}{\pi} \left[\frac{(1-t)\sin nt}{n} \bigg|_0^{\pi} - \int_0^{\pi} (-1) \frac{\sin nt}{n} \, dt \right] \\ &= \frac{2}{n\pi} \int_0^{\pi} \sin nt \, dt = \frac{2}{n\pi} \left[-\frac{\cos nt}{n} \right]_0^{\pi} = \frac{2}{n^2\pi} \left[1 - \cos n\pi \right], \\ a_0 &= \frac{2}{\pi} \int_0^{\pi} (1-t) \, dt = \frac{2}{\pi} \left(\pi - \frac{\pi^2}{2} \right) = 2 \left(1 - \frac{\pi}{2} \right), \end{aligned}$$

and the series is:

$$\left(1-\frac{\pi}{2}\right)+\sum_{n=1}^{\infty}\frac{4}{\pi(2n-1)^2}\cos[(2n-1)t].$$

Graphs:



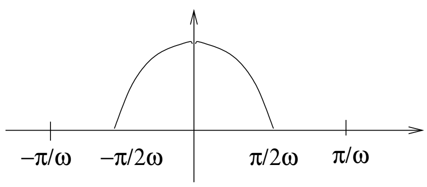


Exercise 6. Find a Fourier series valid for the rectified cosine function:

$$f(t)=\begin{cases}0, & -\frac{\pi}{\omega}<t\le-\frac{\pi}{2\omega}\\ \cos(\omega t), & -\frac{\pi}{2\omega}<t<\frac{\pi}{2\omega}\\ 0, & \frac{\pi}{2\omega}\le t<\frac{\pi}{\omega}\end{cases}, \quad \text{for arbitrary frequency } \omega>0.$$

Answer. We observe two things: first f is *even* and, secondly, the period is $2\pi/\omega$ so that $T=\pi/\omega$ and we write a Fourier series in the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi t}{\pi/\omega}\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\omega t)$$



and

$$a_n = \frac{2}{\frac{\pi}{\omega}} \int_0^{\frac{\pi}{\omega}} f(t)\cos(n\omega t)\,dt = \frac{2\omega}{\pi}\int_0^{\frac{\pi}{2\omega}} \cos(\omega t)\cos(n\omega t)\,dt$$

$$= \frac{2\omega}{\pi}\int_0^{\frac{\pi}{2\omega}} \frac{1}{2}\left[\cos(\omega t + n\omega t) + \cos(n\omega t - \omega t)\right]\,dt$$

$$= \frac{2\omega}{\pi}\int_0^{\frac{\pi}{2\omega}} \left[\cos((n+1)\omega t) + \cos((n-1)\omega t)\right]\,dt$$

$$= \frac{\omega}{\pi}\left[\frac{\sin((n+1)\omega t)}{(n+1)\omega} + \frac{\sin((n-1)\omega t)}{(n-1)\omega}\right]_0^{\frac{\pi}{2\omega}}$$

(note that a_1 will need to be calculated separately!)

$$= \frac{\omega}{\pi}\left[\frac{\sin((n+1)\frac{\pi}{2})}{(n+1)\omega} + \frac{\sin((n-1)\frac{\pi}{2})}{(n-1)\omega}\right].$$

Note

$$\sin\left((n-1)\frac{\pi}{2} + \pi\right) = -\sin\left((n-1)\frac{\pi}{2}\right) = \sin\left((n+1)\frac{\pi}{2}\right).$$

And so

$$a_n = \frac{1}{\pi}\left[\frac{-\sin((n-1)\frac{\pi}{2})}{n+1} + \frac{\sin((n-1)\frac{\pi}{2})}{n-1}\right] = \frac{1}{\pi}\frac{2}{(n^2-1)}\sin\left((n-1)\frac{\pi}{2}\right).$$

But

$\sin((n-1)\pi/2)$	1	0	-1	0	1
n	2	3	4	5	6

So

$$\sin\left((n-1)\frac{\pi}{2}\right) = \begin{cases} 0, & n \text{ odd} \\ (-1)^{\frac{n+2}{2}}, & n \text{ even} \end{cases}$$

and thus

$$a_n = \frac{2}{\pi}\frac{1}{(n^2-1)} \cdot \begin{cases} 0, & n \text{ odd}, \\ (-1)^{\frac{n+2}{2}}, & n \text{ even} \end{cases}.$$

Finally,

$$a_0 = \frac{2\omega}{\pi}\int_0^{\frac{\pi}{2\omega}} \cos\omega t\,dt = \frac{2\omega}{\pi}\left[\frac{\sin\omega t}{\omega}\right]_0^{\frac{\pi}{2\omega}} = \frac{2}{\pi}\sin\left(\frac{\pi}{2}\right) = \frac{2}{\pi},$$

$$a_1 = \frac{2\omega}{\pi}\int_0^{\frac{\pi}{2\omega}} \cos^2\omega t\,dt = \frac{2\omega}{\pi}\int_0^{\frac{\pi}{2\omega}} \frac{1-\cos(2\omega t)}{2}\,dt = \frac{2\omega}{\pi}\cdot\frac{1}{2}\cdot\frac{\pi}{2\omega} = \frac{1}{2}.$$

Thus,

$$f = \frac{1}{\pi} + \frac{1}{2}\cos\omega t + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{\pi(4n^2-1)}\cos(2n\omega t).$$

IV.d *Complex Fourier Series and Fourier Transform*

The introduction of complex numbers allows a different formulation of the Fourier series. From the earlier sections we know that in the interval $-\pi < t < \pi$, we have the representation

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

with

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx. \end{aligned}$$

We have changed the variable from t to x in the integrals for convenience. Recalling once again the identities:

$$\cos nt = \frac{e^{jnt} + e^{-jnt}}{2}, \qquad \sin nt = \frac{e^{jnt} - e^{-jnt}}{2j},$$

we can write

$$\begin{aligned} a_n \cos nt + b_n \sin nt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \Bigg[& \left(\frac{e^{jnx} + e^{-jnx}}{2} \right) \left(\frac{e^{jnt} + e^{-jnt}}{2} \right) \\ & + \left(\frac{e^{jnx} - e^{-jnx}}{2j} \right) \left(\frac{e^{jnt} - e^{-jnt}}{2j} \right) \Bigg] dx \\ = \frac{1}{\pi} \int_{-\pi}^{\pi} & \frac{f(x)}{4} \left[2e^{jn(x-t)} + 2e^{-jn(x-t)} \right] dx \\ = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \right. & \left. \frac{f(x)}{2} e^{jnx} \, dx \right) e^{j(-nt)} + \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{2} e^{-jnx} \, dx \right) e^{jnt}. \end{aligned}$$

Therefore,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$= \frac{a_0}{2} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} f(x)e^{-jnx}\,dx\right)e^{jnt} + \frac{1}{2\pi}\sum_{n=-\infty}^{-1}\left(\int_{-\pi}^{\pi} f(x)e^{-jnx}\,dx\right)e^{jnt}.$$

Writing a_0 in terms of the integral of f we are lead to the *complex Fourier series* for f:

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{jnt} \qquad \text{where} \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-jnt}\,dt.$$

This series represents f in the interval $-\pi < t < \pi$. If we had started with the Fourier series for f which is valid for $-T < t < T$ we would have obtained the complex series:

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{\frac{jn\pi t}{T}} \qquad \text{where} \qquad c_n = \frac{1}{2T}\int_{-T}^{T} f(t)e^{\frac{-jn\pi t}{T}}\,dt.$$

This latter series, of course, furnishes a representation for f valid in the interval $-T < t < T$. We can rewrite this series in the form:

$$f(t) = \sum_{-\infty}^{\infty}\left(\frac{1}{2T}\int_{-T}^{T} f(x)e^{\frac{-jn\pi x}{T}}\,dx\right)e^{\frac{jn\pi t}{T}}$$

since the dummy variable of integration can be changed from t to x for convenience.

Remark. Note that c_n is *complex* in general! It looks like we have to calculate c_n for all the positive n's and then for all the negative n's. But, if $f(t)$ is real (the usual situation) then

$$c_{-n} = \frac{1}{2T}\int_{-T}^{T} f(t)e^{\frac{jn\pi t}{T}}\,dt.$$

So

$$\begin{aligned}\overline{c}_{-n} =& \overline{\frac{1}{2\pi}\int_{-T}^{T} f(t)e^{\frac{jn\pi t}{T}}\,dt} \quad \text{(complex conjugate)}\\ &= \frac{1}{2\pi}\int_{-T}^{T} \overline{f(t)e^{\frac{jn\pi t}{T}}}\,dt = \frac{1}{2\pi}\int_{-T}^{T} f(t)e^{\frac{-jn\pi t}{T}}\,dt = c_n\end{aligned}$$

and we conclude

$$\overline{c}_{-n} = c_n.$$

From this remark, we obtain *Parseval's Theorem.* Let $f(t)$, $g(t)$ be two functions given on $-T < t < T$. Suppose

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{\frac{jn\pi t}{T}}, \qquad g(t) = \sum_{-\infty}^{\infty} d_n e^{\frac{jn\pi t}{T}}.$$

We wish to calculate

$$\begin{aligned}
\int_{-T}^{T} f(t)g(t)\,dt &= \int_{-T}^{T} \left(\sum_{-\infty}^{\infty} c_n e^{\frac{jn\pi t}{T}}\right) g(t)\,dt \\
&= \sum_{-\infty}^{\infty} c_n \int_{-T}^{T} e^{\frac{jn\pi t}{T}} g(t)\,dt \\
&= 2T \sum_{n=-\infty}^{\infty} c_n d_{-n} = 2T \sum_{n=-\infty}^{\infty} c_n \overline{d}_n.
\end{aligned}$$

So (this is the *Theorem*):

$$\frac{1}{2T}\int_{-T}^{T} f(t)g(t)\,dt = \sum_{n=-\infty}^{\infty} c_n \overline{d}_n$$

and, as a consequence,

$$\frac{1}{2T}\int_{-T}^{T} f^2(t)\,dt = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

In particular, the RMS (Root Mean Square) value of f is

$$f_{\mathrm{RMS}} = \sqrt{\frac{1}{2T}\int_{-T}^{T} f^2(t)\,dt} = \sqrt{\sum_{n=-\infty}^{\infty} |c_n|^2}.$$

There is an analogue of this result valid for the ordinary Fourier series. Suppose

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi t}{T} + b_n \sin\frac{n\pi t}{T}\right],$$

$$g(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\alpha_n \cos \frac{n\pi t}{T} + \beta_n \sin \frac{n\pi t}{T} \right].$$

Then

$$\begin{aligned}
\int_{-T}^{T} f(t)g(t)\,dt &= \int_{-T}^{T} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\pi t}{T} + b_n \sin \frac{n\pi t}{T} \right\} \right] g(t)\,dt \\
&= \frac{a_0}{2} \int_{-T}^{T} g(t)dt + \sum_{n=1}^{\infty} \left\{ a_n \left[\int_{-T}^{T} \cos \frac{n\pi t}{T}\, g(t)\,dt \right] + b_n \left[\int_{-T}^{T} \sin \frac{n\pi t}{T}\, g(t)\,dt \right] \right\} \\
&= T \left[\frac{\alpha_0 \alpha_0}{2} + \sum_{n=1}^{\infty} [a_n \alpha_n + b_n \beta_n] \right]
\end{aligned}$$

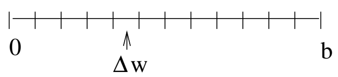
or

$$\frac{1}{2T} \int_{-T}^{T} f(t)g(t)\,dt = \frac{a_0 \alpha_0}{4} + \frac{1}{2} \sum_{n=1}^{\infty} \{a_n \alpha_n + b_n \beta_n\}$$

and

$$\frac{1}{2T} \int_{-T}^{T} f^2(t)\,dt = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} \{a_n^2 + b_n^2\}.$$

As stated above, this series represents f *only in the interval* $-T < t < T$ (*unless* f *is periodic*). In an attempt to obtain a representation for f which is valid for *all* t even if f is not periodic, we let $T \to \infty$ (as follows). Recall first of all that while $\int_0^b g(w)dw$—for some function g—is evaluated by means of antiderivatives, the *definition* of $\int_0^b g(w)dw$ (and thus its physical significance) is given by: divide the interval $0 < w < b$ into N steps of equal size Δw, so that $\Delta w = b/N$, and put $w_n = nb/N = n\Delta w$ for $n = 1, 2, \ldots, N$.



From the sum $\sum_{i=1}^{N} g(w_n)\Delta w$ and recall

$$\int_0^b g(w)\,dw = \lim_{N\to\infty}\left(\sum_{n=1}^{N} g(w_n)\Delta w\right) = \lim_{\Delta w\to 0}\left(\sum_{n=1}^{N} g(n\Delta w)\Delta w\right).$$

Finally,

$$\int_{-b}^{b} g(w)\,dw = \int_{-b}^{0} g(w)\,dw + \int_0^b g(w)\,dw = \lim_{\Delta w\to 0}\left(\sum_{-N}^{N} g(n\Delta w)\Delta w\right).$$

Set $\pi/T = \Delta w$, and note that from the above series we obtain

$$\begin{aligned} f(t) &= \sum_{-\infty}^{\infty} \frac{1}{2T}\int_{-T}^{T} f(x)e^{\frac{jn\pi}{T}(t-x)}\,dx = \sum_{-\infty}^{\infty} \frac{\Delta w}{2\pi}\int_{-T}^{T} f(x)e^{jn\Delta w(t-x)}\,dx \\ &= \sum_{-\infty}^{\infty}\left[\int_{-T}^{T} f(x)e^{jn\Delta w(x-t)}\,dx\right]\frac{\Delta w}{2\pi}. \end{aligned}$$

Note that $\lim_{T\to\infty}(\Delta w) = 0$. If we put

$$g(w) = \frac{1}{2\pi}\int_{-T}^{T} f(x)e^{jw(x-t)}\,dx,$$

then

$$f(t) = \sum_{-\infty}^{\infty} g(n\Delta w)\Delta w$$

and the last expression we have obtained is, by definition, “close” to

$$\int_{-\infty}^{\infty} \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} f(x)e^{jw(t-x)}\,dx\right]dw$$

for T sufficiently large. (Here we have replaced $n\Delta w$ by w and T by ∞.) We thus are lead to the representation

$$f(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} f(x)e^{jw(t-x)}\,dx\right]dw$$

valid for all t. That is, we write:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{jwt}\, dw$$

where

$$g(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-jwx} f(x)\, dx.$$

This is indeed a valid representation for the $f(t)$ usually found in practice. The function g is called the *Fourier transform* of f.

Remarks. (1) We have split 2π as the product of $\sqrt{2\pi} \cdot \sqrt{2\pi}$. This is *not* always done so that sometimes $g(w) = \int_{-\infty}^{\infty} e^{-jwx} f(x)\, dx$, and then $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(w) e^{jwt}\, dt$.

(2) One of the important feature of a transform is that a derivative becomes a product!

(3) Remember that for Laplace transforms, we need $f(t) = 0$ if $t < 0$. This is not the case for Fourier transforms.

We now pass to examples.

Example 1. Find the *complex* Fourier series for f if $f(t) = e^t$ for $-\pi < t < \pi$.

Answer.

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-jnt}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^t e^{-jnt}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-jn)t}\, dt \\ &= \left. \frac{e^{(1-jn)t}}{2\pi(1-jn)} \right|_{-\pi}^{\pi} = \frac{e^{\pi} e^{-jn\pi}}{2\pi(1-jn)} - \frac{e^{-\pi} e^{jn\pi}}{2\pi(1-jn)}. \end{aligned}$$

But

$$e^{\pm jn\pi} = \cos(\pm n\pi) + j \sin(\pm n\pi) = \cos(\pm n\pi) = (-1)^n.$$

Therefore,

$$c_n = \frac{(e^{\pi} - e^{-\pi})(-1)^n}{2\pi(1 - jn)}$$

and

$$f(t) = \sum_{-\infty}^{\infty} \left(\frac{e^{\pi} - e^{-\pi}}{2\pi} \right) \frac{(-1)^n}{(1 - nj)} e^{jnt}.$$

Example 2. Find the Fourier transform for f if

$$f(t) = \begin{cases} 1, & \text{for } 0 < t < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Answer.

$$\begin{aligned} g(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-jwt} f(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-jwt} \cdot 1\, dt \\ &= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-jwt}}{-jw} \right|_0^1 = \frac{1}{j\sqrt{2\pi}\, w} \left[1 - e^{-jw} \right], \end{aligned}$$

if $w \neq 0$. If $w = 0$,

$$g(0) = \frac{1}{\sqrt{2\pi}} \int_0^1 1\, dt = \frac{1}{\sqrt{2\pi}}.$$

Example 3. Let $f(t) = t$, $g(t) = |t|$ on $-\pi < t < \pi$. Use the Fourier series to calculate $\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(t) dt$.

Answer.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(t)\, dt = \frac{a_0}{2} \cdot \frac{\alpha_0}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \{ a_n \alpha_n + b_n \beta_n \}$$

where

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{ a_n \cos nx + b_n \sin nx \}$$

$$g = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}\{\alpha_n \cos nx + \beta_n \sin nx\}.$$

So we only need to calculate the Fourier series. Now t is odd, so $a_n = 0$,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} t \sin nt\, dt = \frac{2}{\pi}\left[\left.\frac{-t\cos nt}{n}\right|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nt\, dt\right] \\ &= \frac{2(-\pi)(-1)^n}{\pi n} = \frac{2}{n}(-1)^{n+1}, \end{aligned}$$

i.e.,

$$t = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nt.$$

While $|t|$ is even and we have $b_n = 0$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi}|t|\cos nt\, dt = \frac{2}{\pi}\int_0^{\pi} t\cos nt\, dt \\ &= \frac{2}{\pi}\left[\left.\frac{t\sin nt}{n}\right|_0^{\pi} - \int_0^{\pi}\frac{\sin nt}{n}\, dt\right] \\ &= \frac{2}{\pi}\left.\frac{\cos nt}{n^2}\right|_0^{\pi} = \frac{2}{\pi n^2}[(-1)^n - 1] = \frac{2}{\pi n^2}\cdot\begin{cases} 0, & n \text{ even} \\ -2, & n \text{ odd}, \end{cases} \end{aligned}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}|t|\, dt = \frac{2}{\pi}\int_0^{\pi} t\, dt = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi.$$

So

$$|t| = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}[(-1)^n - 1]\cos nt$$

and thus

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(t)\, dt = \frac{1}{4}\underbrace{0}_{a_0}\cdot\underbrace{\frac{\pi}{2}}_{\alpha_0} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\underbrace{0}_{a_n}\cdot\underbrace{\frac{2}{\pi n^2}[(-1)^n - 1]}_{\alpha_n} + \underbrace{\frac{2}{n}(-1)^{n+1}}_{b_n}\cdot\underbrace{0}_{\beta_n}\right) = 0.$$

But we know this is obvious: t is odd, $|t|$ is even, so $t|\,t\,|$ is odd and $\int_{-\pi}^{\pi} t\,|t|\, dt = 0$.

IV.e *Differentiation and Integration of Fourier Series*

Suppose

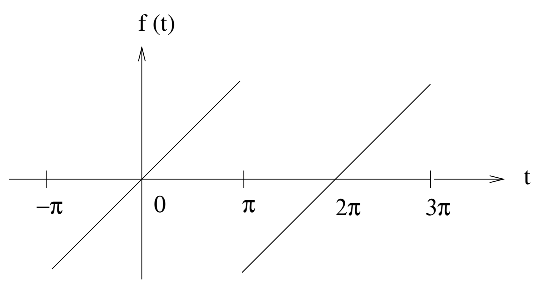
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\pi t}{T} + b_n \sin \frac{n\pi t}{T} \right\}$$

with a_0, a_n, b_n as given earlier, for $-T < t < T$, where f is a periodic function of period $2T$. For functions encountered in practice, we can integrate this expression term by term,

$$\int_{t_0}^{t_1} f(t)\,dt = \frac{a_0}{2}(t_1 - t_0) + \sum_{n=1}^{\infty} \left\{ \frac{a_n \sin \frac{n\pi t}{T}}{\frac{n\pi}{T}} - b_n \frac{\cos \frac{n\pi t}{T}}{\frac{n\pi}{T}} \right\} \Bigg|_{t=t_0}^{t_1}.$$

The situation, as far as differentiation is concerned, is somewhat more complicated. We can always differentiate the series term by term, but remember that we can write Fourier series even for functions with jumps (i.e., which do not have a derivative everywhere). So term by term differentiation gives $f'(t)$ only if (1) $f(t)$ is continuous (i.e., *no jumps anywhere*, even at $t = \pm T$!) and (2) $f'(t)$ also has a Fourier series. Condition (2) is no problem in practice, but condition (1) can be, even away from the jumps (if any)! To see this, consider a standard example:

Example 1. $f(t) = t$ for $-\pi < t < \pi$, $f(t)$ is periodic with period 2π. The graph of $f(t)$ is as shown, so that $f(t)$ is *not* continuous: there are jumps at $t = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$.



Now the Fourier series for f is

$$f(t) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nt}{n}.$$

In particular for any t between $-\pi$ and π we have

$$t = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nt}{n}.$$

If we differentiate term by term and this was indeed the derivative of t, then we would obtain

$$1 = 2\sum_{n=1}^{\infty}(-1)^{n+1}\cos nt$$

by differentiating both sides. This would be true in particular at $t = 0$, but $\cos 0 = 1$ so

$$1 = 2\sum_{n=1}^{\infty}(-1)^{n+1}\ !$$

But this is impossible, since

$$2\left(\sum_{n=1}^{\infty}(-1)^{n+1}\right) = 2(1 - 1 + 1 - 1 + 1 - \cdots)$$

oscillates and does *not* converge to 1 or anything else! Observe that here $f'(t) = 1$ for $-\pi < t < \pi$, but f' does *not* exist at $\pm\pi$. Condition (1) is thus important and if (1) and (2) are satisfied, then

$$f'(t) = \sum_{n=1}^{\infty}\left\{a_n\frac{n\pi}{T}\left(-\sin\frac{n\pi t}{T}\right) + b_n\frac{n\pi}{T}\cos\frac{n\pi t}{T}\right\}.$$

Example 2. Let f be as in Example 1. Find $\int_{\alpha}^{t} f(s)\,ds$, for $-\pi < \alpha < t < \pi$.

Answer.

$$f(t) = t = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nt}{n}.$$

So

$$\begin{aligned}
\int_{\alpha}^{t} f(s)\,ds &= \sum_{n=1}^{\infty}\int_{\alpha}^{t} 2(-1)^{n+1}\frac{\sin ns}{n}\,ds \\
&= \sum_{n=1}^{\infty} 2(-1)^{n+1}\left[-\frac{\cos ns}{n^2}\right]_{\alpha}^{t} \\
&= \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n^2}\left[\cos n\alpha - \cos nt\right].
\end{aligned}$$

On the other hand,

$$\int_{\alpha}^{t} f(s)\,ds = \int_{\alpha}^{t} s\,ds = \frac{t^2-\alpha^2}{2}.$$

So we conclude

$$\frac{t^2-\alpha^2}{2} = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n^2}[\cos n\alpha - \cos nt].$$

We can rewrite this expression as:

$$\frac{t^2}{2} = \frac{\alpha^2}{2} + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n^2}\cos n\alpha - \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n^2}\cos nt$$

or

$$\frac{t^2}{2} = C - \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n^2}\cos nt$$

where

$$C = \frac{\alpha^2}{2} + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n^2}\cos n\alpha.$$

We can look at the right hand side as the Fourier series of $t^2/2$ over $-\pi < t < \pi$. So if we wish we can find C as the "$a_0/2$" of the expansion for the function $g(t) = t^2/2$, i.e.,

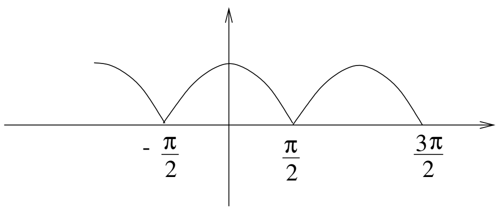
$$C = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{t^2}{2}\,dt.$$

Example 3. Find the Fourier series for the derivative of the rectified cosine function:

$$f(t) = \cos t, \qquad \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2}; \qquad f(t) = f(t+\pi),$$

by differentiating term by term.

Answer. Observe that $f(t)$ has no jumps.



We may therefore obtain the Fourier series for $f(t)$ and differentiate term by term to get the series for $f'(t)$. Now $f(t)$ is even, so $b_n = 0$ and

$$\begin{aligned}
a_n &= \frac{1}{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cos 2nt\, dt \\
&= \frac{2}{\pi}\int_{-\pi 2}^{\frac{\pi}{2}} \left[\frac{e^{jt}+e^{-jt}}{2}\right]\left[\frac{e^{j2nt}+e^{-j2nt}}{2}\right] dt \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[e^{j(2n+1)t} + e^{-j(2n-1)t} + e^{j(2n-1)t} + e^{-j(2n+1)t}\right] dt \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\cos(2n+1)t + \cos(2n-1)t\right] dt \\
&= \frac{1}{\pi}\left[\frac{\sin(2n+1)t}{2n+1} + \frac{\sin(2n-1)t}{2n-1}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}
\end{aligned}$$

$$= \frac{2}{\pi}\left[\frac{\sin(2n+1)\frac{\pi}{2}}{2n+1} + \frac{\sin(2n-1)\frac{\pi}{2}}{2n-1}\right].$$

Observe that

n	1	2	3	4	5
$\sin(2n+1)\frac{\pi}{2}$	-1	1	-1	1	$\cdots$
$\sin(2n-1)\frac{\pi}{2}$	1	-1	1	-1	$\cdots$

So

$$a_n = \frac{2}{\pi}\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right](-1)^n = \frac{2(-1)^n}{\pi}\,\frac{(-2)}{(4n^2-1)}.$$

I.e.,

$$a_n = \frac{4(-1)^{n+1}}{\pi(4n^2-1)}.$$

Also

$$a_0 = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\, dt = \frac{2}{\pi}\sin t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{4}{\pi}$$

and

$$f = \frac{2}{\pi} + \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{\pi(4n^2-1)}\cos 2nt.$$

Thus

$$f'(t) = \sum_{n=1}^{\infty}\frac{4(-1)^n 2n}{\pi(4n^2-1)}\sin 2nt.$$

To practice, we can check this. After all, $f' = -\sin t$ for $-\pi/2 < t < \pi/2$, so is

$$\sin t = -\sum_{n=1}^{\infty}\frac{8(-1)^n n}{\pi(4n^2-1)}\sin 2nt\ ?$$

Now $\sin t$ is odd, and so $a_n = 0$, but

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t \sin 2nt\, dt \\
&= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{e^{jt}-e^{-jt}}{2j}\right]\left[\frac{e^{2njt}-e^{-2njt}}{2j}\right] dt \\
&= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{(2nt+t)j}-e^{-(2nt-t)j}-e^{(2nt-t)j}+e^{-j(2nt+t)}}{4j^2}\, dt \\
&= -\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [2\cos(2nt+t)-2\cos(2nt-t)]dt \\
&= -\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\cos(2n+1)t-\cos(2n-1)t]dt \\
&= -\frac{1}{\pi}\left[\frac{\sin(2n+1)t}{2n+1}-\frac{\sin(2n-1)t}{2n-1}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}.
\end{aligned}$$

We calculated $\sin(2n+1)t$ and $\sin(2n-1)t$ before. So

$$b_n = -\frac{2}{\pi}\left[\frac{1}{2n+1}+\frac{1}{2n-1}\right](-1)^n = \frac{(-1)^{n+1}}{4\pi}\cdot 2\cdot 4\cdot \frac{4n}{(4n^2-1)} = \frac{8n(-1)^{n+1}}{\pi(4n^2-1)}.$$

That is,

$$\sin t = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}8n}{\pi(4n^2-1)} \sin 2nt$$

as expected.

IV.f *Orthogonal Systems and Generalized Fourier Series*

So far we have obtained representations for $-T < t < T$ for "arbitrary" functions $f(t)$ in terms of the trigonometric functions $\sin \frac{n\pi t}{T}$ and $\cos \frac{n\pi t}{T}$, $n = 1, 2, \ldots$. It is often convenient to represent f in terms of *other* functions, particularly for distributed parameter problems, and we now indicate how this can be done. We begin with some theoretical considerations. Let $g_m(t)$, $g_n(t)$ be two functions on the interval $a < t < b$. We define:

$$(g_m, g_n) = \int_a^b g_m(t) g_n(t)\, dt.$$

Example 1. Let $g_m(t) = t$, $g_n(t) = t^2$, $a = 0$, $b = 1$. Then

$$(g_m, g_n) = \int_a^b g_m\, g_n\, dt = \int_0^1 (t)(t^2)\, dt = \frac{1}{4}.$$

Example 2. Let $g_m(t) = \sin t$, $g_n(t) = \cos t$, $a = -\pi$, $b = \pi$. Then

$$(g_m, g_n) = \int_{-\pi}^{\pi} \sin t \cos t\, dt = \left.\frac{\sin^2 t}{2}\right|_{-\pi}^{\pi} = 0.$$

The properties of the operation (,) which we have introduced are analogous to those of the inner (or dot) product of 3-dimensional vectors. You may recall that two vectors are *perpendicular* (or orthogonal) iff their dot product is zero. Analogously we say that a collection of functions $g_1, g_2, g_3, \ldots$ is *orthogonal in the interval* $a < t < b$ iff $(g_m, g_n) = 0$ for any two different functions g_m, g_n. This is simply *terminology by analogy.* It does *not* mean that functions are "arrows," just like vectors!

Example 3. $g_1 = \sin t$, $g_2 = \cos t$, $g_3 = 1$, form an orthogonal set on $-\pi < t < \pi$

since

$$(g_1, g_2) = 0 \qquad \text{(as shown above)},$$
$$(g_1, g_3) = \int_{-\pi}^{\pi} (\sin t)(1)\, dt = -\cos t\Big|_{-\pi}^{\pi} = 0,$$
$$(g_2, g_3) = \int_{-\pi}^{\pi} (\cos t)(1)\, dt = \sin t\Big|_{-\pi}^{\pi} = 0.$$

Example 4. Let g_1, g_2, g_3 be as before. These do *not* form an orthogonal set in the interval $0 < t < \pi/2$ since

$$(g_1, g_3) = \int_0^{\frac{\pi}{2}} \sin t\, dt \neq 0.$$

These two examples indicate that a set of functions may be orthogonal in one interval and not in another.

Given a 3-d vector $\mathbf{a}$ it is known that the *length* (or *magnitude* or *norm*) of $\mathbf{a}$ is given by $\sqrt{\mathbf{a}\cdot\mathbf{a}}$. Analogously we define the *norm* or *magnitude* of the function g_n by its RMS value (except we neglect the length of the interval):

$$\|g_n\| = \sqrt{(g_n, g_n)} = \sqrt{\int_a^b g_n^2(t)\, dt}\,.$$

Example 5. Let $g_1 = 1$, $g_2 = t$, $g_3 = t^2$, $a = 0$, $b = 1$. Then

$$\|g_1\| = \sqrt{\int_0^1 1^2\, dt} = 1,$$
$$\|g_2\| = \sqrt{\int_0^1 t^2\, dt} = \sqrt{\frac{1}{3}},$$
$$\|g_3\| = \sqrt{\int_0^1 (t^2)^2\, dt} = \sqrt{\frac{1}{5}}.$$

Given a set of functions $g_1, g_2, g_3, \ldots,$ we say that it is *orthonormal* iff (1) the functions are orthogonal (i.e., $(g_n, g_m) = 0$ for $n \neq m$), and (2) the norm of each function is 1 (i.e., $(g_n, g_n) = 1$).

Example 6. Consider the functions $g_1 = 1$, $g_2 = \sin t$, $g_3 = \cos t$ in the interval $-\pi < t < \pi$. As noted above, this is an orthogonal set, since $(g_n, g_m) = 0$, $(n \neq m)$. It is *not* an orthonormal set since

$$\|g_1\| = \sqrt{(g_1, g_1)} = \sqrt{\int_{-\pi}^{\pi} 1\, dt} = \sqrt{2\pi} \neq 1,$$

$$\|g_2\| = \sqrt{(g_2, g_2)} = \sqrt{\int_{-\pi}^{\pi} \sin^2 t\, dt} = \sqrt{\pi} \neq 1,$$

$$\|g_3\| = \sqrt{(g_3, g_3)} = \sqrt{\int_{-\pi}^{\pi} \cos^2 t\, dt} = \sqrt{\pi} \neq 1.$$

However, we *can obtain an orthonormal set* from g_1, g_2, g_3 by dividing each function by its norm, *just like we got unit vectors from vectors.* In fact let

$$h_1 = \frac{g_1}{\|g_1\|} = \frac{1}{\sqrt{2\pi}},$$

$$h_2 = \frac{g_2}{\|g_2\|} = \frac{\sin t}{\sqrt{\pi}},$$

$$h_3 = \frac{g_3}{\|g_3\|} = \frac{\cos t}{\sqrt{\pi}}.$$

Then $(h_i, h_j) = 0$ (check this!) and

$$\|h_1\| = \sqrt{(h_1, h_1)} = \sqrt{\int_{-\pi}^{\pi} \frac{1}{2\pi}\, dt} = 1,$$

$$\|h_2\| = \sqrt{(h_2, h_2)} = \sqrt{\int_{-\pi}^{\pi} \frac{\sin^2 t}{\pi}\, dt} = 1,$$

$$\|h_3\| = \sqrt{(h_3, h_3)} = \sqrt{\int_{-\pi}^{\pi} \frac{\cos^2 t}{\pi}\, dt} = 1.$$

Given an orthogonal set $g_1, g_2, \ldots$, we may attempt to represent a function $f(t)$ in terms of linear combinations of $g_1, g_2, \ldots$. That is, we attempt to write

$$f(t) = \sum_{n=1}^{\infty} c_n g_n(t)$$

with the constants $c_1, c_2, \ldots$ suitably chosen. To see how the c_n should be chosen, proceed as follows: Multiply by g_m on both sides and integrate:

$$(f, g_m) = \int_a^b f g_m \, dt = \int_a^b \sum c_n g_n g_m \, dt = \sum c_n \int_a^b g_n g_m \, dt = \sum c_n (g_n, g_m),$$

but

$$(g_n, g_m) = \begin{cases} 0, & n \neq m, \\ \|g_m\|^2, & n = m \end{cases}.$$

Thus,

$$(f, g_m) = c_m \|g_m\|^2 \qquad \text{and} \qquad c_m = \frac{(f, g_m)}{\|g_m\|^2}.$$

This tells us how the c_i must be chosen.

There is another difficulty. Since we started with any orthogonal system $g_1, g_2, \ldots$, there may not be enough functions in the system to represent f (even though the c_n are chosen by the above rule). For example consider the system made up of *only* the three functions $g_1 = 1$, $g_2 = \cos t$, $g_3 = \sin t$ which we know is orthogonal for $-\pi < t < \pi$. Even if we choose c_1, c_2, c_3 by the above rule, we cannot represent the function $f(t) = t$ in terms of $c_1 g_1 + c_2 g_2 + c_3 g_3$, since $t \neq c_1 + c_2 \cos t + c_3 \sin t$ for all t in $-\pi < t < \pi$. To see this just choose several values of t and note that no constants c_1, c_2, c_3 can work for all the t's. Other functions $g_4, g_5, \ldots$ must be included in the system. Hence if we are given an orthogonal system we must somehow ensure that there are "enough" functions in it so that we can represent all the functions in which we are interested. Such systems are called *complete.*

We know one such system. It is: 1, $\sin nt$, $\cos nt$ for $n = 1, 2, \ldots$ and $-\pi < t < \pi$. It is this system which was used to construct the Fourier series above and the coefficients were given precisely by the above formula for the c_m (although written differently).

Given a complete orthogonal system $g_1, g_m, \ldots$ we can write the representation

$$f = \sum_{m=1}^{\infty} c_m g_m \quad \text{with} \quad c_m = \frac{(f, g_m)}{\|g_m\|^2}.$$

This is called the *generalized Fourier series* for f. The problem of deciding when we have a complete system of orthogonal functions is beyond the scope of the course. In practice most of them arise as the collection of all eigenfunctions of a boundary value problem.

We also encounter systems of functions $g_1, g_2, \ldots$ which are *not* orthogonal, but such that there is a fixed positive function $p(t)$ such that

$$\int_a^b p(t) g_m(t) g_n(t)\, dt = 0, \qquad m \neq n.$$

We then say that the system $g_1, g_2, \ldots$ is *orthogonal with respect to the weight function* $p(t)$.

Given a system $g_1, g_2, \ldots$, which is orthogonal with respect to $p(t)$ and is also complete, we attempt the representation

$$f(t) = \sum_{n=1}^{\infty} c_n g_n(t).$$

To see how the c_n should now be chosen we proceed as follows: Multiply both sides by pg_m and integrate

$$\int_a^b p(t)f(t)g_m(t)\,dt = \sum_{n=1}^{\infty} c_n \left(\int_a^b p(t)g_n(t)g_m(t)\,dt \right)$$

but $\int_a^b p(t)g_n(t)g_m(t)\,dt = 0$ for $n \neq m$, and therefore

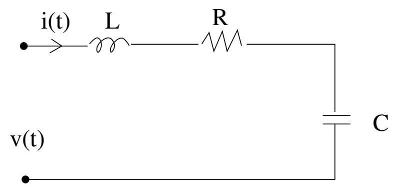
$$\int_a^b p(t)f(t)g_m(t)\,dt = c_m \int_a^b p(t)g_m^2(t)\,dt$$

and

$$c_m = \frac{\int_a^b p(t)f(t)g_m(t)\,dt}{\int_a^b p(t)g_m^2(t)\,dt}.$$

IV.g *An Application: Resonance Effects and Harmonics (Optional)*

We conclude with an application of how some of the Fourier series results can be used. Specifically we investigate the effects of harmonics when a circuit is driven by non-sinusoidal wave forms. To be specific consider the simple series LRC circuit shown:

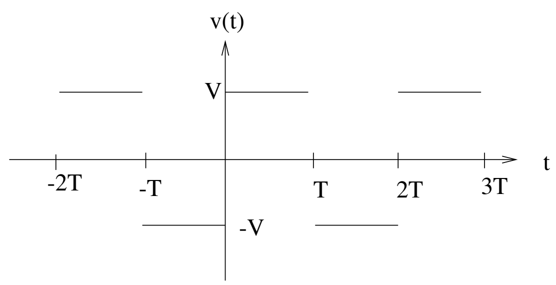


with small resistance R and for which we can disregard transients.

We assume the circuit is driven by a square wave:

$$v(t) = V \cdot \begin{cases} 1, & 0 < t < T \\ -1, & -T < t < 0 \end{cases}$$

extended by periodicity so that $v(t)$ is as shown



Our purpose is to study $i(t)$ as T varies. We could attempt to do this by Laplace transform methods but this does not seem a good way for our purposes. Observe first that we can expand $v(t)$ as a Fourier sine series, since v is odd. Specifically: $a_n = 0$, and

$$b_n = \frac{2}{T} \int_0^T \sin \frac{n\pi t}{T}\, dt = \frac{2}{n\pi} \left[-\cos \frac{n\pi t}{T} \right]_{t=0}^{T}$$

$$= \frac{2}{n\pi}[1 - (-1)^n] = \begin{cases} \frac{4}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}.$$

So

$$v(t) = V \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin \frac{(2n-1)\pi t}{T},$$

and we can view $v(t)$ as a sum of series. We are thus led to calculating the response of the circuit to a sine input, say $v_1(t) = V \sin \omega t$. To practice, let us do this by differential equations. We have

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(r)\, dr = V \sin \omega t.$$

Differentiating gives

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \omega V \cos \omega t.$$

Since we disregard transients, we attempt a solution $i(t) = A \sin \omega t + B \cos \omega t$, with A, B to be found. Substitution into the equation gives

$$\left(\frac{1}{C} - \omega^2 L\right)[A \sin \omega t + B \cos \omega t] + R\omega[A \cos \omega t - B \sin \omega t] = \omega V \cos \omega t$$

or (equating $\sin \omega t$, $\cos \omega t$ coefficients on both sides)

$$A\left[\frac{1}{C} - L\omega^2\right] - R\omega B = 0$$

$$R\omega A + B\left[\frac{1}{C} - L\omega^2\right] = \omega V.$$

That is,

$$B(\omega) = \frac{\omega V[\frac{1}{C} - L\omega^2]}{[\frac{1}{C} - L\omega^2]^2 + R^2\omega^2}$$

$$A(\omega) = \frac{R\omega^2 V}{[\frac{1}{C} - L\omega^2]^2 + R^2\omega^2}$$

where we have written $A(\omega)$, $B(\omega)$ to indicate that A, B depend on ω. We recall and these formulas indicate, that the resonance frequency is given by $\omega = 1/\sqrt{LC}$ *when the circuit is driven by* $\sin \omega t$*!* We return to our case. Observe that the frequency of the driving square wave voltage is $\omega_0 = \pi/T$, so

$$v(t) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin[(2n-1)\omega_0 t],$$

i.e., $v(t)$ is the (infinite) sum of sines and thus

$$\begin{aligned} i(t) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \{&A((2n-1)\omega_0) \sin[(2n-1)\omega_0 t] \\ &+ B((2n-1)\omega_0) \cos[(2n-1)\omega_0 t]\}. \end{aligned}$$

We can calculate the RMS value of i in terms of the coefficients:

$$\begin{aligned} \frac{1}{2T} \int_{-T}^{T} i^2(t)\, dt &= \sum_{n=1}^{\infty} \frac{4^2}{(2n-1)^2\pi^2} \{A^2((2n-1)\omega_0) + B^2((2n-1)\omega_0)\} \\ &= \sum_{n=1}^{\infty} \frac{4^2}{(2n-1)^2\pi^2} \frac{[(2n-1)\omega_0 V]^2}{[\frac{1}{C} - L(2n-1)^2\omega_0^2] + R^2\omega_0^2(2n-1)^2} \\ &= \sum_{n=1}^{\infty} \frac{16}{\pi^2} \cdot \frac{\omega_0^2 V^2}{[\frac{1}{C} - L(2n-1)^2\omega_0^2] + R^2\omega_0^2(2n-1)^2}. \end{aligned}$$

Recall that $\omega_0 = \pi/T$. Suppose that for some *small* n_0, we have $(2n_0 - 1)\omega_0 = 1/\sqrt{LC}$. That is, one of the harmonics of ω_0 (but *not* ω_0 itself) is the resonant frequency. Then since every term in the sum is positive, we have

$$\frac{1}{2T} \int_{-T}^{T} i^2(t)\, dt \geq \frac{16}{\pi^2} \cdot \frac{\omega_0^2 V^2}{R^2\omega_0^2(2n_0-1)^2} = \frac{16V^2}{\pi^2 R^2 (2n_0-1)^2}.$$

If R is small, the RMS value of $i(t)$ is very large even though the frequency ω_0 of the driving voltage is a fraction of the resonant frequency one would expect, specifically, $\omega_0 = \frac{1}{\sqrt{LC}} \cdot \frac{1}{(2n_0-1)}$. The key is that the square wave is a sum of sines, and while

ω_0 is *not* $1/\sqrt{LC}$, **one** of the sines composing $v(t)$ is indeed of this frequency. It is this component that causes the resonance.

The same phenomenon is observed when a circuit is driven by other non-sinusoidal inputs: triangular, saw tooth, etc.