Math 309 Spring - Summer 2018

Mathematical Techniques for Electrical Engineers

Solutions to Midterm Examination II

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Question 1. Let $w = \dfrac{1-z}{1+z}$, show that

$$\mathrm{Re}(w) = \frac{1-|z|^2}{|1+z|^2}$$

for $z \neq -1$.

Solution: We have $\overline{w} = \dfrac{1-\overline{z}}{1+\overline{z}}$, so that

$$2\mathrm{Re}(w) = w + \overline{w} = \frac{1-z}{1+z} + \frac{1-\overline{z}}{1+\overline{z}} = \frac{(1-z)(1+\overline{z}) + (1+z)(1-\overline{z})}{(1+z)(1+\overline{z})} = \frac{2-2|z|^2}{(1+z)(\overline{1+z})} = \frac{2-2|z|^2}{|1+z|^2},$$

and

$$\mathrm{Re}(w) = \frac{1-|z|^2}{|1+z|^2}.$$

Question 2. Show that the function

$$v(x,y) = e^{x^2-y^2} \cdot \sin 2xy$$

is harmonic and find an analytic function $f(z)$ such that $v(x,y)$ is the imaginary part of $f(z)$.

Solution: Note that if $z = x+iy$, then $z^2 = x^2 - y^2 + 2ixy$, so that

$$e^{z^2} = e^{x^2-y^2} e^{2ixy} = e^{x^2-y^2}(\cos 2xy + i \sin 2xy),$$

that is,

$$f(z) = e^{z^2} = e^{x^2-y^2}\cos 2xy + ie^{x^2-y^2}\sin 2xy,$$

and $v(x,y) = e^{x^2-y^2} \cdot \sin 2xy$ is the imaginary part of the analytic function $f(z) = e^{z^2}$.

Question 3. Show directly that if w is any value of the mutliple valued function

$$f(z) = -i \log\left[iz + \sqrt{1-z^2}\right],$$

then $\sin w = z$.

Solution: If $w = -i \log \left[iz + \sqrt{1-z^2} \right]$, then

$$\begin{aligned}
\sin w &= \frac{e^{i(-i\log[iz+\sqrt{1-z^2}])} - e^{-i(-i\log[iz+\sqrt{1-z^2}])}}{2i} \\
&= \frac{e^{\log[iz+\sqrt{1-z^2}]} - e^{-\log[iz+\sqrt{1-z^2}]}}{2i} \\
&= \frac{1}{2i}\left[iz + \sqrt{1-z^2} - \frac{1}{iz+\sqrt{1-z^2}} \right] \\
&= \frac{1}{2i}\left[\frac{-z^2 + 2iz\sqrt{1-z^2} + 1 - z^2 - 1}{iz+\sqrt{1-z^2}} \right] \\
&= \frac{1}{2i}\left[\frac{2iz\sqrt{1-z^2} - 2z^2}{iz+\sqrt{1-z^2}} \right] \\
&= \frac{1}{2i}\left[\frac{2iz(\sqrt{1-z^2} + iz)}{iz+\sqrt{1-z^2}} \right] \\
&= \frac{2iz}{2i} = z,
\end{aligned}$$

and $\sin w = z$.

Question 4. Find all solutions of $\cosh z = \dfrac{1}{2}$.

Hint: $\cosh z = \cosh x \cos y + i \sinh x \sin y$.

Solution: Note that

$$\cosh z = \cosh x \, \cos y + i \, \sinh x \, \sin y = \frac{1}{2}$$

if and only if

$$\begin{aligned}
\cosh x \, \cos y &= \frac{1}{2} \\
\sinh x \, \sin y &= 0.
\end{aligned}$$

Now, if $\sin y = 0$, then $\cos y = \pm 1$ and this implies that $\cosh x = \pm\dfrac{1}{2}$, which is impossible, since $\cosh x \geq 1$ for all real numbers x. Therefore, $\sin y \neq 0$, and from the second equation we must have $\sinh x = 0$, so that $x = 0$. From the first equation we have $\cos y = \dfrac{1}{2}$, so that $y = \pm\dfrac{\pi}{3} + 2n\pi$.

Therefore, $\cosh z = \dfrac{1}{2}$ if and only if

$$z = i\left(\pm\frac{\pi}{3} + 2n\pi\right)$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Question 5. For $z = x + iy \in \mathbb{C}$, let $f(z) = \cos x \, \cos y + i \sin x \, \sin y$.

(a) Find all points (x, y) for which the Cauchy-Riemann equations are satisfied.

(b) Find all points $z \in \mathbb{C}$ at which f is differentiable and compute $f'(z)$ for those points.

(c) Find all points $z \in \mathbb{C}$ at which f is analytic.

SOLUTION:

(a) Here $u(x, y) = \cos x \cos y$ and $v(x, y) = \sin x \sin y$, so that

$$\frac{\partial u}{\partial x} = -\sin x \cos y \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \sin x \cos y,$$

while

$$\frac{\partial u}{\partial y} = -\cos x \sin y \qquad \text{and} \qquad \frac{\partial v}{\partial x} = \cos x \sin y.$$

Thus,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

for all x and y, while

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{if and only if} \qquad \sin x \cos y = 0.$$

Therefore, the Cauchy-Riemann equations hold at (x_0, y_0) if and only if

$$x_0 = m\pi \quad \text{for some integer } \ m,$$

or

$$y_0 = \frac{(2n+1)\pi}{2} \quad \text{for some integer } \ n,$$

that is, if and only if

$$(x_0, y_0) \in D = \big\{(x, y) \,\big|\, x = m\pi, \ m \in \mathbb{Z}, \ y \in \mathbb{R}\big\} \cup \big\{(x, y) \,\big|\, x \in \mathbb{R}, \ y = \tfrac{(2n+1)\pi}{2}, \ n \in \mathbb{Z}\big\}.$$

(b) Since u and v have continuous partial derivatives at all points (x, y), then $f'(z)$ exists only at those points $(x_0, y_0) \in D$ where the Cauchy-Riemann equations hold, and

$$f'(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = -\sin x_0 \cos y_0 + i \cos x_0 \sin y_0$$

for $(x_0, y_0) \in D$.

(c) The function $f(z)$ is analytic nowhere, since the set D where $f'(z)$ exists has no interior points.

Question 6. Let a and b be positive real numbers. Find the real and imaginary parts

of the integral

$$I = \int_0^{\pi} e^{(a+ib)t} \, dt$$

and use them to evaluate the real integrals

$$\int_0^{\pi} e^{ax} \cos bx \, dx \qquad \text{and} \qquad \int_0^{\pi} e^{ax} \sin bx \, dx.$$

SOLUTION: We have

$$I = \int_0^\pi e^{(a+ib)t}\,dt = \frac{1}{a+ib}e^{(a+ib)t}\Big|_0^\pi$$

$$= \frac{a-ib}{a^2+b^2}\left(e^{(a+ib)\pi}-1\right)$$

$$= \frac{a-ib}{a^2+b^2}\left(e^{a\pi}\cos b\pi + ie^{a\pi}\sin b\pi - 1\right)$$

$$= \frac{(a-ib)\left[(e^{a\pi}\cos b\pi - 1) + ie^{a\pi}\sin b\pi\right]}{a^2+b^2}$$

$$= \frac{a\left(e^{a\pi}\cos b\pi - 1\right) + be^{a\pi}\sin b\pi}{a^2+b^2} + i\,\frac{ae^{a\pi}\sin b\pi - b\left(e^{a\pi}\cos b\pi - 1\right)}{a^2+b^2}.$$

Since

$$I = \int_0^\pi e^{(a+ib)t}\,dt = \int_0^\pi e^{ax}\cos bx\,dx + i\int_0^\pi e^{ax}\sin bx\,dx,$$

we have

$$\int_0^\pi e^{ax}\cos bx\,dx = \frac{a\left(e^{a\pi}\cos b\pi - 1\right) + be^{a\pi}\sin b\pi}{a^2+b^2}$$

and

$$\int_0^\pi e^{ax}\sin bx\,dx = \frac{ae^{a\pi}\sin b\pi - b\left(e^{a\pi}\cos b\pi - 1\right)}{a^2+b^2}.$$

10 pts	1	
10 pts	2	
10 pts	3	
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10 pts	5	
10 pts	6	
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