Math 309 - Spring-Summer 2017

Solutions to Problem Set # 9

Completion Date: Friday July 7, 2017

Question 1.

Show in two ways that the sequence

$$z_n = -2 + i\,\frac{(-1)^n}{n^2} \qquad (n = 1, 2, \dots)$$

converges to -2.

Solution:

(a) We have

$$|z_n + 2| = \left|\frac{i(-1)^n}{n^2}\right| = \frac{1}{n^2} \to 0$$

as $n \to \infty$.

(b) Also, $z_n = x_n + iy_n$ where $x_n = -2$ and $y_n = \dfrac{(-1)^n}{n^2}$ for $n \geq 1$, and

$$x_n \to -2 \qquad \text{and} \qquad y_n \to 0$$

as $n \to \infty$, and again $z_n \to -2 + 0 \cdot i = -2$ as $n \to \infty$.

Question 2.

Let r_n denote the moduli and Θ_n the principal values of the arguments of the complex numbers

$$z_n = -2 + \frac{i(-1)^n}{n^2}$$

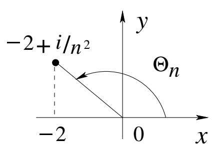
for $n \geq 1$. Show that the sequence r_n $(n = 1, 2, \dots)$ converges but that the sequence Θ_n $(n = 1, 2, \dots)$ does not.

Solution: If $z_n = -2 + \dfrac{i(-1)^n}{n^2}$ for $n \geq 1$, then

$$r_n = |z_n| = \sqrt{4 + \frac{1}{n^4}} \to \sqrt{4} = 2$$

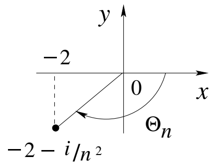
as $n \to \infty$. However, if $\Theta_n = \mathrm{Arg}(z_n)$, $-\pi < \Theta_n \leq \pi$ for $n \geq 1$, then:

For n even, we have $\Theta_n = \pi - \sin^{-1}\left(\dfrac{1}{\sqrt{1+4n^4}}\right)$



and $\Theta_n \to \pi$ as $n \to \infty$ through even values.

For n odd, we have $\Theta_n = -\pi + \sin^{-1}\left(\dfrac{1}{\sqrt{1+4n^4}}\right)$



and $\Theta_n \to -\pi$ as $n \to \infty$ through odd values.

Therefore, $\lim\limits_{n\to\infty} r_n = 2$, but $\lim\limits_{n\to\infty} \Theta_n$ doesn't exist.

Question 3.

(a) Show that if the sequence $\{z_n\}_{n\geq 1}$ converges, then $(z_n - z_{n-1}) \to 0$ as $n \to \infty$.

(b) Let $z_0 \neq 0$. Show that the sequence $\{(z/z_0)^n\}_{n\geq 1}$ diverges if $|z| = |z_0|$ and $z \neq z_0$.

Hint: For $|z| = |z_0|$, show first that

$$\left|\left(\frac{z}{z_0}\right)^n - \left(\frac{z}{z_0}\right)^{n-1}\right| = \left|\frac{z}{z_0} - 1\right| > 0,$$

and use the result of part (a).

Solution:

(a) Suppose that $\lim\limits_{n\to\infty} z_n$ exists and equals α, then given $\epsilon > 0$, there exists a positive integer n_0 such that $|z_n - \alpha| < \epsilon/2$ whenever $n - 1 > n_0$. Therefore,

$$|z_n - z_{n-1}| = |z_n - \alpha + \alpha - z_{n-1}| \leq |z_n - \alpha| + |z_{n-1} - \alpha| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whenever $n > n - 1 > n_0$. That is, $\lim\limits_{n\to\infty} |z_n - z_{n-1} - 0| = 0$, so that $\lim\limits_{n\to\infty} (z_n - z_{n-1}) = 0$.

(b) From the hint and part (a), it is not possible for limit to exist, since this would imply that

$$\left|\frac{z}{z_0} - 1\right| = 0,$$

which contradicts the fact that $z \neq z_0$.

Question 4.

Show that

$$\text{if} \qquad \lim_{n\to\infty} z_n = z, \qquad \text{then} \qquad \lim_{n\to\infty} |z_n| = |z|.$$

Solution: If $\lim\limits_{n\to\infty} z_n = z$, then from the back end of the triangle inequality, we have

$$\big||z_n| - |z|\big| \leq |z_n - z|,$$

and given $\epsilon > 0$, choose n_0 so that $|z_n - z| < \epsilon$ for all $n \geq n_0$, then

$$\big||z_n| - |z|\big| \leq |z_n - z| < \epsilon$$

for all $n \geq n_0$, and therefore $\lim\limits_{n\to\infty} |z_n| = |z|$.

The converse is **not** true, for example, if

$$z_n = 1 + i(-1)^n$$

for $n \geq 1$, then

$$|z_n| = \sqrt{1+1} = \sqrt{2},$$

for all $n \geq 1$, so that $\lim\limits_{n\to\infty} |z_n| = \sqrt{2}$ exists, but the sequence $\{z_n\}_{n\geq 1}$ does not converge (why?).

Question 5.

Obtain the Maclaurin series representation

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z| < \infty).$$

Solution: The Maclaurin series expansion for $z\cosh(z^2)$ follows from the Maclaurin series expansion for $\cosh z$, namely

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \quad |z| < \infty$$

by replacing z by z^2 to get $\cosh(z^2) = \sum\limits_{n=0}^{\infty} \dfrac{z^{4n}}{(2n)!}$, and then multiplying by z to get

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}, \quad |z| < \infty.$$

Question 6.

Obtain the Taylor series

$$e^z = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z-1| < \infty)$$

for the function $f(z) = e^z$ by

(a) using $f^{(n)}(1) \quad (n = 0, 1, 2\ldots)$;

(b) writing $e^z = e^{z-1}e$.

Solution:

(a) If $f(z) = e^z$, then $f^{(n)}(z) = e^z$ for all $n \geq 0$, so that $f^{(n)}(1) = e$ for all $n \geq 0$, and therefore

$$f(z) = e^z = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

for $|z-1| < \infty$.

(b) Also, replacing z by $z-1$ in the Maclaurin series for e^z, we have

$$e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!},$$

and

$$e^z = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

for $|z-1| < \infty$.

Question 7.

Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9} \cdot \frac{1}{1+(z^4/9)}.$$

Ans: $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1} \qquad (|z| < \sqrt{3})$.

Solution: From the geometic series we have

$$f(z) = \frac{z}{9} \cdot \frac{1}{1+(z^4/9)} = \frac{z}{9} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n}}{9^n}$$

for $|z|^4 < 9$, that is,

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{9^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{3^{2n+2}}$$

for $|z| < \sqrt{3}$.

Question 8.

With the aid of the identity

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right),$$

expand $\cos z$ into a Taylor series about the point $z_0 = \pi/2$.

Solution: The Maclaurin series for $\sin z$, valid for all $z \in \mathbb{C}$ is

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!},$$

and replacing z by $\left(z - \frac{\pi}{2}\right)$, and using the identity above, we have

$$\cos z = -\sum_{n=0}^{\infty} \frac{(-1)^n \left(z - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \left(z - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}$$

for all $z \in \mathbb{C}$.

Question 9.

What is the largest circle within which the Maclaurin series for the function $\tanh z$ converges to $\tanh z$? Write the first two nonzero terms of that series.

SOLUTION: Since

$$\tanh z = \frac{\sinh z}{\cosh z},$$

then the largest circle within which $\tanh z$ is analytic is the one whose radius equals the distance from the origin to the closest zero of $\cosh z$.

Now, since

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y = 0$$

if and only if

$$x = 0 \qquad \text{and} \qquad y = \left(n + \tfrac{1}{2}\right)\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

the zeros of $\cosh z$ are

$$z = i\left(n + \tfrac{1}{2}\right)\pi, \quad n = 0, \pm 1, \pm 2, \ldots.$$

The zeros of $\cosh z$ closest to the point $z = 0$ are

$$z_0 = \frac{\pi i}{2} \qquad \text{and} \qquad z_{-1} = -\frac{\pi i}{2},$$

and $f(z) = \tanh z$ is analytic in the interior of the disk $|z| = \frac{\pi}{2}$.

Since $f(z) = \tanh z = -f(-z)$, then only odd powers of z appear in the Maclaurin series for $f(z)$, and

$$f'(z) = \frac{1}{\cosh^2 z} \qquad \text{and} \qquad f'(0) = 1.$$

Since

$$f''(z) = -\frac{2\sinh z}{\cosh^3 z} = -2\tanh z \,\mathrm{sech}^2 z.$$

then

$$f'''(z) = \frac{-2(1 - 2\sinh^2 z)}{\cosh^4 z} \qquad \text{and} \qquad f'''(0) = -2.$$

Therefore, the first two nonzero terms of the Maclaurin series for $f(z) = \tanh z$ are

$$f(z) = \tanh z = 1 \cdot z - \frac{2}{3!} \cdot z^3 + \cdots = z - \frac{z^3}{3} + \cdots$$

Question 10.

Show that when $z \neq 0$,

$$\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots.$$

SOLUTION: We have

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

for $|z| < \infty$, and therefore, if $z \neq 0$, then

$$\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots$$

for $0 < |z| < \infty$.

Question 11.

Represent the function

$$f(z) = \frac{z+1}{z-1}$$

(a) by its Maclaurin series, and state where the representation is valid;

(b) by its Laurent series in the domain $1 < |z| < \infty$.

Ans: (a) $-1 - 2\sum_{n=1}^{\infty} z^n \quad (|z| < 1)$; (b) $1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n}$.

SOLUTION:

(a) For the Maclaurin series, if $|z| < 1$, then

$$f(z) = \frac{z+1}{z-1} = -\frac{1+z}{1-z} = -(1+z)(1+z+z^2+z^3+\cdots) = -(1+z+z^2+z^3+\cdots) - (z+z^2+z^3+\cdots),$$

and

$$f(z) = -(1+2z+2z^2+2z^3+\cdots) = -1 - 2\sum_{n=1}^{\infty} z^n$$

for $|z| < 1$.

(b) For the Laurent series, if $1 < |z| < \infty$, then

$$f(z) = \frac{z+1}{z-1} = \frac{z+1}{z\,(1-1/z)} = \frac{1+z}{z} \cdot \frac{1}{1-1/z},$$

and

$$f(z) = \left(1 + \frac{1}{z}\right)\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right],$$

that is,

$$f(z) = 1 + \frac{2}{z} + \frac{2}{z^2} + \frac{2}{z^3} + \cdots = 1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n}$$

for $|z| > 1$.

Question 12.

Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

SOLUTION: We want the series expansion of $\frac{z}{(z-1)(z-3)}$ **inside** the circle $|z-1| = 2$, excluding the point $z = 1$.

Look at

$$\frac{z}{z-3} = \frac{z-3+3}{z-3} = 1 + \frac{3}{z-3} = 1 + \frac{3}{z-1-2} = 1 - \frac{3}{2-(z-1)} = 1 - \frac{3}{2} \cdot \frac{1}{1 - \dfrac{z-1}{2}}$$

for $z \neq 3$.

For $|z-1|<2$, we have

$$\frac{z}{z-3} = 1 - \frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n,$$

so that for $0<|z-1|<2$, we have

$$\frac{z}{(z-1)(z-3)} = \frac{1}{z-1} - \frac{3}{2}\cdot\frac{1}{z-1} - \frac{3}{4}\sum_{n=1}^{\infty}\left(\frac{z-1}{2}\right)^{n-1},$$

that is,

$$\frac{z}{(z-1)(z-3)} = -\frac{1}{2(z-1)} - 3\sum_{m=0}^{\infty}\frac{(z-1)^m}{2^{m+2}}$$

for $0<|z-1|<2$.

Question 13.

Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

Ans: $\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1} + \frac{1}{z} \quad (0<|z|<1)$; $\qquad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{z^{2n+1}} \quad (1<|z|<\infty)$.

Solution:

(a) For $0<|z|<1$, we have

$$f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z}\left[1 - z^2 + z^4 - z^6 + z^8 - + \cdots\right],$$

that is,

$$\frac{1}{z(1+z^2)} = \frac{1}{z} - z + z^3 - z^5 + z^7 - z^9 + \cdots$$

for $0<|z|<1$.

(b) For $|z|>1$, we have

$$f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z^3}\cdot\frac{1}{1+\dfrac{1}{z^2}} = \frac{1}{z^3}\left[1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right],$$

that is,

$$\frac{1}{z(1+z^2)} = \frac{1}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} - \frac{1}{z^9} + \cdots$$

for $|z|>1$.