Math 309 - Spring-Summer 2017

Solutions to Problem Set # 7

Completion Date: Friday June 23, 2017

Question 1.

Evaluate the following integrals:

$$\text{(a)} \int_1^2 \left(\frac{1}{t} - i\right)^2 dt; \qquad \text{(b)} \int_0^{\pi/6} e^{i2t}\, dt; \qquad \text{(c)} \int_0^{\infty} e^{-zt}\, dt \quad (\text{Re } z > 0).$$

Ans: (a) $-\frac{1}{2} - i \ln 4$; (b) $\frac{\sqrt{3}}{4} + \frac{i}{4}$; (c) $\frac{1}{z}$.

SOLUTION:

(a) We have

$$\begin{aligned}\int_1^2 \left(\frac{1}{t} - i\right)^2 dt &= \int_1^2 \left(\frac{1}{t^2} - \frac{2i}{t} + i^2\right) dt = \int_1^2 \left(\frac{1}{t^2} - 1\right) dt - 2i \int_1^2 \frac{dt}{t} \\ &= -\frac{1}{t}\bigg|_1^2 - t\bigg|_1^2 - 2i \ln t\bigg|_1^2 = -\left(\frac{1}{2} - 1\right) - (2-1) - 2i(\ln 2 - \ln 1),\end{aligned}$$

and $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt = -\frac{1}{2} - 2i \ln 2 = -\frac{1}{2} - i \ln 4.$

(b) We have

$$\begin{aligned}\int_0^{\pi/6} e^{i2t}\, dt &= \int_0^{\pi/6} (\cos 2t + i \sin 2t)\, dt = \int_0^{\pi/6} \cos 2t\, dt + i \int_0^{\pi/6} \sin 2t\, dt \\ &= \frac{1}{2} \sin 2t\bigg|_0^{\pi/6} + i\left(-\frac{1}{2}\cos 2t\right)\bigg|_0^{\pi/6} = \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 0\right) - \frac{i}{2}\left(\frac{1}{2} - 1\right)\end{aligned}$$

and $\int_0^{\pi/6} e^{i2t}\, dt = \frac{\sqrt{3}}{4} + \frac{i}{4}.$

Note that

$$\begin{aligned}\int_0^{\pi/6} e^{i2t}\, dt &= \frac{1}{2i} e^{i2t}\bigg|_0^{\pi/6} = \frac{1}{2i}\left[e^{i\pi/3} - 1\right] \\ &= -\frac{i}{4} + \frac{\sqrt{3}}{4} - \frac{1}{2i} = \frac{i}{4} + \frac{\sqrt{3}}{4}\end{aligned}$$

is much easier!

(c) If $M > 0$, we have

$$\int_0^M e^{-zt}\, dt = \int_0^M e^{-(x+iy)t}\, dt = \int_0^M e^{-xt} \cdot e^{-iyt}\, dt = \int_0^M e^{-xt} \cos yt\, dt - i \int_0^M e^{-xt} \sin yt\, dt$$

and letting $M \to \infty$,

$$\int_0^\infty e^{-zt}\,dt = \int_0^\infty e^{-xt}\cos yt\,dt - i\int_0^\infty e^{-xt}\sin yt\,dt \tag{1}$$

where both integrals on the right converge since $x = \text{Re } z > 0$.

Now, since

$$\frac{d}{dt}\left(e^{-zt}\right) = -ze^{-zt},$$

then for $M > 0$, we have

$$\int_0^M e^{-zt}\,dt = -\frac{1}{z}e^{-zt}\Big|_0^M = \frac{1}{z}\left(1 - e^{-Mz}\right),$$

and since

$$\left|e^{-Mz}\right| = e^{-Mx}\cdot\left|e^{-Miy}\right| = e^{-Mx} \to 0$$

as $M \to \infty$ provided $x > 0$, then

$$\int_0^\infty e^{-zt}\,dt = \lim_{M\to\infty}\int_0^M e^{-zt}\,dt = \frac{1}{z} \tag{2}$$

provided $x = \text{Re z} > 0$.

Equating real and imaginary parts of (1) and (2), we get

$$\int_0^\infty e^{-xt}\cos yt\,dt = \frac{x}{x^2+y^2} \qquad \text{and} \qquad \int_0^\infty e^{-xt}\sin yt\,dt = -\frac{y}{x^2+y^2},$$

which should look familiar!

Question 2.

Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta}\,e^{-in\theta}\,d\theta = \begin{cases} 0 & \text{when} \quad m \neq n, \\ 2\pi & \text{when} \quad m = n. \end{cases}$$

Solution: If m, n are integers, with $m \neq n$, then

$$\begin{aligned}\int_0^{2\pi} e^{im\theta}\cdot e^{-in\theta}\,d\theta &= \int_0^{2\pi} e^{i(m-n)\theta} = \frac{1}{i(m-n)}e^{i(m-n)\theta}\Big|_0^{2\pi} \\ &= \frac{1}{i(m-n)}\left[e^{i(m-n)2\pi} - e^{i(m-n)0}\right] = \frac{1}{i(m-n)}\left[1-1\right]\end{aligned}$$

since $e^{i(m-n)2\pi} = e^{i0} = 1$ (the exponential function is periodic with period $2\pi i$).

Therefore,

$$\int_0^{2\pi} e^{im\theta} \cdot e^{-in\theta}\, d\theta = 0$$

if $m \neq n$.

Also, if $m = n$, then $e^{im\theta} \cdot e^{-in\theta} = 1$, so that

$$\int_0^{2\pi} e^{im\theta} \cdot e^{-in\theta}\, d\theta = \int_0^{2\pi} 1\, dt = 2\pi$$

if $m = n$.

Question 3.

According to the definition of integrals of complex-valued functions of a real variable,

$$\int_0^{\pi} e^{(1+i)x}\, dx = \int_0^{\pi} e^x \cos x\, dx + i \int_0^{\pi} e^x \sin x\, dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

Ans: $-(1+e^{\pi})/2, \quad (1+e^{\pi})/2.$

Solution: We have

$$\int_0^{\pi} e^{(1+i)x}\, dx = \int_0^{\pi} e^x \cos x\, dx + i \int_0^{\pi} e^x \sin x\, dx,$$

and integrating we have

$$\int_0^{\pi} e^{(1+i)x}\, dx = \frac{1}{1+i} e^{(1+i)x}\Big|_0^{\pi} = \frac{1}{1+i}\left(e^{(1+i)\pi} - 1\right),$$

so that

$$\int_0^{\pi} e^{(1+i)x}\, dx = \frac{1-i}{2}\left[e^{\pi} \cdot e^{i\pi} - 1\right] = -\frac{1-i}{2}\left(e^{\pi} + 1\right).$$

Equating real and imaginary parts, we have

$$\int_0^{\pi} e^x \cos x\, dx = -\frac{1}{2}\left(e^{\pi} + 1\right)$$

and

$$\int_0^{\pi} e^x \sin x\, dx = +\frac{1}{2}\left(e^{\pi} + 1\right).$$

Question 4.

Use parametric representations for the contour C, or legs of C, to evaluate

$$\int_C f(z)\,dz$$

when $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of

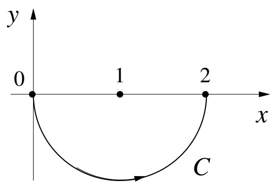
(a) the semicircle $z = 1 + e^{i\theta} \quad (\pi \le \theta \le 2\pi)$;

(b) the segment $0 \le x \le 2$ of the real axis.

Ans: (a) 0; (b) 0.

SOLUTION:

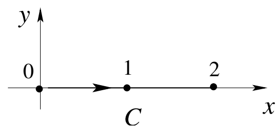
(a) If we parametrize the semicircle as $z = 1 + e^{i\theta}$, $\pi \le \theta \le 2\pi$, then we trace out the semicircle in the counterclockwise direction from the point $(0, 0)$ to the point $(2, 0)$.



We have

$$\int_C (z-1)\,dz = \int_\pi^{2\pi} e^{i\theta} \cdot ie^{i\theta}\,d\theta = i\int_\pi^{2\pi} e^{2i\theta}\,d\theta = \frac{1}{2}e^{2i\theta}\Big|_\pi^{2\pi} = 0.$$

(b) If C is the arc from $z = 0$ to $z = 2$ consisting of the segment $0 \le x \le 2$ of the real axis, then we parametrize C as $z = t$, $0 \le t \le 2$.



We have

$$\int_C (z-1)\,dz = \int_0^2 (t-1)\,dt = \frac{1}{2}(t-1)^2\Big|_0^2 = \frac{1}{2}\left[1^2 - 1^2\right] = 0.$$

Question 5.

Use parametric representations for the contour C, or legs of C, to evaluate

$$\int_C f(z)\,dz$$

when $f(z)$ is defined by the equations

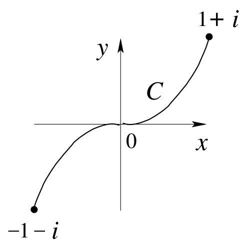
$$f(z) = \begin{cases} 1 & \text{when} \quad y < 0, \\ 4y & \text{when} \quad y > 0, \end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

Ans: $2 + 3i$.

SOLUTION: We parametrize the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$ as

$$z = t + it^3, \quad -1 \le t \le 1.$$



Since f is piecewise continuous, we have

$$\begin{aligned} \int_C f(z)\,dz &= \int_{-1}^{0} 1 \cdot (1 + 3it^2)\,dt + \int_0^1 4t^3(1 + 3it^2)\,dt \\ &= t\Big|_{-1}^{0} + it^3\Big|_{-1}^{0} + t^4\Big|_0^1 + 2it^6\Big|_0^1 \\ &= (0 - (-1)) + i\,(0 - (-1)) + (1 - 0) + 2i(1 - 0) \end{aligned}$$

and

$$\int_C f(z)\,dz = 2 + 3i.$$

Question 6.

Use parametric representations for the contour C, or legs of C, to evaluate

$$\int_C f(z)\, dz$$

when $f(z)$ is the branch

$$z^{-1+i} = \exp\left[(-1+i)\log z\right] \qquad (|z| > 0,\ 0 < \arg z < 2\pi)$$

of the indicated power function, and C is the positively oriented unit circle $|z| = 1$.

Ans: $i\left(1 - e^{-2\pi}\right)$.

SOLUTION: If we parametrize C as

$$z = e^{it}, \quad 0 < t < 2\pi,$$

then $dz = ie^{it}\, dt$ and

$$\begin{aligned}\int_C z^{-1+i}\, dz &= \int_C \exp\left[(-1+i)\log z\right]\, dz \\ &= \int_0^{2\pi} e^{(-1+i)it} i\, e^{it}\, dt \\ &= i\int_0^{2\pi} e^{-t}\, dt \\ &= -ie^{-t}\Big|_0^{2\pi},\end{aligned}$$

and

$$\int_C z^{-1+i}\, dz = i\left(1 - e^{-2\pi}\right).$$

Question 7.

Let C_0 denote the circle of radius R centered at z_0, $|z - z_0| = R$, taken counterclockwise. Use the parametric representation $z = z_0 + Re^{i\theta}$ $(-\pi \le \theta \le \pi)$ for C_0 to derive the following integration formulas:

(a) $\displaystyle\int_{C_0} \frac{dz}{z - z_0} = 2\pi i;$

(b) $\displaystyle\int_{C_0} (z - z_0)^{n-1}\, dz = 0 \quad (n = \pm 1, \pm 2, \dots).$

SOLUTION:

(a) We have

$$\int_{C_0} \frac{dz}{z - z_0} = \int_{-\pi}^{\pi} \frac{1}{R} e^{-i\theta} \cdot i\, Re^{i\theta}\, d\theta = i\int_{-\pi}^{\pi} 1\, d\theta = 2\pi i.$$

(b) We have $z - z_0 = Re^{i\theta}$, $-\pi \le \theta \le \pi$, and $dz = Rie^{i\theta}\,d\theta$, and

$$\begin{aligned}\int_{C_0} (z-z_0)^{n-1}\,dz &= \int_{-\pi}^{\pi} R^{n-1} e^{i(n-1)\theta} \cdot i\,Re^{i\theta}\,d\theta = \int_{-\pi}^{\pi} R^n i\, e^{in\theta}\,d\theta \\ &= R^n i \cdot \frac{e^{in\theta}}{ni}\bigg|_{-\pi}^{\pi} = \frac{R^n}{n}\left[e^{n\pi i} - e^{-n\pi i}\right] = 0,\end{aligned}$$

so that

$$\int_{C_0} (z-z_0)^{n-1}\,dz = 0$$

for $n = \pm 1, \pm 2, \ldots$.

Question 8.

Let C_R be the circle $|z| = R$ ($R > 1$), described in the counterclockwise direction. Show that

$$\left|\int_{C_R} \frac{\text{Log } z}{z^2}\,dz\right| < 2\pi\left(\frac{\pi + \ln R}{R}\right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

Solution: On C_R, we have $z = Re^{i\theta}$, $-\pi \le \theta \le \pi$, and

$$\text{Log } z = \ln R + i\theta, \quad -\pi < \theta < \pi,$$

so that

$$\frac{\text{Log } z}{z^2} = \frac{\ln R + i\theta}{R^2 e^{2i\theta}}$$

on C_R, and

$$\left|\frac{\text{Log } z}{z^2}\right| = \frac{|\ln R + i\theta|}{R^2} \le \frac{\ln R + |\theta|}{R^2} < \frac{\ln R + \pi}{R^2} = M$$

on C_R.

Therefore,

$$\left|\int_{C_R} \frac{\text{Log } z}{z^2}\,dz\right| < M \cdot \int_{C_R} |z'(t)|\,dt = M \cdot L$$

where $L = 2\pi R$ is the length of the contour, and

$$\left|\int_{C_R} \frac{\text{Log } z}{z^2}\,dz\right| < \left(\frac{\ln R + \pi}{R}\right) \cdot 2\pi.$$

Using l'Hospital's rule, since

$$\lim_{R\to\infty} \left(\frac{\ln R + \pi}{R}\right) = 0,$$

then

$$\lim_{R\to\infty} \int_{C_R} \frac{\text{Log } z}{z^2}\,dz = 0.$$